Selling to Advised Buyers*

Andrey Malenko†

Anton Tsoy‡

January 11, 2016

Abstract

In many cases, agents that make purchase decisions are uninformed and rely on the advice of biased experts. For example, when contemplating an acquisition, the board of the bidder relies on the advice of the managerial team when deciding what offer to make for the target. In this paper, we study how to sell assets to such “advised buyers” if the goal is (i) to maximize revenues; (ii) to maximize allocative efficiency. In static mechanisms, such as first- and second-price auctions, advisors communicate a coarsening of information and a version of the revenue equivalence theorem holds. In contrast, dynamic mechanisms, such as multiple-round auctions, result in more informative communication between buyers and their advisors, which leads to more efficient allocations. Whether this leads to higher revenues depends on the direction of the bias. When advisors are biased toward overpaying, an ascending-price auction dominates any static mechanism in terms of both efficiency and expected revenues. When advisors are biased towards underpaying, a descending-price auction dominates static mechanisms in terms of efficiency but often results in lower revenues.

*We thank Alessandro Bonatti, Peter DeMarzo, Doug Diamond, Glenn Ellison, Bob Gibbons, Navin Kartik, Nadya Malenko, Jonathan Parker, Konstantin Sorokin, Jean Tirole, Juuso Toikka, and participants at the MIT Theory Lunch, AMMA 2015, 2015 Stony Brook International Conference on Game Theory, EIEF, Finance Theory Group (UT Austin), Mechanism Design Workshop at NES for helpful comments.

†MIT Sloan School of Management. Email: amalenko@mit.edu.

‡Einaudi Institute of Economics and Finance. Email: tsoianton.ru@gmail.com.
1 Introduction

In many applications, agents that make purchase decisions have limited information about their valuations of the assets for sale. As a consequence of limited information, they often rely on the advice of informed experts, who however often have misaligned preferences. Consider the following examples:

1. A firm competing for a target in a takeover contest. While the board of directors typically has formal authority over submitting offers for the target, the firm’s managers are more informed about the valuation of the target. The managers, however, could be prone to overbidding because of career concerns and empire building preferences. A similar conflict of interest occurs if the board consults an investment banker.

2. Bidding in spectrum auctions. Large telecommunication companies bidding in spectrum auctions have research teams in charge of preparing for the auction, producing valuation estimates of auctioned frequencies, and advising the top management and board of directors on bidding. These research teams may have different incentives as winning the auction could give a positive signal to the market and help the research team attract future business.

3. Suppliers competing in procurement. When a construction company competes on cost for a project in a procurement auction, managers who will work on a project are privately informed about the actual cost of the project for the firm, while the top management of the company makes offers for the project. If the cost at which the construction company determines the budget that managers will operate with, they have a bias for overstating the cost.

4. Realtors in real estate transactions. A potential buyer of a house gets advice from a realtor about what offer to make. The realtor has information about the value of the house that the buyer does not have. The realtor may be biased toward overpaying, since she is compensated as a percentage of the transaction price and conditional on the transaction happening.

These examples have a common element: There is a separation of authority over bidding decisions and information about the buyer’s valuation. We call such players “advised buyers” and ask the following question in the paper: Do agency problems between buyers and their advisors affect how the seller should design the sale process? We analyze this question both
from the position of maximizing expected revenues, which is likely the goal if the designer is the seller, and from the position of allocative efficiency, which could be a more important concern than revenues if the designer is the government.

We study a canonical setting where the seller has an asset for sale to auction among a number of potential buyers who have independent private values. But we depart from the canonical setting in one crucial aspect: We assume that each potential buyer is a bidder-advisor pair, where the bidder is the party with formal authority over bidding decisions (e.g., the board of the firm) but without information about her valuation, while the advisor (e.g., the firm’s manager) is the party that knows the valuation of the bidder but has a conflict of interest. Our initial focus is on the case where advisors have a bias toward overbidding, that is, given the same information, the advisor’s maximum willingness to pay exceeds that of the bidder she advises by $b > 0$. Later we consider the case where the advisor’s bias is toward underbidding, which could be more relevant in procurement.

This overbidding bias captures empire-building motives or career concerns in the examples above. Before the bidder submits an offer, the advisor communicates with the bidder via a game of cheap talk. Specifically, if the sale process consists of a single round, there is only one round of communication. In contrast, if the sale process consists of multiple rounds, the advisor can communicate with the bidder at each stage of the auction. In this environment, there is an interesting interaction between communication and the design of the sale process. On one hand, the amount of information transmitted from each advisor to her bidder affects bids submitted in the auction and through them the efficiency and revenue of each auction format. On the other hand, the auction format affects the incentives of advisors to reveal their information to their bidders.

We analyze equilibria of the model under the NITS (no incentive to separate) condition adapted from Chen, Kartik, and Sobel (2008). In fact, even if the NITS condition does not select the unique equilibrium in the communication game, our analysis about the comparison of auctions does not require any further refinement. We first study static (i.e., single-round) auctions. As one could expect from the standard game of “cheap talk” (Crawford and Sobel, 1982), communication in this case takes a partition form: all types of the advisor are partitioned into intervals and types in each interval induce the same bid. Even though our game is not a special case of Crawford and Sobel (1982), since the payoffs of the buyer and her advisor are endogenous, their logic directly applies here. Despite the endogenous cheap-talk

\[1\]

Intuitively, the NITS condition says that the weakest type (i.e., the lowest valuation if the advisor has an overbidding bias) has the option to credibly reveal herself at any stage of the auction, which puts a lower bound on her expected payoff at any stage.
game, we show that efficient equilibria of the static auction still have a partition structure with an upper bound on the number of partition intervals, and that the most informative equilibrium satisfies the NITS condition. We prove a version of the revenue equivalence theorem for static auctions: modulo the existence of efficient equilibria in a particular static auction, all efficient equilibria of any static auction bring the same expected revenue and generate the same communication. In other words, in the class of static sale procedures, it is not possible to manipulate the rules of the auction to extract extra revenues or induce better communication. Thus, among static auctions, advising relationships do not matter for the auction design, at least in the canonical environment of independent private values.

Surprisingly, this conclusion drastically changes if the asset is sold via dynamic mechanisms, such as multi-round auctions. To see this, consider an ascending-bid (English) auction, where the price continuously increases until only one bidder remains in the auction who then pays the price at which the previous bidder dropped out. From the position of a potential buyer, the ascending-bid auction is a stopping time problem: At what price level to drop out. Information transmission is perfect at the bottom of the type distribution and there is pooling at the top in a sense that types at the top induce the same bid. Because of the superior information transmission, the English auction generally outperforms any static auction both in terms of efficiency and expected revenue. In particular, the revenue equivalence does not hold between static and dynamic auctions.

The key distinction of the English auction is that the advisor can reveal information over time. Under the simplest communication protocol, the advisor reveals her information to the bidder right before the advisor’s optimal quitting time. Under such a communication protocol, perfect information transmission is possible for types at the bottom of the distribution for the following reason. If the bidder observes her values, then it is optimal for bidders to quit the auction when the running price equals her value. Because of the overbidding bias, the advisor prefers to quit the auction later than the bidder. If the advisor perfectly reveals the value at her optimal quitting time, then it is optimal for the bidder to immediately quit the auction. Indeed, at this point the bidder is already past her break-even price and any further delay will result into a higher chance of winning at a price that brings negative profit. Because in the English auction the bidder is restricted to submit only bids higher than the current auction price, types at the bottom of the distribution are able to communicate their private information perfectly and induce the bidder to quit at their optimal price.

However, even in the English auction, information cannot be transmitted perfectly for all types when the support of the distribution of values is finite. As the price of the auction
approaches the highest possible valuation of the asset, the uncertainty of each bidder about his value decreases. At some point, he can accurately predict his value as well as the fact that he will overpay for the asset if he wins, because the advisor waits until the advisor’s optimal price to quit. Therefore, the bidder will always quit when her uncertainty is sufficiently reduced before types of the advisor at the top reveal themselves.

The information transmission affects the efficiency of auction formats. Because of imperfect information transmission, static auctions are necessarily inefficient, as ties occur with positive probability. At the same time, in the English auction the information transmission for types at the bottom of the distribution is perfect, hence allocation is more efficient for these types compared to static auctions. It turns out that even taking into account the pooling at the top, the English auction is always more efficient than any static auction, as no static auction makes these types at the top separate even partially.

The information transmission also affects the revenue of auction formats. Under the assumption that the distribution of valuations satisfies the monotone hazard ratio property, the expected revenue of the auction is higher in the English auction than in any static auction format. Hence, when advisors are biased toward overbidding, there is no trade-off between revenue and efficiency: the English auction dominates any static format on both dimensions. This fact has an important practical implication. While in many contexts the expected revenue is the primary objective of the seller, in other contexts, such as FCC auctions and privatizations, efficiency can be an equally important or even the primary goal. Moreover, the bias for overbidding is relevant in many applications, because of the empire-building and career concerns described above. Our results suggest that the English auction is the preferred method of selling assets in this environment no matter whether the seller is concerned about efficiency, revenue or both.

The intuition for the higher revenue comes from the fact that the seller would prefer to sell directly to advisors, as they have a higher willingness to pay for the asset. However, because bidders have formal authority over bidding and advisors can only affect them through the information they provide, the equilibrium bids reflect a mix of interests of bidders and advisors, and so, are lower. The English auction is an auction format that allows the seller to essentially eliminate bidders and sell directly to advisors, as bids are optimal for advisors.

Do dynamic auction also dominate static auctions if advisors are biased towards underbidding? Surprisingly, the answer is no: the comparison of dynamic and static auctions is ambiguous in this case. If the bidder knew his value, he would submit a bid that wins with higher probability than an optimal bid of the advisor. Hence, with bias toward underbidding,
the Dutch auction that restricts bidders to submit bids not higher than the current price of the auction allows for a better information transmission. We construct an equilibrium of the Dutch auction that exhibits pooling at the bottom and perfect information transmission at the top of the distribution. This equilibrium is more efficient than any efficient equilibrium of any static auction, but it can bring lower expected revenue to the seller. The reason for this is that when the advisor is biased toward underbidding, selling directly to advisors no longer guarantees the highest expected revenue, as advisors have lower willingness to pay. Because of that, it is possible that the seller benefits from imperfect communication between the advisor and the bidder, as it results into an upward bias of bids relative to the bids submitted directly by advisors.

This paper is related to two strands of the literature: The literature on the comparison of auction formats and the literature on communication of non-verifiable information. First, it is related to the literature on communication of non-verifiable information (cheap talk), pioneered by Crawford and Sobel (1982). This literature usually focuses on exogenous payoffs of players from the decision made by the decision-maker (buyer in our context) and exogenous timing of the game (typically, one round of communication). In contrast, the payoffs and the game itself are endogenous. In particular, we show how by using communication over time, the seller can make communication between bidders and their advisors more efficient, which sometimes (but not always) leads to higher revenues. The result that decisions over the timing lead to very different equilibria than static decisions due to irreversibility of time is due to Grenadier, Malenko, and Malenko (2015). The new features of our model is that there are multiple sender-receiver (bidder-advisor) pairs, and the design of the game is strategic by the seller. A number of papers study cheap talk models with other, less related to ours, dynamic aspects of communication.2

Second, the paper is related to the mechanism design literature on the comparison of auction formats. The central result in the literature is the celebrated revenue equivalence theorem in the independent private values setting (Myerson 1981; Samuelson, 1981). As we show, it continues to hold when bidders are “advised buyers” and sale mechanisms are static, but breaks down when they are multi-round. The revenue equivalence theorem can fail for other reasons, such as affiliation of values (Milgrom and Weber, 1982), bidder asymmetries (Maskin and Riley, 2000), or budget constraints (Che and Gale, 1998). To our knowledge, we are the first to study the problem of the design of sale procedures when potential buyers are advised by informed experts. A few of papers study related aspects. Compte and Jehiel

---

(2007) show that the ascending-bid auction brings higher revenues than static auctions when bidders can acquire information about their values. The key conditions for this result are the asymmetry of bidders in information endowments and knowledge of the number of remaining bidders. Our paper provides a different rationale for multiple-round auctions, which does not rely on bidder asymmetries and knowledge of the number of remaining bidders. Burkett (2015) studies a principal-agent relationship in the auction context when the principal decides on the budget of the biased agent who submits the bid. He shows that the ranking of first- and second-price auctions usually agrees with those from models without budget constraints. Our result on revenue equivalence for static auctions is related to his. Finally, several papers study other effects of cheap talk communication in a mechanism design and/or trading environments. Matthews and Postlewaite (1989) study pre-play communication in a two-person double auction. Ye (2007) and Quint and Hendricks (2013) study two-stage auctions, where the actual bidding is preceded by the indicative stage, which is a form of cheap talk between bidders and the seller. Kim and Kircher (2015) study how auctioneers with private reservation values compete for potential bidders by announcing cheap-talk messages. Several papers study the role of cheap-talk communication in non-auction trading environments.\(^3\)

The structure of the paper is the following. Section 2 introduces the model and illustrates our main findings with a simple example. Section 4 characterizes equilibria of static auctions and establishes a version of the revenue equivalence for static auctions. Section 5 characterizes equilibria of the English auction under the NITS condition when bidders have overbidding bias, and shows that the English auction outperforms any static auction. Section 6 analyzes the case of advisors’ preferences for underbidding. Section 7 shows that the effect of the advisors’ bias on the efficiency and revenues is quantitatively significant. Section 8 concludes. Key proofs are provided in the text and the rest of the proofs are relegated to Appendix.

2 Model

Consider the standard setting with independent private values. There is a single indivisible asset for sale. The value of the asset to the seller is normalized to zero. There are \(N\) ex-ante identical potential buyers (bidders). The valuation of bidder \(i\), \(v_i\), is an i.i.d. draw from distribution with c.d.f. \(F\) and p.d.f. \(f\). The distribution \(F\) has full support on \([v, \bar{v}]\) with \(0 \leq v < \bar{v} \leq \infty\). In the analysis, we will frequently refer to the distribution of valuation of

the strongest opponent of a bidder. We denote by \( \hat{v} \) the maximum of \( N - 1 \) i.i.d. random variables distributed according to \( F \) and its c.d.f. by \( G: G(\hat{v}) = F(\hat{v})^{N-1} \).

The novelty of our setup is that each bidder \( i \) does not know his valuation \( v_i \), but consults an advisor who does. Let advisor \( i \) denote the advisor to bidder \( i \). Advisor \( i \) knows \( v_i \), but has no information about valuations of other bidders except for their distribution \( F \), which is common knowledge. While advisor \( i \) knows \( v_i \), she is biased relative to the bidder. Specifically, the payoffs from acquiring the asset by bidder \( i \) are

\[
\text{Bidder } i : v_i - p, \quad (1)
\]

\[
\text{Advisor } i : v_i + b - p, \quad (2)
\]

where \( b \) is the bias of the advisor. The value that all players get from not acquiring the asset is zero. Bias \( b \) is commonly known.\(^4\) Our primary focus is on the preference of advisors for overbidding, \( b > 0 \), as it is most prominent in applications. In Section 6, we also consider the case of \( b < 0 \), which shares several similarities with the case of \( b > 0 \), but also differs from it in a number of important aspects.

Our formulation (1) – (2) captures the empire building motives described in the introduction. For example, consider a publicly traded firm bidding for a target. The board of the firm has formal authority over the bidding process, maximizes firm value, but does not know valuation \( v_i \). Suppose that the CEO of the firm knows \( v_i \), but is biased. Specifically, if the CEO owns fraction \( \alpha \) of the stock of the company and gets a private benefit of \( B \) from acquiring the target and managing a larger company, her payoff is \( \alpha(v_i - p) + B \). Normalizing this payoff by \( \alpha \) and denoting \( b = \frac{B}{\alpha} \), we obtain (1) – (2).

In this paper, we compare how different selling mechanisms affect the seller’s expected revenue and the allocative efficiency. Several formats are commonly used in practice and studied in the academic literature:

1. **Second-price auction (SPA)**. Bidders simultaneously submit sealed bids, and the bidder with the highest bid wins the auction and pays the second-highest bid.

2. **First-price auction (FPA)**. Bidders simultaneously submit sealed bids, and the bidder with the highest bid wins the auction and pay her bid.

3. **Ascending-price (English) auction**. The seller continuously increases the price \( p \), which we refer to as the *running price*, starting from zero. At each moment, each

\(^4\)For many of our results it is sufficient to assume that \( b \) is commonly known by bidders and advisors, while the seller knows only the sign of the bias.
bidder decides whether to continue participating or to *quit* the auction. Once a bidder quits, she cannot re-enter the auction. Once only one bidder remains, she is declared the winner and pays the price at which the last of her opponents quit the auction.

4. **Descending-price (Dutch) auction.** The seller continuously decreases the price \( p \), which we refer to as the *running price*, starting from a high enough level. At each moment, each bidder decides whether to *stop* the auction. The first bidder who stops the auction is declared the winner and pays the price at which she stopped the auction.

In all of these auction formats, if a tie occurs, the winner is drawn randomly from the set of tied bidders. We study a rich class of static auctions formally described in Section 4, but restrict attention to the ascending-price and descending-price auctions among dynamic mechanisms.

Communication between bidders and their advisors is modeled as game of cheap talk. If the auction format is static (i.e., it consists of a single round of bidding), the timing of the game is as follows:

1. Advisor \( i \) sends a private message \( \tilde{m}_i \in M \) to bidder \( i \) where \( M \) is some infinite set of messages.

2. Having observed message \( \tilde{m}_i \), bidder \( i \) chooses her action, i.e., what bid \( \beta_i \) to submit.

3. Given all submitted bids \( \beta_1, ..., \beta_N \), the asset is allocated and payments are made according to the rule specified by the auction.

If the auction format is dynamic (i.e., it consists of multiple rounds of bidding), the advisor sends a message to the bidder before each round of bidding. In ascending-price and descending-price auctions, we index stages by corresponding running prices \( p \).

A *(private) history* of bidder \( i \) at the beginning of stage \( p \) consists of all bidders’ actions and all messages sent by advisor \( i \) in the previous rounds. In static auctions, there is a single round of bidding, so the history is empty. A history at stage \( p \) in the ascending-bid auction consists of the current running price \( p \), the set of bidders remaining in the game, stages at which some of the bidders dropped out in the past, and messages sent by advisor \( i \) up to stage \( p \). A history at stage \( p \) in the descending-bid auction consists of simply the current running price \( p \) and messages sent by advisor \( i \) up to stage \( p \). Let \( \mathcal{H} \) denote the set of all possible histories.

A strategy of advisor \( i \) is a measurable mapping \( m_i : [\bar{v}, \bar{v}] \times \mathcal{H} \rightarrow M \) from the set of possible valuations and histories into a message sent to bidder \( i \). A strategy of bidder \( i \) is a
mapping \( a_i : \mathcal{H} \times M \rightarrow A \) from the history and current message into the action chosen by the bidder. We focus on pure strategies in dynamic auctions, but allow for mixing by bidders in static auctions. In static auctions, \( A \) consists of all possible mixtures over bids. In the English/Dutch auction, \( A = \{0, 1\} \) consists of a decision to quit the auction or continue.

The equilibrium concept is the Perfect Bayesian Equilibrium in Markov strategies (PBEM) where the state consists of the auction stage \( p \) and a bidder’s posterior belief about her valuation \( v_i \).\(^5\) Since all bidders are symmetric, we focus on symmetric equilibria in which strategies \( m_i \) and \( a_i \) do not depend on \( i \). Thus, we suppress index \( i \) in the notation.

In dynamic auctions, the timing of actions becomes relevant. A convenient class of equilibria is the one in which the advisor gives a real-time recommendation of the auction to the bidder, such that the bidder follows her advisor’s recommendation on equilibrium path.

**Definition 1.** A PBEM in the dynamic auction is in online strategies if \( m : [v, \tilde{v}] \times \mathcal{H} \rightarrow A \) and \( a(h, \tilde{a}) = \tilde{a} \) for all \( h \in \mathcal{H} \) and all \( \tilde{a} \) in the image of \( m(\cdot, h) \).

Let us stress that action recommendations happen in real time. For example, the online strategy in an ascending-bid auction is such that the advisor recommends whether to quit or continue bidding at each current price level \( p \). In particular, the strategy in which the advisor makes only one recommendation at the beginning of the auction is not an online strategy.

The following lemma states that restriction to online strategies is without loss of generality.

**Lemma 1.** For any PBEM there is another outcome-equivalent PBEM in online strategies.

There is in general a multiplicity of equilibria both in cheap-talk and auction games. To make meaningful comparison across auction formats, we rely on the following refinements. First, we assume that bidders play weakly dominant strategies if such strategies exist. In particular, it guarantees that in the second-price auction bidders bid their expected values. Second, we impose the “no incentive to separate” (NITS) condition, adapted from Chen, Kartik, and Sobel (2008), to select among equilibria in the communication game between bidders and advisors. Call type \( v_w \equiv v \) the weakest type of advisor.\(^6\) According to the NITS

---

\(^5\)The state that we choose does not capture the whole payoff-relevant history which in addition includes the number of remaining bidders in the auction. The reason for this restriction is a technicality that arises in continuous-time games. When the running price in the English auction changes continuously, the outcome of the auction may be indeterminate, which is a common problem of formulating games in continuous time (Simon and Stinchcombe, 1989).

\(^6\)Similarly, if \( b < 0 \), then \( v_w \equiv \tilde{v} \) is the weakest type of advisor.
condition, the weakest type has an option to credibly reveal herself if she wants. Thus, an equilibrium violates the NITS condition if the payoff of the weakest type is less than what she would get from revealing herself to the seller. Intuitively, when an advisor is biased for overbidding, it is natural to assume that the recommendation to bid little would be perceived as credible by the bidder. Chen, Kartik, and Sobel (2008) show that NITS can be justified by perturbations of the cheap-talk game with non-strategic players and costs of lying.

For dynamic selling mechanisms, such as ascending- and descending-bid auctions, we require that the NITS condition holds at every stage of the game. Specifically, let \( v^p_w \equiv \inf \{ v | v \in \text{supp}(\mu^p) \} \) be the weakest remaining type of the advisor at stage \( p \), where \( \mu^p \) for a posterior belief of the bidder about her value at stage \( p \). Similarly to Chen, Kartik, and Sobel (2008), an equilibrium violates the NITS condition if at stage \( p \), the advisor of type \( v^p_w \) is better off claiming that she is the weakest remaining type than playing her equilibrium strategy:

**Definition 2.** An equilibrium satisfies the NITS condition if for any \( p \), type \( v^p_w \) of the advisor weakly prefers her equilibrium strategy to the action optimally chosen by the bidder at stage \( p \) who knows that her value is \( v^p_w \).

We refer to an equilibrium as babbling if regardless the realization of the valuation, each bidder always plays the same strategy. We refer to an equilibrium of the static auction as the most informative if it induces the largest number of actions. As we show later, the most informative equilibrium in the static auctions always satisfies NITS. However, NITS need not select the unique equilibrium, and for the comparison of auction formats, we do not need a selection beyond NITS.

### 3 An Example: Two Bidders and Uniform Distribution

We start the analysis by working out a simple example that illustrates the results of the paper: How multiple-round auctions differ from single-round auctions, when bidders rely on informed advisors, and why the direction of the conflict of interest between bidders and advisors is crucial for the optimal design of the sale process. In this example, there are two bidders \( (N = 2) \), each valuation is an i.i.d. draw from the uniform distribution over \([0, 10]\),

---

7Similarly, if \( b < 0 \), then \( v^p_w \equiv \sup \{ v | v \in \text{supp}(\mu^p) \} \) is the weakest type of advisor at stage \( p \).
Figure 1: Thresholds in the partition equilibrium of the second-price auctions. Type $\omega_k$ of advisor is indifferent between pooling with types in $[\omega_{k-1}, \omega_k)$ by sending message $m_k$ and types in $[\omega_k, \omega_{k+1})$ by sending $m_{k+1}$. The difference between messages $m_k$ and $m_{k+1}$ is that $m_{k+1}$ wins for sure against types in $[\omega_{k-1}, \omega_k)$ and ties against types in $[\omega_{k-1}, \omega_k)$, while $m_k$ ties with types in $[\omega_{k-1}, \omega_k)$ and looses against types in $[\omega_{k-1}, \omega_k)$. 

and the advisors’ bias is $b = 1$ (when the bias is for overbidding) or $b = -1$ (when the bias is for underbidding).

**Overbidding bias ($b = 1$).** First, consider the second-price auction. Because of the bias, the advisor cannot credibly communicate the valuation to the bidder, and the equilibrium must have a partition structure. Consider the conditions that characterize an equilibrium with $N$ partitions, $[\omega_0, \omega_1], \ldots, [\omega_{N-1}, \omega_N]$, with $\omega_0 = 0$ and $\omega_N = 10$. Given the advisor’s message that conveys that the valuation is in the $k^{th}$ partition, the best response of the bidder is to bid the updated expected valuation, $(\omega_{k-1} + \omega_k) / 2$. This bid is the winning bid with probability one, if the valuation of the rival bidder is below $\omega_{k-1}$, with probability 50%, if the valuation of the rival bidder is between $\omega_{k-1}$ and $\omega_k$, and with probability zero, if it is above $\omega_k$. By inducing the bidder to bid $(\omega_k + \omega_{k+1}) / 2$ instead of $(\omega_{k-1} + \omega_k) / 2$, the advisor increases the probability of winning against types $[\omega_{k-1}, \omega_k]$ from 50% to one and against types $[\omega_k, \omega_{k+1}]$ from zero to 50%. This implies that for the cut-off type of the advisor $\omega_k$, the additional payoff from a higher probability of winning against types $[\omega_{k-1}, \omega_k]$ is equal to the cost from overpaying for the asset when the bidder wins against types $[\omega_k, \omega_{k+1}]$:

$$\frac{\omega_k - \omega_{k-1}}{10} \left( \omega_k + b - \frac{\omega_{k-1} + \omega_k}{2} \right) = \frac{\omega_{k+1} - \omega_k}{10} \left( \frac{\omega_k + \omega_{k+1}}{2} - \omega_k - b \right), \ k = 1, \ldots, N - 1.$$ 

This indifference condition simplifies to

$$\omega_{k+1} = 2\omega_k - \omega_{k-1} + 2b, \ k = 1, \ldots, N - 1.$$ 

When $b = 1$, the most informative equilibrium has three partitions, $[0, 1\frac{1}{3}], [1\frac{1}{3}, 4\frac{2}{3}]$, and $[4\frac{2}{3}, 10]$. The corresponding bids are $2\frac{2}{3}$, 3, and $7\frac{1}{3}$ (see Figure 2). Since the lowest bid is
below $b = 1$, this equilibrium satisfies the NITS condition: The weakest type of the advisor ($v = 0$) is better off communicating that the valuation is in $[0, \frac{1}{2}]$ than credibly revealing that $v = 0$. There exist two other equilibria: one with two partitions ($[0, 4]$ and $[4, 10]$) and the babbling equilibrium. Since the lowest bid (2 in the former case; 5 in the latter) exceeds $b = 1$, these equilibria do not satisfy the NITS condition. Indeed, the weakest type of the advisor ($v = 0$) is better off credibly revealing that $v = 0$ and ensuring that she never wins.

Next, consider the ascending-bid auction. Now a bidder faces a stopping time problem: At each price $p$, she decides whether to quit the auction or stay for a little longer. When $b = 1$, there exists the following equilibrium in this auction. Suppose that an advisor with type $v$ plays the threshold strategy of recommending to stay in the auction, if $p < v + 1$, and to quit once $p$ hits $v + 1$. Given this strategy, what is the best response of the bidder? If a bidder gets the recommendation to quit at price $p \in [1, 11]$, she infers that the valuation is equal to $v = p - 1$. Since the current price $p$ is already past the valuation of the bidder $p - 1$, the best response of the bidder is to quit the auction immediately. If a bidder has received a sequence of recommendations to continue bidding, she trades off the value of waiting for more information against the possibility of the overpaying for the asset. As the running price of the auction $p$ increases, the bidder’s posterior belief on the valuation ([$p - 1, 10$]) shrinks, meaning that the residual value of the advisor’s private information declines with price $p$. Therefore, the best response of the bidder to the advisor recommending to stay in
the auction is to stay in the auction, as long as \( p \leq \hat{p} \), given by

\[
0 = \mathbb{E}[v|v \geq \hat{p} - 1] - \hat{p},
\]

which implies \( \hat{p} = 9 \). Intuitively, \( \hat{p} = 9 \) is exactly the price at which the bidder is indifferent between winning the auction and getting the valuation of 9 on average (when the auction reaches this price, the bidder’s posterior is that \( v \in [8, 10] \)) and quitting it.

It turns out that this is the unique equilibrium satisfying the NITS criterion. To see this, consider why the equilibrium analogous to the equilibrium with three partitions in the second-price auction violates the NITS condition in the ascending-bid auction. In this equilibrium, after the price passes \( p = \frac{2}{3} \), the lowest type that remains in the auction is \( v = \frac{1}{3} \). According to the equilibrium, no bidder drops out after \( p = \frac{2}{3} \) until \( p \) reaches 3. This implies that the advisor with type \( v = \frac{1}{3} \) gets a negative expected payoff, since he wins with probability 50% at price \( p = 3 \), if the rival bidder’s type is in \( [1\frac{1}{3}, 4\frac{2}{3}] \). If the lowest type of the advisor \( v = \frac{1}{3} \) could credibly reveal itself before \( p \) reaches 3, he would do it, since it would lead to the bidder quitting the auction immediately. Hence, the equilibrium with three partitions does not satisfy the NITS condition. By the same logic, any equilibrium that satisfies the NITS condition has the property of separation up to a cut-off. The equilibrium with cut-off \( p=9 \) is the unique such equilibrium in this example.

As we have seen, the equilibrium in the ascending-bid auction is very different from the equilibrium in the second-price auction. In the former, a bidder bids up to \( \min \{v + 1, 9\} \), when its valuation is \( v \). In the latter, a bidder bids \( \frac{2}{3} \) if \( v \in [0, 1\frac{1}{3}] \), 3 if \( v \in [1\frac{1}{3}, 4\frac{2}{3}] \), and \( 7\frac{1}{3} \) if \( v \in [4\frac{2}{3}, 10] \). What does this imply for the comparison of revenues and efficiency? It is easy to see that the ascending-bid auction is more efficient: Not only there is a separation of types up to \( v = 8 \), but the pooling interval \([8, 10]\) is lower than the pooling interval in the top partition in the second-price auction \([4\frac{2}{3}, 10]\). Indeed, the expected valuation of the winning bidder is \( 6\frac{49}{75} \) in the ascending-bid auction and \( 6\frac{47}{135} \) in the second-price auction. Not only the ascending-bid auction is more efficient, but it also generates higher expected revenues than the second-price auction: \( 4\frac{23}{75} \) versus \( 3\frac{85}{135} \). The comparison of revenues is not obvious at first glance, since one distribution of bids does not dominate the other. Nevertheless, higher expected revenues in the ascending-bid auction is a general result.

**Underbidding bias** (\( b = -1 \)). Consider the case of an under-bidding bias of advisors, \( b = -1 \). In this case, the most informative equilibrium in the second-price auction has partitions \([0, 5\frac{1}{3}]\), \([5\frac{1}{3}, 8\frac{2}{3}]\), and \([8\frac{2}{3}, 10]\). This is the unique equilibrium satisfying the NITS
Unlike with $b = 1$, the ascending bid auction does not have the equilibrium in which advisors separate themselves up to a cut-off. To see this, suppose that an advisor with type $v$ plays the threshold strategy of recommending to stay in the auction, if $p < v - 1$, and to quit, otherwise. If the bidder gets the recommendation to quit at price $\tilde{p} \in (0, 9]$, she updates her valuation to $\tilde{p} + 1$. Her best response is thus to stay in the auction until the price hits $\tilde{p} + 1$. Expecting that the bidder will not follow his recommendation, the advisor is better off deviating from recommending the bidder to quit at $p = v - 1$. In other words, truthful communication (even up to a cut-off) is inconsistent with equilibrium for the same reason, it is inconsistent with equilibrium in the second-price auction. As in the second-price auction, all equilibria have partition structure, and the unique equilibrium satisfying the NITS condition has three partitions. Thus, when $b = -1$, the ascending-bid auction is strategically equivalent to the second-price auction, like in the model without advisors, but unlike when $b = 1$. This asymmetry arises because the bidder can only adjust bids in one direction: She can improve past bids (wait until a higher price), but she cannot do the reverse, that is, go back in time and quit at the lower price.

However, the descending-price (Dutch) auction with $b = -1$ has some similarities to the ascending-bid auction with $b = 1$. In the descending-price auction, a bidder faces a stopping time problem: At each price $p$, she chooses whether to accept it and buy the asset at price $p$ or whether to wait until a marginally lower price and risk losing the auction. Let us construct an equilibrium, which is similar to the equilibrium in the ascending-bid auction for the case of $b = 1$. Conjecture that the bidder plays a strategy of not stopping the auction, unless the advisor recommends to do it or unless the price hits some lower threshold $p$. Given this strategy, the optimal price at which the advisor with type $v$ sends a recommendation to quit, $σ(v)$, satisfies:

$$ σ(v) = \arg \max_{p \geq v} (v - 1 - p) σ^{-1}(p) ,$$

which represents the familiar trade-off between buying the asset for a lower price and having a lower probability of winning. Let $v^*$ denote the lowest type that recommends to stop the auction before $p$, and pick $v^*$ and $p$ so that the bidder is indifferent between buying the asset or not when the price hits $p$, given her belief at that point:

$$ E[v|v \leq v^*] = p = v^* - 1 .$$

The equilibrium with two partitions ($[0, 6]$ and $[6, 10]$) violates the NITS criterion, because the weakest type (now the highest type, $v = 10$) is better off credibly revealing himself. This is because the advisor with type $v = 10$ prefers to win when she faces a rival that bids 8. For the same reason, the equilibrium with one partition $[0, 10]$ violates the NITS criterion.
Combining with (3), we obtain \( v^* = 2, \ p = 1, \) and \( \sigma(v) = \frac{v^2 - 2v + 4}{2v} \). If a bidder receives a recommendation from her advisor to stop the auction at price \( \tilde{p} \in (1, 4.2) \), she infers that the valuation is \( \sigma^{-1}(\tilde{p}) \). Since \( \tilde{p} \) is already below the bidder’s optimal stopping point, her best response is to stop the auction immediately.\(^9\) If a bidder has received a sequence of recommendations to continue staying in the auction, she trades off the value of waiting for more information against the possibility of losing the auction. As the running price \( p \) goes down, the bidder’s posterior belief about the valuation, \( [0, \sigma^{-1}(p)] \) shrinks, and her best response is to wait for the recommendation of the advisor to stop until \( p \) gets too low, which happens to be \( p = 1 \) in this example.

Can we conclude that the descending-price auction is more efficient and generates higher revenues than the second-price auction when advisors are biased for underbidding, like we did with the ascending-price auction in the case of an overbidding bias? The answer to the first question is a “yes”, but to the second one is a “no.” The descending-bid auction is indeed more efficient than the second-price auction: Not only there is a separation of types in \([2, 10]\), but the pooling interval \([0, 2]\) is smaller than the pooling interval in the bottom partition in the second-price auction, \([0, 5\frac{1}{3}]\). Thus, the descending-bid auction is efficient if the highest valuation is above 2, and results in a lower probability of misallocation, if it is below 2. Indeed, the expected valuation of the winning bidder is \(7\frac{2}{13}\) in the descending-price auction and \(6\frac{47}{135}\) in the second-price auction. However, the descending-price auction generates lower revenues than the second-price auction: approximately 2.7 versus 3\(\frac{88}{135}\). Thus, the second-price auction dominates if the goal of the designer is expected revenues, but the descending-price auction dominates if the goal is efficiency.

The opposite implications for expected revenues occur for the following reason. When advisors are biased for overbidding, the seller’s goal of higher expected revenues is aligned with the bias of advisors. In contrast, when advisors are biased for underbidding, the bias goes in the opposite direction from the seller’s goal of higher expected revenues.

### 4 Static Auctions

We now proceed to establishing the results illustrated in the previous section in the general setup. This section shows that the revenue equivalence theorem extends to the setting when the interests of bidders and advisors are not aligned (\( b \neq 0 \)), if selling mechanisms are static in the sense that they admit only a single round of communication between bidders and

\(^9\)The fact that the advisor’s optimal stopping price is below the bidder’s optimal price follows from the single-crossing property of payoff function (3).
their advisors. For static mechanisms, we characterize equilibrium information transmission and show that there is necessarily an efficiency loss due to imperfect communication. In the next sections, we show that dynamic mechanisms lead to very different implications both in terms of information transmission and generated revenues.

We first show that many static auctions are equivalent to the second-price auction. Consider the subgame in which bidders have received particular messages from their advisors. After each bidder gets a message from her advisor, she updates her best estimate of her valuation. We refer to the bidder $i$’s expected value of the asset conditional on receiving message $\tilde{m}_i$ from her advisor as her type $\theta_i \equiv \mathbb{E}[v_i|\tilde{m}_i] \in [\underline{v}, \overline{v}]$. Denote by $F_\theta$ the distribution of each bidder’s types generated in equilibrium.

We restrict attention to auctions that take a form of a direct mechanism. In the direct mechanism, each bidder $i$ announces her type $\tilde{\theta}_i \in [\underline{v}, \overline{v}]$ and to each profile of announcements $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_N)$ corresponds the probability of allocation $q_i(\tilde{\theta})$ to bidder $i$ and transfer $t_i(\tilde{\theta})$ from bidder $i$ such that $q_i(\tilde{\theta}) \geq 0$ and $t_i(\tilde{\theta}) \geq 0$ for all $i$ and $\sum_{i=1}^N q_i(\tilde{\theta}) = 1$. We restrict attention to truthful equilibria in direct mechanisms in which each bidder announces her type truthfully. In addition, we require that $q_i$ does not depend on the identity of bidders and so, there are functions $q$ and $t$ such that $q_i(\theta_1, \ldots, \theta_i, \ldots, \theta_N) = q(\theta_i, \theta_{-i})$ and $t_i(\theta_1, \ldots, \theta_i, \ldots, \theta_N) = t(\theta_i, \theta_{-i})$ for all $(\theta_1, \ldots, \theta_N)$ and all $i$.

Observe that we require that the set of possible announcement is $[\underline{v}, \overline{v}]$ as opposed to being the support of $F_\theta$. In other words, the seller does not condition the offered mechanism on the details of the communication between bidders and their advisors. The next lemma is a standard result in the mechanism design literature (see e.g. Holmstrom, 1979) and it shows that many direct mechanisms are equivalent to the second-price auction.

**Lemma 2.** Any direct mechanism such that

1. for all $i = 1, \ldots, N$ and all $(\theta_1, \ldots, \theta_N) \in [\underline{v}, \overline{v}]^N$

   $$q(\theta_i, \theta_{-i}) = \begin{cases} \frac{1}{n}, & \text{if } \theta_i \in \max\{\theta_1, \ldots, \theta_N\} \text{ and } n \equiv |\{j : \theta_j = \max\{\theta_1, \ldots, \theta_N\}\}|, \\ 0, & \text{otherwise}; \end{cases}$$

2. truthful reporting is a weakly dominant strategy;

3. loosing bidders pay zero;

is the second price auction in a sense that both have the same $q$ and $t$. Moreover, there exists an equilibrium of the second price auction which generates the same expected revenue and
the same distribution of bidders’ types $F_\theta$.

In this paper, we are interested in whether the auction format affects information transmission and through it expected revenues and efficiency. Lemma 2 tells us that it does not if one restricts attention to static auctions with the same allocation rule and rent to the lowest type. For example, one cannot expect to generate a better information transmission or higher revenues by switching from the second-price auction to the first-price auction or all-pay auction.

Lemma 2 provides a useful analytic tool. The second-price auctions are easier to analyze as they allow for a simple bidding equilibrium in weakly dominant strategies. In contrast, the equilibrium of the first-price auction with discrete types of bidders requires mixing. As we will see next, discrete types naturally arise in the communication between the bidder and the advisor. We restrict attention in the analysis to the second-price auction and Lemma 2 shows that our characterization and comparison results are valid for a wide class of static auctions.

**Theorem 1.** Suppose $\bar{v} < \infty$. The communication strategy in the second-price auction is characterized as follows. There exists a positive integer $K$ such that for all $1 \leq K \leq \bar{K}$, there exists an equilibrium in which types of advisor $v \in [\omega_{k-1}, \omega_k)$ induce the same action of the bidder and signal to the bidder that bidder’s value is equal to $m_k = \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]$. Thresholds $(\omega_k)_{k=1}^K$ are determined as follows:

$$G(\omega_{k-1}, \omega_k)(1 - \Lambda_k)(\omega_k + b - m_k) + G(\omega_k, \omega_{k+1})\Lambda_{k+1}(\omega_k + b - m_{k+1}) = 0. \quad (5)$$

where

$$\Lambda_k = \frac{1}{G(\omega_{k-1}, \omega_k)} \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{1}{n + 1}. \quad (6)$$

Theorem 1 shows that in static auctions, the misalignment of interests on the bidder’s side results into a coarsening of the information transmitted from the advisor to the bidder. In particular, this implies that with positive probability the object is allocated inefficiently when $b \neq 0$. Theorem 1 is a counter-part of Theorem 1 in Crawford and Sobel (1982). However, our result does not follow from their result. In our game, the cheap-talk game is endogenous. Each bidder and advisor play a cheap-talk game in which actions are bids.

---

10Here and further, when a random variable $v$ is distributed according to $F$, we use a short-hand notation $F(a, b)$ for $\mathbb{P}(v \in [a, b]) = F(b) - F(a)$. 

---

18
The attractiveness of each bid for the bidder and advisor is endogenous and depends on how opponents bid in the auction. The bidding behavior of opponents depends on the information communicated between opponent bidders and their advisors. Theorem 1 shows that main insights from the cheap-talk literature are still true even when the cheap-talk game is endogenously determined in equilibrium by the communication between opponents and their advisors.

Equation (5) reflects the incentive of threshold types $\omega_k$ in the second-price auction. Notice that $\Lambda_k$ is the expected probability of winning a tie when the bidder submits bid $m_k$. Type $\omega_k$ is indifferent between sending message $m_k$ and $m_k+1$. In the second-price auction, the bidder pays the second highest bid. Therefore, strategies bring different payoffs only when the bidder faces a highest opponent of type in the interval $(\omega_k-1, \omega_k)$ or in the interval $[\omega_k, \omega_{k+1})$. The first term in equation (5) represents the benefit from submitting a higher bid. A higher bid $m_{k+1}$ increases the probability of winning a tie from $\Lambda_k$ to 1. The second term in equation (5) is the cost associated with a higher bid. Sending message $m_{k+1}$, the advisor risks winning the auction at price $m_{k+1}$. Since the costs and the benefits, should be equalized for threshold types, the advisor with type $\omega_k$ prefers not to buy at a higher price $m_{k+1}$.

It will be useful to derive the necessary condition for informative communication in static auctions.

**Corollary 1.** A necessary condition for a non-babbling equilibrium is

$$b \leq \mathbb{E}v - v.$$ 

*It is also sufficient when $N = 2$.*

Chen, Kartik, and Sobel (2008) shows that in the standard cheap-talk model, NITS always exists and selects equilibria that are sufficiently informative (induce a high number of actions). In particular, under some conditions, NITS selects the most informative equilibrium of the cheap-talk. We next verify that this result is also true in our model.

**Theorem 2.** The most informative equilibrium of the second-price auction satisfies the NITS condition.

The proof of Lemma 2 adapts the argument in Chen et al. (2008) showing that if there is an equilibrium in the cheap-talk game with $K$ actions induced in equilibrium that fails to satisfy NITS, then there is also an equilibrium with $K + 1$ induced actions. Again we cannot
apply their result directly, as the cheap-talk game between the bidder and the advisor is endogenous. Their result implies that in our model for a fixed equilibrium, we can construct a different cheap-talk equilibrium for one bidder and her advisor that is more informative. However, this need not be an equilibrium of the model, as once we change the cheap-talk equilibrium of bidders, this changes the cheap-talk game played and hence, this will not be an equilibrium of the model.

5 Ascending-Bid Auction

This section characterizes PBEMs of the ascending bid (English) auction and shows that the ascending bid auction dominates any static auction in terms of information transmission, efficiency and revenue.

5.1 Characterization

This subsection shows that all PBEMs of the English auction satisfying NITS are in delegation strategies defined as follows.

Definition 3. Strategies of players are delegation strategies if for some $v^*$:

- the advisor sends message “quit” when the running price equals $v + b$;
- the bidder quits if either the running price is above $v^* + b$ or she receives message “quit”.

If players follow delegation strategies, advisor’s types below $v^*$ perfectly reveal themselves over time, while types above $v^*$ pool with each other. If the advisor were submitting the bids, then it is a weakly dominant strategy for her to quit when $p = v + b$. Thus, the bidder essentially delegates bidding to the advisor with the restriction that the bid is below $v^* + b$.

The next theorem shows that all equilibria in the English auction satisfying NITS are in delegation strategies.

Theorem 3. Suppose $b > 0$. Any PBEM in the English auction that satisfies the NITS condition is in delegation strategies with cutoff $v^*$ characterized as follows:

1. $v^*$ satisfies

   
   \begin{align}
   v^* + b &= \mathbb{E}[v|v \geq v^*], \text{ if } v < v^* < \infty, \quad (6) \\
   v^* + b &\geq \mathbb{E}[v|v \geq v^*], \text{ if } v^* = v. \quad (7)
   \end{align}
2. Let \( v_0^* = v, \ v_{K+1}^* = v, \) \( v^*_1 < \ldots v^*_k < \cdots < v^*_K \) be all solutions to the equation (6). Then \( v^* \) equals to some \( v_k^* \in \{v_0^*, \ldots, v_{K+1}^*\} \) such that for all \( v_j^*, \ j < k, \) and all \( n = 1, \ldots, N - 1:\)

\[
\int_{v_j^*}^{v_k^*} (1 - F(s)) (\mathbb{E}[v|v > s] - s - b) dG_n(s) \geq 0,
\]

where \( G_n \) is the distribution of the maximum of \( n \) i.i.d. random variables distributed according to \( F. \)

Unlike in static auctions, the English auction admits equilibria in delegation strategies. First, observe that the communication strategy is optimal for the advisor. The advisor induces the bidder to quit either at her optimal price \( v + b \) if \( v \leq v^* \), or at price \( v^* + b \) if \( v > v^* \), which is still better for the advisor compared to any price below \( v^* + b \). Second, the dynamic nature of the English auction makes it optimal for the bidder to follow the advice. The advisor reveals the value to the bidder only when the running price equals advisor’s optimal quitting price. Because of the overbidding bias, the bidder gets negative utility if she wins at the current or any future running price. Therefore, the bidder prefers to quit immediately after the advisor’s recommendation to quit (and getting utility zero), rather than to wait longer and face the risk of winning at a price that exceeds her value. The cutoff \( v^* \) is chosen so that at any stage \( p < v^* + b \), the option value to the bidder of staying in the auction and waiting for the advisor’s recommendation is above zero (the value of quitting immediately and loosing).

In general, there are PBEMs in the English auction that are not in delegation strategies. Without the NITS condition, the English auction is strategically equivalent to the second price auction as in the standard auction setting. In particular, for any equilibrium of the second price auction, there exists an outcome-equivalent PBEM equilibrium of the English auction. To construct such equilibrium, we can simply specify that types in \([\omega_k - 1, \omega_k)\) that send message \( m_k \) in the second-price auction, in the English auction, send the same message in the beginning of the auction from which the bidder infers her expected value \( m_k \) and quits when the running price reaches \( m_k \).

Theorem 3 shows the NITS condition effectively rules out partition equilibria (as well as any other equilibria in non-delegation strategies). To see this, consider a partition equilibrium described in Theorem 1. In the beginning of the game type \( \omega_k \) is willing to submit price \( m_{k+1} \) as it increases her probability of winning against types in \([\omega_{k-1}, \omega_k)\), despite the
risk of winning at a higher price $m_{k+1}$. However, after the running price exceeds $m_k$, the benefits of submitting higher bid disappear, and only costs remain. Thus, at this stage, type $\omega_k$ would prefer to reveal herself, this way inducing the bidder to quit immediately, which contradicts the NITS condition.

Conditions on $v^*$ in Theorem 3 reflect the option value to the bidder of following the advisor’s recommendation. Suppose that at stage $p$ the lowest remaining type is $v_p$ and there are $n$ other bidders remaining in the auction. The bidder wins the auction if her value is higher than the highest value $s$ of the remaining bidders, in which case she pays $s + b$. Then her expected payoff is $\mathbb{E}[v|v > s] - s - b$. Integrating over all $s$, we get that the option value to the bidder equals

$$
\frac{1}{(1 - F(v_p))^n} \int_{v_p}^{v^*} (1 - F(s)) (\mathbb{E}[v|v > s] - s - b) dG_n(s).
$$

(9)
The bidder waits for the recommendation as long as this option value is positive. Condition (6) ensures that the bidder does not want to stop listening to the advisor slightly earlier or later,\footnote{If $v^* + b < \mathbb{E}[v|v \geq v^*]$, then the bidder would prefer to quit slightly later, while she would prefer to quit slightly earlier if $v^* + b > \mathbb{E}[v|v \geq v^*]$.} and condition (8) ensures that the option value stays positive up until $p = v^* + b$.

For most commonly-used parametric families of distributions conditions on $v^*$ in Theorem 3 can be significantly simplified. Let us introduce the mean residual lifetime function

$$MRL(s) = \mathbb{E}[v|v \geq s] - s,$$

which is well studied in statistics (see Bagnoli and Bergstrom, 2005). Many commonly-used distributions have monotone $MRL$. Function $MRL$ is decreasing for such distribution as normal, logistic, extreme value, Weibull, gamma, power distribution with power greater than one, as well as their truncations from above or below.\footnote{Log-concavity of density $f$ or log-concavity of reliability function $1 - F$, which are preserved by truncations, are sufficient for a weakly decreasing $MRL$ (Bagnoli and Bergstrom, 2005).} For Pareto and log-normal distribution truncated from below at 1, $MRL$ increasing.\footnote{Log-convexity of the density is sufficient for a weakly increasing $MRL$ (An, 1998).}

We next characterize PBEMs in the English auction for distributions with monotone $MRL$. After that we will return to the general characterization in Theorem 3 to discuss how equilibria look like for general distributions.
When $MRL$ is strictly decreasing, there is $v^*$ solving (6) if and only if $E v - v \geq b$, and it is unique whenever it exist (see see Figure 3a). Since $MRL$ crosses $b$ from above, $E[v|v \geq s] - s - b$ is positive for all $s < v^*$ and so, the option value (9) is positive for all $v_p < v^*$ which leads to the following proposition.

**Proposition 1.** Suppose $b > 0$ and $MRL$ is strictly decreasing. The unique PBEM of the English auction satisfying the NITS condition is in delegation strategies. Moreover, $v^* < \bar{v}$ and the equilibrium is informative if and only if $E v \geq \bar{v} + b$.

Proposition 1 shows that pooling at the top in the uniform example holds for a general class of distributions. When $MRL$ is strictly decreasing, the bidder does not wait until all types of the advisor reveal themselves and at some point quits the auction before learning perfectly her value. Over the course of the auction, the bidder learns information about her value even if the advisor does not send any messages. The fact that there was no message so far indicates that the value cannot be lower than the running price minus bias $b$. When $\tau$ is finite, after a certain time, the bidder knows that the value is close to $\bar{v}$. If she wins the auction, she will pay a price close to $\bar{v} + b$ and hence, it is very likely that she will overpay for the good. Similarly, when $\bar{v} = \infty$, the expected value from winning the auction grows slower than the price under the strictly decreasing $MRL$. As a result, the bidder prefers to quit earlier and there is an interval of values at the top that she never learns.

Before proceeding to increasing $MRL$, let us first consider a knife-edge case of the exponential distribution for which $MRL$ is constant.\(^{15}\)

**Proposition 2.** Suppose $F$ is exponential with parameter $\lambda$ and $b \neq \frac{1}{\lambda}$. The unique PBEM of the English auction satisfying the NITS condition is in delegation strategies with:

---

\(^{15}\)The Online Appendix contains a detailed analysis of the second-price and English auctions for the exponential distribution.
• $v^* = \infty$ when $b < \frac{1}{\lambda}$;
• $v^* = v$ when $b > \frac{1}{\lambda}$.\(^{16}\)

The equilibria in the exponential case have quite different structure from the decreasing $MRL$. Due to the memoryless property of the exponential distribution, if it is optimal for the bidder to continue waiting for the advisor’s recommendation in the beginning of the auction, then it is optimal to do so in any stage of the auction. Thus, the equilibrium has a bang-bang property: communication is either fully reveling or completely uninformative. The next proposition shows that this is true for a more general class of distributions.\(^{17}\)

**Proposition 3.** Suppose $b > 0$, $MRL$ is strictly increasing on $[v, \infty)$, and let

$$\bar{b} \equiv \sup \left\{ b > 0 : \int_v^\infty (1 - F(s))(\mathbb{E}[v|v > s] - s - b)dG_n(s) \geq 0 \right\},$$

for all $n = 1, \ldots, N - 1$. Then all PBEM of the English auction satisfying the NITS condition are in delegation strategies with $v^*$ characterized as follows:

• $v^* = \infty$ when $b \in [0, \mathbb{E} v - v]$;
• $v^* = v$ or $v^* = \infty$ when $b \in [\mathbb{E} v - v, \bar{b}]$;
• $v^* = v$ when $b \in (\bar{b}, \infty)$.

When $MRL$ is strictly increasing, there is at most one solution $\bar{v}$ to (6), however, it cannot be an equilibrium cutoff $v^*$ (see Figure 3b). The reason is that since $\mathbb{E}[v|v \geq s] - s$ crosses $b$ from below at $\bar{v}$, if $v^* = \bar{v}$ the option value to the bidder of waiting for advisor’s recommendation is negative for $v < \bar{v}$. Therefore, the equilibrium is either fully separating ($v^* = \infty$) or babbling ($v^* = 0$). Intuitively, in the beginning of the auction, winning is a bad news, as the bidder gets negative utility if she wins. As the auction continues, eventually, the bidder gets positive utility from winning, as $\mathbb{E}[v|v \geq s]$ increases faster than the price $s + b$ that the bidder pays in case she wins. The bidder is willing to follow the advice of the bidder if the benefits of winning later in the auction outweigh the risk of winning early in the auction. Hence, the condition on $b$: when $b$ is sufficiently low, there is a fully informative equilibrium. As $b$ increases, at some point, the babbling equilibrium is possible, and a for sufficiently high $b$, babbling equilibrium is the only equilibrium. Notice that the babbling

\(^{16}\)When $b = \frac{1}{\lambda}$, there is a continuum of PBEMs in delegation strategies with $v^* \in [0, \infty]$.

\(^{17}\)Observe that it is necessary for $MRL$ to be increasing that $\overline{v} = \infty$. Indeed, if $\overline{v} < \infty$, then $\lim_{s \to \overline{v}} \mathbb{E}[v|v > s] - s = 0 < \mathbb{E} v - v$.  

24
equilibrium is an equilibrium only for sufficiently large $b$, for which the lowest type $v$ gets positive utility $\frac{1}{N}(v+b-Ev)$ from winning at the babbling price and so, the NITS condition is satisfied.

Let us now return to the general characterization of PBEMs in the English auction in Theorem 3. For general distributions, there can be multiple solutions to equation (6) which are, together with $v$ and $\nu$, are candidates for $v^*$. Condition (8) ensures that the option value to the bidder of following advisor’s recommendation given by (9) is positive for all $v_p < v^*$. The integral (9) can be split into several integrals with limits of integration given by $(v_k^*)_{k=0}^{K+1}$. Since (9) should hold for every $v_p$ up to $v^*$, only solutions to (6), in which $MRL$ crosses $b$ from above are possible candidates for the equilibrium cutoff. Moreover, the option value (9) is the smallest at the solutions to (6) where $MRL$ crosses $b$ from below, so to find all equilibria it is sufficient to check whether the option is positive only at those solutions.

As an illustration, consider the $MRL$ function depicted in Figure 3c. There are three solutions $v_1^*$, $v_2^*$, and $v_3^*$ to equation (6). By Theorem 3, there can be at most five equilibria satisfying NITS in this situation. $v_0^*$ is not an equilibrium cutoff, as it fails the NITS ($Ev > v + b$), and $v_4^* = \nu$ is not an equilibrium cutoff, as $MRL$ is below $b$ at $\nu$. Neither is $v_2^*$, as $MRL$ crosses $b$ from below at $v_2^*$. Hence, only candidates for the equilibrium cutoff are $v_1^*$ and $v_3^*$. There is an equilibrium with cutoff $v_1^*$ if and only if the integral (9) for $v_p = v_1^*$ is positive. There is an equilibrium with cutoff $v_3^*$ if and only if the integral (9) for $v_p = v_3^*$ is positive.

Observe that the condition for informative equilibria in Propositions 1 and 2 is the same as the necessary condition for informative equilibria for static auctions (see Corollary 1). In particular, for the case of weakly decreasing $MRL$ and $N = 2$, there is an informative equilibrium in the English auction if and only if there is an informative equilibrium in the static auction. However, generally there can be informative (and even fully revealing) equilibria in the English auction even when the most informative equilibrium of a static auction is babbling. In particular, proposition 3 allows the existence of non-babbling equilibria even when $b > Ev - v$ and all equilibria of the static auctions are babbling as long as (10) holds. To give a concrete example, suppose $N = 2$ and $F(v) = 1 - \left(\frac{1}{v}\right)^2$ is a Pareto distribution on $[1, \infty)$. We have $Ev - v = 1$ and so all equilibria in static auctions are babbling whenever $b > 1$. We can compute (10) as follows

$$\int_1^\infty \frac{2(\hat{v} - b)}{v^5} d\hat{v} = 2 \int_1^\infty d \left( \frac{1}{v^4} - \frac{b}{v^5} \right) = 2 \int_1^\infty d \left( \frac{b}{4v^4} - \frac{1}{3v^5} \right) = \frac{4 - 3b}{6},$$

\footnote{Observe that for Pareto distribution $f(v) = \frac{2}{v^2}$ and $Ev[v > \hat{v}] = \hat{v}^2 \int_\hat{v}^\infty \frac{2}{v^2} dv = 2\hat{v}.}$
which is positive whenever \( b < \frac{4}{3} \). Hence, for \( b \in (1, \frac{4}{3}) \) there exists an informative equilibrium, even though all equilibria of static auctions are babbling. The reason is that the term \( \mathbb{E}[v|v > v^*] = 2v^* \) grows faster than \( v^* + b \) so that for \( b < \frac{4}{3} \), (10) holds.

5.2 Auction Comparison

We next compare the English auction with static auctions. We start with the efficiency and informativeness comparison.

An equilibrium of an auction is \textit{more informative} than an equilibrium of a different auction, if the partition of advisor types generated by the former is finer than the partition generated by the latter. An equilibrium of an auction is \textit{more efficient} than an equilibrium of a different auction, if the value of the winner in it is higher for any realization of values. We say that an equilibrium is strictly more informative (efficient), whenever for some realization of values the partition of types is strictly finer (the value of the winner is strictly higher).

Propositions 2 and 3 show that the efficiency and informativeness comparison is clear in the case of weakly increasing \( MRL \). When there is an informative equilibrium in the English auction, it attains perfect communication and always allocates the asset to the bidder with the highest value. Thus, it dominates any equilibrium of any other auction format. Theorem 3 shows that in general (in particular, when \( MRL \) is decreasing), there can be pooling at the top which distorts the efficiency. This leads to a loss of efficiency and depending on the size of this pooling region, it is possible that the equilibrium of the static auction is more efficient if it generates a more efficient outcomes for types above \( v^* \). The next theorem shows that this is not the case. More strongly, the superior efficiency stems from a superior information transmission.

\begin{theorem}
Suppose \( b > 0 \). Then any PBEM of the English auction satisfying NITS is more informative and more efficient than any equilibrium of the second-price auction, and it is strictly more informative and efficient whenever it is non-babbling.
\end{theorem}

Theorem 4 shows that there is no partition generated by a static auction that is finer than the partition generated by the English auction. That is, there is no \( \omega_k > v^* \) where \( \omega_k \) and \( v^* \) are as in Theorems 1 and 3. This implies that the English auction generates a finer information partition than any static auction and hence, is more efficient.

The argument for Theorem 4 can be sketched as follows. Suppose that there exists an equilibrium of the static auction such that \( \omega_{k-1} = v^* < \omega_k < \omega_{k+1} = \overline{v} \). For simplicity, also
assume that $N = 2$. Then equation (5) implies that

$$\frac{1}{2} F(\omega_{k-1}, \omega_k)(\omega_k + b - m_k) = -\frac{1}{2} F(\omega_k, \omega_{k+1})(\omega_k + b - m_{k+1}),$$

or

$$\omega_k + b = \frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})} m_k + \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})} m_{k+1} = \mathbb{E}[v | v \geq v^*].$$

However, this contradicts the fact that $v^* < \omega_k$ solves (6). Intuitively, if there were a variation in bids among types above $v^*$, any type above $v^*$ would strictly prefer to submit a higher bid and increase her chances of winning against types below. This happens because $v^*$ is already sufficiently close to $\overline{v}$ and price $m_{k+1}$ does not vary much from price $m_k$.

We next turn to the revenue. Denote by $\varphi(v) \equiv v + b - \frac{1-F(v)}{f(v)}$ the virtual valuation of advisor.

**Theorem 5.** Suppose $b > 0$ and $\varphi$ is increasing. Then any PBEM of the English auction satisfying NITS brings higher revenue than any equilibrium of the second-price auction satisfying NITS, and it brings a strictly higher revenue whenever it is non-babbling.

The key insight of Theorem 5 is that we can view the problem of the seller of extracting maximal revenue as the a problem of designing a mechanism that extracts rents from informed advisors. In this case, the fact that there is a communication puts a restriction on the set of mechanisms that are the seller can implement. However, one can still use the envelope formula in Myerson (1981) to write the revenue as follows:

$$N \left( \mathbb{E}[\varphi(v)p(v)] - U_A(0) \right),$$

where $p(v)$ is the probability that type $v$ wins the auction and $U_A(0)$ is the expected utility of type 0 from the auction. In equation (11), only $p(\cdot)$ and $U_A(0)$ depend on the format of the auction. By Lemma 2, it is sufficient to compare the English auction with the second-price auction. The higher efficiency of the English auction (by Theorem 1) and increasing virtual valuation implies that the first term in (11) is higher than in any static auction, while the NITS guarantees that the rent of the lowest type is positive in static auctions, while it is zero in the English auction.

While superior efficiency of the English auction because of the better information transmission is intuitive, it is a priori not clear if the English auction should also bring higher expected revenue. If types in some interval pool and induce the same bid in the second-price auction, then it can potentially increase the revenue of the seller. As we have already seen, at
least some types above $\omega_k$ get negative utility from winning at price $m_{k+1}$, but with positive probability they end up winning the asset at this price. We show that despite this occasional overpaying, it does not occur often enough to reduce significantly the information rents of advisors. The key in ensuring this is the NITS condition. As an illustration, let us return to consider the exponential example. In the babbling equilibrium of the second-price auction all types pool and bidders submit bids $\frac{1}{\lambda}$, which results in the revenue. At the same time, the English auction brings revenue $E[\min\{v_1, v_2\}] + b = \frac{1}{2\lambda} + b$ and so for $b < \frac{1}{2\lambda}$, the babbling equilibrium of the second-price auction brings higher revenue. In this case, a significant amount of low types make a bid that exceeds their value in the second-price auction. This way the seller extracts an extra revenue. However, for $b < \frac{1}{2\lambda}$, babbling equilibrium fails to satisfy the NITS conditions.

For distributions with increasing MRL we can go beyond the comparison with static auctions and completely solve the problem of optimal and efficient auction design.

**Proposition 4.** Suppose $0 < b < E_v - v$ and MRL is increasing on $[v, \infty)$. Then the unique PBEM of the English auction satisfying NITS is fully efficient. Moreover, the unique PBEM satisfying NITS of the English auction with a reservation price $\varphi^{-1}(0)$ is optimal.

Proposition 4 follows immediately from our characterization in Proposition 3. When MRL is increasing and the conflict of interest is not too high, because the communication is fully revealing, the English auction implements the efficient outcome. Further, in the standard auction setting, Myerson (1981) shows that generally a second price auction with a reservation price is an optimal mechanism. In particular, the second price auction where bids are submitted directly by advisors is an optimal mechanism from extracting rents from informed advisors. Proposition 3 can be easily modified to allow for a reservation price by simply assuming that the seller start increasing price starting from $r$. Hence, for distributions with increasing MRL, the English auction implements also the optimal outcome, and no other auction format can increase the revenues. We conjecture that the English auction dominates all other auction formats (not necessarily static) in terms of efficiency and revenue. In order to show this, one needs to show that no other auction format either static or dynamic can attain a finer communication for types above $v^*$.

### 5.3 Comparative Statics

The characterization of equilibria in the English auction yields interesting comparative statics.
Proposition 5. Suppose MRL is strictly decreasing. Then the following hold.

1. The efficiency of the English auction is decreasing in $b$.

2. The revenue of the English auction is increasing in $b$ in the neighborhood of $b = 0$ and decreasing in $b$ in the neighborhood of $b = \bar{b}$ where $\bar{b} \equiv E[v] - \mu$.

3. For any bias $b > 0$, there exists $N(b)$ and $\varepsilon(b) > 0$ such that for all $N > N(b)$, the seller prefers bias $b - \varepsilon(b)$ to bias $b$.


Naturally, as the conflict of interest decreases, the pooling region decreases and equilibria become more informative and efficient. Interestingly, the revenue is generally non-monotone and the seller prefers moderate values of the advisor’s bias. In particular, for low values of the bias, the seller prefers to increase the bias, while for sufficiently high values of the bias the seller prefers to lower the bias. Intuitively, there are two contrary forces that affect the revenue as $b$ increases. On the one hand, as the bias increases, the distribution of advisors’ values increases in the sense of first-order stochastic dominance, and hence, the profit of the seller increases. On the other hand, the bias reduces the informativeness of the communication, as it lowers $v^*$ (see equation (6)). Initially the seller benefits from the increase in the bias, as it shift the distribution of advisors’ values to the right. However, as the bias increases, the cost of reduced communication eventually outweighs the benefit from a first-order stochastic shift of advisors’ values. In the uniform example discussed above, the revenue of the seller equals $E[\min\{v_1, v_2, v^*\}] + b = \frac{1}{100} v^*(100 - 10v^* + \frac{1}{3}v^{*2}) + b$ and equation (6) implies $v^* = \max\{0, 10 - 2b\}$. Therefore, the revenue has an inverse-U-shaped and attains maximum for $b = \frac{5}{\sqrt{2}} \approx 3.54$.

By the third statement of Proposition 5, as the environment becomes more competitive, the seller prefers a lower bias of advisors. The intuition is that as $N$ increases, it becomes more likely that the second order statistic is above $v^*$. However, pooling above $v^*$ does not allow the seller to fully benefit from such a shift in the distribution of $\hat{v}$. Therefore, with the increase in $N$, the seller prefers lower biases as their allow for a finer discrimination of higher types.

By the last statement of Proposition 5, when $MRL$ is decreasing, the number of bidders does not affect the communication. This is quite striking, as the following argument seems right at first sight. When $N$ goes to infinity, the highest value is close to the second highest value. Therefore, the bidder almost certainly overpays $b$ and so, should get a negative
expected utility from following the recommendation of the advisor. This reasoning however
does not account for the pooling at the top, and in particular, the realizations of types when
the second highest value is below $v^*$, but the highest value is above $v^*$.\footnote{This can be clearly seen in the uniform example in subsection 3 with $N$ bidders. The option value of
the bidder at stage 0 is given by

$$
\int_0^{v^*} \left( \frac{v}{10} \right)^{N-1} (v - b - \mathbb{E}[s|s < v])dv + \int_{v^*}^{10} \left( \frac{v}{10} \right)^{N-1} (v - b - \frac{1}{2}v^*)dv,
$$

where the first term accounts for the realizations of the bidder’s type below $v^*$, while the second term for
realizations of type above $v^*$. The argument above accounts for the first, but not the second term. The
first term is indeed negative and converges to zero at speed $\left( \frac{v^*}{10} \right)^{N-1} \frac{1}{N}$, while the first term is positive
and converges to zero at a slower speed of $\left( \frac{v^*}{10} \right)^{N-1}$. Thus, even when $N \to 0$, the bidder’s value does not
become negative, although it converges to zero, as the distribution of the second-highest type gets more and
more concentrated at $\overline{v}$.}

Interestingly, in the general case, increased competition reduces the scope of information
transmission. Equation (6) for $v^*$ does not depend on $N$, but condition (8) becomes more
stringent as $N$ increases, as it needs to hold for a larger set of $n$s. The reduction in the
communication can happen because the value of the advice for the bidder depends on the
competitiveness of the auction, and this dependence can be non-monotone. When $n$ is larger,
the value of the highest opponent bidder is higher in the sense of first-order stochastic
dominance of distribution $G_n$. However, the integrand in (8) need not be a monotone
function when $MRL$ is non-monotone, and so, the option can both increase and decrease as
$n$ increases.

Proposition 5 also sheds light on the communication in static auctions. In static auctions,
the dependence of the communication on $N$ is more convoluted than in the English auction.
The number of bidders $N$ enters recursion (39) in a complicated way and from it, it is not clear
how $N$ affects the communication partition. However, from Theorem 4, the communication
partition in the English auction is finer than the partition generated by any static auction.
This implies that the communication in static auctions does not become perfect as we increase
the competitiveness of the auction, which is a priori not obvious from recursion (39).

5.4 Discussion

This section discusses the generality of the characterization and the role of commitment.

**Generality** The characterization in Theorem 3 can be extended in several directions. First,
Theorem 3 can be immediately generalized with obvious changes of notations to the case of
bidders with different biases $b_i$ and value distributions $F_i$.

Second, the analysis of the English auction does not change with the introduction of the reserve price which essentially boils down to the truncation of the distribution of values. It is a standard result in the auction literature that the reserve prices can increase the auction revenue (Myerson, 1981). In fact, in the English auction the optimal reserve price can be easily computed: it is simply the minimum of the optimal reserve price $\varphi^{-1}(0)$ and $v^* + b$.

Third, the characterization does not require the seller to know the magnitude of $b$. It is sufficient that it is common knowledge that there is a conflict of interest on the bidder’s side and $b > 0$.

The generality of Theorem 3 is in contrast with the characterization of equilibria in static auctions in Theorem 1 which becomes more complicated for asymmetric bidders and reserve prices. For example, in asymmetric auctions threshold types are generally different for different bidders and pinned down by a complex system of recursive equations. Similarly, while one can easily compute an optimal reserve price in the English auction, this is not the case in static auctions. Indeed, if the seller restricts bids in the second price auction to be above some $r$, then this affects the equilibrium communication. After the introduction of the reserve price, the distribution of values is $F(\cdot | v \in [r, \bar{v}])$ and generally the partition of types generated in equilibrium changes, which in turn changes which types tie with each other in equilibrium. Hence, determining the optimal revenue price is more difficult in the static auction. We find the simplicity and robustness of the English auction to be an appealing property, which is in stark contrast with the complexity of the static auctions.

Let us mention that the dynamic auction formats have advantages over static formats in the presence of the advisors' bias only when there are several bidders, but does not hold in negotiations where there is only one buyer. First, when there is only one buyer, it is efficient to allocate to this buyer by posting a price $\bar{v}$, as $\bar{v} \geq 0$. Second, under the increasing virtual valuation $\varphi$ and $\mathbb{E}[v|v > \varphi^{-1}(0)] \geq \hat{v}$, in the negotiation, it is optimal to post a price $\varphi^{-1}(0)$. Indeed, such a price is optimal when the seller sells directly to the advisor. Moreover, if the advisor simply tells the bidder whether to buy or not at price $\varphi^{-1}(0)$, under $\mathbb{E}[v|v > \varphi^{-1}(0)] \geq \varphi^{-1}(0)$, it is optimal for the bidder to follow the advisor's recommendation. Intuitively, when there is only one buyer, a coarse information is sufficient to implement both the efficient and optimal allocations. Thus, there is no advantage of using dynamic auction formats. When there are several buyers, the seller needs to extract finer information about values to implement both efficient and optimal outcomes. Therefore, there is a benefit in using the dynamic auction formats, as they enable a better information
transmission.

Commitment We assume that there is no formal contract that allows the bidder to com-
mitt to a certain bidding strategy. Interestingly, in the English auction, the bidder essentially
delegates bidding to the advisor with the cap on the bid level. A natural question is if the
bidder could offer a contract to the advisor, would it differ from the delegation contract.20

Specifically, consider the following auction with contracts. Suppose that each bidder can
commit to a contract that specifies a mapping from advisor’s type announcements into bids
in the auction.21 In the beginning, bidders commit to such contracts. Then advisors com-
municate their private information, and bidders bid in the auction abiding to their contract.
The following proposition shows that offering a delegation contract is an equilibrium of this
game.

Proposition 6. Suppose MRL is decreasing and \( \inf_{v \in [\underline{\pi}, \overline{\pi}]} (\ln f(v))' \geq -b \). Strategies de-
scribed in Theorem 3 also constitute an equilibrium of the English or second-price auction
with contracts.

The proof follows from a general analysis of the delegation problem in Amador and
Bagwell (2013).22 Proposition 6 states that if all bidders in the English auction follow the
advisor’s recommendation up to threshold \( v^* \), then the bidder would not gain from the
ability to commit to an optimal contract. In particular, committing to a coarse information
transmission is not optimal, and the bidder prefers to use a finer information of the advisor
even though it comes at cost of implementing an optimal bid of the advisor. Notice that the
strategic equivalence of the English and second-price auctions is restored when bidders can
commit to contracts.

There is normally a drastic difference between the outcomes with and without commit-
tment. The second-price auction is as a typical example. Without commitment only coarse
communication is possible, while the optimal contract with commitment is a partial dele-
gation. Surprisingly, in the auction setting such a difference disappears once we consider
dynamic auctions. Our analysis suggests what type of commitment the bidder needs to get
an optimal outcome. In the English auction, although the bidder does not have a full com-
mitment, he has a partial commitment not to lower bids after the information is revealed
and this is enough to implement an optimal contract.

20 In the literature on delegation, such a contract is called the *interval delegation*.
21 By the revelation principle, restricting attention to direct mechanisms is without loss of generality.
22 Their analysis should be augmented, as their concavity assumptions on payoff functions are not satisfied
in the English auction.
6 Bias toward Underbidding

Motivated by the empire-building and career concerns, we focused so far on the bias toward overbidding. This section considers the case $b < 0$ when advisors are biased toward underbidding that could be more relevant in procurement auctions.\(^{23}\) As in the case of overbidding bias, dynamic auction formats can attain better information transmission and higher efficiency than static auctions, but unlike the case of overbidding bias, can lead to a lower revenue.

The analysis of static auctions is similar to the case of bias toward overbidding. In particular, the characterization of equilibria in Theorem 1 and the version of the revenue equivalence in Theorems 2 still hold. Thus, the communication strategy has a partition structure and one can focus on the second-price auction in the analysis of static conditionally efficient auctions.

We now turn to dynamic auctions. When $b < 0$, if the bidder knew the value, then she would submit a higher bid than the advisor. Then the English auction does not have an advantage over static auctions, as it only restricts the bidder to submit bids lower than the running price. However, now the Dutch auction can allow for a better information transmission, as it restricts the bidder from submitting bids above the running price. A relevant statistics of the distribution is the mean-advantage-over-inferiors $\text{MAI}$ defined as

$$\text{MAI}(s) \equiv s - \mathbb{E}[v|v \leq s].$$

Most of the commonly used distributions have strictly increasing $\text{MAI}$.\(^{24}\) The next theorem constructs a partially revealing PBEM of the Dutch auction.

**Theorem 6.** Suppose $b < 0$ and $\text{MAI}$ is strictly increasing. Let $v^*$ be the solution to

$$\mathbb{E}[v|v < v^*] = v^* + b, \quad (12)$$

where $v^* = \bar{v}$ if equation (12) does not have a solution. There exists PBEM of the Dutch auction characterized by $\{\sigma(\cdot), v^*\}$ as follows. The advisor of type $v > v^*$ sends message “stop” when the running price $p$ reaches $\sigma(v) \equiv \mathbb{E}[\max\{\hat{v}, v^*\} + b|\hat{v} < v]$ and the advisor of type $v < v^*$ sends “stop” when the running price $p$ reaches $\sigma(v^*)$. The bidder immediately

---

\(^{23}\)The bias toward underbidding is also relevant in takeover contests when the management has the “quiet life” preference (Bertrand and Mullainathan, 2003): incorporating additional business requires additional effort from managers and managers prefer not to increase the size of the firm.

\(^{24}\)In particular, all distributions with a strictly concave log of c.d.f. have increasing $\text{MAI}$ (see Bagnoli and Bergstrom (2005) for related results and a list of distributions with log-concave c.d.f.).
stops the auction after she receives the message “stop” or when the running price $p$ reaches $\sigma(v^*)$.

As with the overbidding bias, the equilibrium constructed in Theorem 6 is partially revealing and quite different from the partition equilibrium in static auctions. The partially perfect communication is possible because the Dutch auction restricts bidders not to increase the bids. Because of the underbidding bias of the advisor, the optimal price of stopping the auction for the bidder is higher than for the advisor. Thus, it is optimal for her to stop immediately after she gets a recommendation from the advisor. Function $\sigma(v)$ in Theorem 6 is the equilibrium bidding strategy in the Dutch auction if bids were submitted directly by advisors, and it is implemented for $v > v^*$. However, unlike the case $b > 0$, the pooling happens at the bottom of the distribution, not at the top. The reason for this is that at a certain stage, the uncertainty of the bidder about her value is sufficiently reduced. Then the bidder prefers stopping the auction immediately to guarantee the victory, rather than trying to win at a lower price, but facing the risk of loosing the auction.

We next turn to the comparison of auction formats.

**Theorem 7.** Suppose $b < 0$. The PBEM of the Dutch auction in Theorem 6 is more efficient than any conditionally efficient equilibrium of any static auction.

As in the case of overbidding bias, because of the superior information transmission, the Dutch auction is more efficient than any static auction. This is true despite that there is a pooling at the bottom in the Dutch auction. We show that generally the pooling region below $v^*$ is always smaller than any first element of the partition equilibrium in any static auction. As an illustration, in the exponential example, equation (12) can be simplified to

$$\frac{v^*}{1 - e^{-\lambda v^*}} = \frac{1}{\lambda} - b,$$

from which one can see that $v^* < x$ where $x$ is defined in Proposition ??.

The left panel of Figure 5b depict the expected value of the winner in the two auction formats when $\lambda = \frac{1}{5}$. The Dutch auction is more efficient than the second-price auction and the gap in the efficiency increases as the size of the bias increases.

Unlike the case $b > 0$, the revenue comparison is ambiguous when the advisor is biased toward underbidding. We can proceed as in Theorem 5 to break down the expected revenue into two parts as in (11): the part increasing with the auction efficiency and the part

\[ e^{\lambda v} \left( e^{\lambda v} - 1 \right) > e^{2\lambda v} + 1 \]

which indeed holds.
decreasing with the rent of the lowest type. Because of the higher efficiency the first term in (11) is higher in the Dutch auction. However, the rent of the lowest type is lower in the second-price auction. Indeed, because the first element of the partition in the second-price auction is larger than the pooling region in the Dutch auction ($\omega_1 > v^*$), type 0 has a lower probability of winning in the Dutch auction and he also pays a lower price conditional on winning. Therefore, the second-price auction may bring higher revenue than the Dutch auction, if the second term in (11) dominates the first term. In fact, in our exponential example, the revenue of the Dutch auction is lower for all $b < 0$ (see the right panel of Figure 5b).

Finally, notice that the NITS condition does not play any role in the revenue comparison. When $b < 0$, if the bidder asked value $v$ and thought that the advisor tells the truth, then all types of the advisor would bias their reports downwards and nobody would prefer to tell $v$. Thus, when $b < 0$, the weakest type of advisor $v_{w}^{p}$ is the highest type remaining in the game at stage $p$. With this change in the specification of the weakest type, the definition of the NITS condition for $b < 0$ is the same as in Definition 2. Since NITS puts restrictions on the utility of the highest type, while in the revenue comparison, the utility of the lowest type matters, the NITS condition does not play as important role as in the case of $b > 0$.

7 Quantitative Example: Auctions of Companies

In this section, we assess the quantitative implications of our analysis, applying the model to auctions of companies. The analysis of Section 4 proves that the ascending-bid auction dominates static mechanisms in both efficiency and revenues. However, these results do not imply that the difference is meaningful quantitatively. Suppose that each bidder $i$ is a firm, consisting of the board of directors and the manager. The board has formal authority over submitting bids but has no information about firm’s valuation of the target $v_i$, except for the prior distribution. The manager knows $v_i$, but has a bias $b > 0$ for overpayment.

To get a plausible value of $b$, we use the following argument. There is a strong empirical evidence that the compensation of CEO and other top executives is increasing in the absolute size of the firm. This dependence leads to their bias for overpaying for the target. On the other hand, overpaying for the target results in the destruction of firm value and ultimately in a poor performance of the acquirer’s stock price. Since the wealth of top managers is sensitive to their company’s stock price, there is a limit to which they are willing to

---

26The advisor of type $v$ gets negative profit in both auctions, and pays $m_1 = E[v|v < \omega_1]$ in the second-price auction, and $v^* + b = E[v|v \leq v^*]$ in the Dutch auction. Since $\omega_1 > v^*$, the latter price is smaller.
overpay for the target. Bias $b$ is the point at which the positive effect on compensation of higher firm size exactly is exactly offset by the negative effect on compensation due to firm value destruction. To get the estimate of $b$, we use CEO compensation regressions from Harford and Li (2007) and the characteristics of the typical deal from Betton, Eckbo, and Thorburn (2008). Since the market leverage ratio of the median target is 13.1% and the median ratio of the deal size to the acquirer’s assets is 31%, the ratio of the deal size to the acquirer’s equity for a typical deal is $31\% \times \frac{1}{0.869} = 35.67\%$. Since the median acquisition premium is 39%, the ratio of the pre-deal target’s equity to the acquirer’s equity is $35.67\% \times \frac{1}{1.39} = 25.66\%$. Assume that after the acquisition, the sales of the combined company increase by the same amount, i.e., by 25.66% in perpetuity. Using the estimate of Harford and Li (2007), this increase in sales leads to an increase in the acquirer’s CEO compensation by $0.435 \times \log (1 + 0.2566) = 4.32\%$ every year. In addition, acquiring the target is associated with an increase in the CEO compensation of 3.7%, irrespectively of the increase in sales, in the year of the acquisition. Thus, the positive effect of acquiring the target on CEO compensation for a typical deal is 8.02% in the year of the deal and 4.32% in every subsequent year. Using the expected tenure of 6 years and discounting at 10%, the present value of the positive effect is 22.52% of the CEO’s annual compensation. On the other hand, overbidding by $b$ (normalizing the pre-acquisition value of the target’s equity to one), reduces the acquirer’s equity value by $b \times 0.2566$. Since the portfolio value of equity incentives is 9.5 times the CEO annual pay (Table II in Harford and Li, 2007), the negative effect on the CEO wealth is $9.5 \times b \times 25.66\%$ of the CEO’s annual compensation. The estimate of overpayment bias $b$ is thus $b = \frac{0.2252}{9.5 \times 0.2566} = 9.2\%$.\footnote{This value is likely an underestimate, since it ignores non-financial benefits of the acquirer’s CEO, such as the preference for power and empire-building, and since the sales of the combined firm may exceed the sum of the individual companies’ sales due to synergies.} For example, if the value of target under its current ownership is $1billion and the true value of the target to the acquirer is $1.4 billion, the maximum willingness to pay for the target by the CEO is $1.492 billion.

To get a plausible distribution of valuations, we use the estimates from Gorbenko and Malenko (2014). We normalize the value of the target under its current management to one. Using data on bids and assuming lognormal distribution, Gorbenko and Malenko (2014) estimate that the valuations of strategic bidders are distributed with parameters $\mu = 0.167$ and $\sigma = 0.258$. We use this distribution, truncated at one, for the distribution of valuations in our numerical example. We assume that there are $N = 4$ bidders.

The results are presented in Table 1. First, consider the ascending-bid auction. The
unique equilibrium satisfying the NITS criterion is that each bidder increases her bid until her advisor recommends to stop doing so. In other words, the estimated value of $b$ is low enough to imply informative communication. Second, consider static auctions (for concreteness, the second-price auction). The most informative equilibrium in this case consists of three partitions, $[1, 1.12]$, $[1.12, 1.38]$, and $[1.38, \infty]$. The corresponding expected valuations are 1.06, 1.24, and 1.64. In the second-price auction, each advisor communicates that the valuation is in one of the three partitions, and the bidder submits one of the three expected values. The comparison of expected revenues is striking. The expected takeover premium is 49% in the ascending-bid auction, which is 23% higher than the expected takeover premium in static auctions (21%). The comparison of efficiency is less striking, but the difference is also sizable: The expected valuation of the winning bidder is 1.65 in the ascending-bid auction, but 1.57 in the static auction. As the comparison of expected bidders' payoffs illustrates, an increase in revenues largely occurs because of the more aggressive bidding among bidders. Overall, we conclude that the result that the ascending-bid auction is more efficient and brings more revenues than static auctions is quantitatively very large, at least for the application of auctions of companies.

8 Conclusion

This paper studies the interaction between the information transmission and bidding in auctions. In static auctions, the revenue-equivalence result holds giving in particular equivalence of the first- and second-price auctions. However, dynamic auctions, such as the English and the Dutch auctions, are generally more efficient than static auctions. This happens because in dynamic auctions the set of bids available to the bidder shrinks. Therefore, by sending the information later in the game, the advisor can induce the bidder to choose a more favorable action and hence, would provide a more refined information to the bidder. Moreover, the English auction also dominates static auctions in terms of revenue when advisors are biased.
toward overbidding, the case most relevant empirically. This paper characterize equilibria in different auction formats and shows the efficiency/revenue comparison.

A Appendix

Proofs for Section 2

Proof of Lemma 1. Specify new online strategies $m'$ and $a'$ as follows. Let $m'(v, h) = a(h, m(v, h))$ and $a'(h, \tilde{a}) = \tilde{a}$ for all $h \in \mathcal{H}$ and all $\tilde{a}$ in the image of $m'(\cdot, h)$. For any $h$, fix an action $\tilde{a}(h)$ in the image of $m'(\cdot, h)$. For any recommendation that does not belong to the image of $m'(\cdot, h)$, the bidder interprets this deviation as a recommendation of action $\tilde{a}(h)$. Hence, it is sufficient to guarantee that advisors do not deviate to on-path recommendations. Clearly, strategy profiles $m'$ and $a'$ generate the same outcome. The proof that they constitute an equilibrium is provided in the text.

Proofs for Section 4

Proof of Lemma 2. The first part of the lemma follows directly from Theorem 3.2 in Milgrom, 2004. Let $\bar{q}(\theta) = \mathbb{E}[q(\theta, \theta_{-i})|\theta]$ be type $\theta$’s expected probability of allocation and $\bar{t}(\theta) = \mathbb{E}[t(\theta, \theta_{-i})|\theta]$ be type $\theta$’s expected payment in the truthful equilibrium of some direct mechanism satisfying the assumption of the lemma, where expectations are taken with respect to $\theta_{-i}$ distributed i.i.d. according to $F_\theta$. Consider a strategy $m$ of the advisor in such a direct mechanism. If advisors follow strategy $m$ in the second-price auction, then functions $\bar{q}$ and $\bar{t}$ in the second-price auction are the same as in the direct mechanism. This implies that the strategy $m$ also constitutes an equilibrium in the second-price auction, as the set of deviations that the advisor can induce in the second-price auction is the same as in the direct mechanism.

Proof of Theorem 1. To any profile of bids $\vec{a} = (a_i)_{i \in \mathcal{N}}$ corresponds an allocation $(q_1(\vec{a}), \ldots, q_N(\vec{a}))$ such that $\sum_{i=1}^{N} q_i(\vec{a}) = 1$ and transfers $(t_1(\vec{a}), \ldots, t_N(\vec{a}))$. Denote by $q(a_i) \equiv \mathbb{E}[q_i(a_i, a_{-i})]$ and $t(a_i) \equiv \mathbb{E}[t_i(a_i, a_{-i})]$ the expected probability of allocation and transfer, respectively, from action $a_i$, where expectations are taken fixing strategies of other bidders and advisors $m_{-i}$ and $a_{-i}$. Bidder $i$ chooses a bid from $A$ given that her expected value is $\theta_i = \mathbb{E}[v_i|a_i]$. Let where $Q = \{q(a_i), a_i \in A\}$ and $t(q) = \min_{a_i, q = q(a_i)} t(a_i)$. Then the bidder and the advisor play the cheap-talk game with payoffs given by

\begin{align*}
\text{Bidder} & : \quad qv - t(q), \\
\text{Advisor} & : \quad q(v + b) - t(q).
\end{align*}
Since the mixed derivatives of (13) and (14) are positive, the set of types of the advisor that induce the same probability of allocation $q$ is an interval. Therefore, to characterize equilibria of the second-price auction, we need to determine incentives of threshold types of the advisor $\omega_k$. Consider any such type $\omega_k$. In the second-price auction, a message is simply an expected value of the bidder $m_k$. Let $\hat{m}$ be the message of the highest bidder among $N-1$ opponents of the bidder. From submitting a message $m_k$, type $\omega_k$ gets utility

$$E[\omega_k + b - \hat{m} | \hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k) \Lambda_k (\omega_k + b - m_k).$$

From submitting a message $m_{k+1}$, type $\omega_k$ gets utility

$$E[\omega_k + b - \hat{m} | \hat{v} < \omega_{k-1}] + G(\omega_{k-1}, \omega_k) (\omega_k + b - m_k) + G(\omega_k, \omega_{k+1}) \Lambda_{k+1} (\omega_k + b - m_{k+1}).$$

Type $\omega_k$ should be indifferent between the two which gives equation (5).

**Claim 1.** If $\omega_{k+1} = \omega_k$, then either $k = 0$ or $k = K$.

**Proof:** Suppose by contradiction that for some $0 < k < K$, $\omega_{k+1} = \omega_k$. This implies that $H(\omega_k, \omega_{k+1}) = 0$ and so, from (5), $H(\omega_{k-1}, \omega_k)(1-\Lambda_{k-1})(\omega_k + b - m_k) = 0$ and $H(\omega_{k+1}, \omega_{k+2}) \Lambda_{k+2} (\omega_{k+1} + b - m_{k+2}) = 0$. This implies that $\omega_k + b = m_k$ and $\omega_{k+1} + b = m_{k+2}$. But only the first equality can hold if $b < 0$ and only the second equality can hold if $b > 0$, contradiction. q.e.d.

**Claim 2.** There exists $\varepsilon > 0$ such that for all $k$, either $\omega_{k+1} - \omega_k > \varepsilon$ for $0 < k < K$.

**Proof:** It follows from (5) that whenever $\omega_{k-1} < \omega_k < \omega_{k+1}$, we have

$$\omega_k + b > E[v | v \in [\omega_{k-1}, \omega_k]]$$

and

$$\omega_k + b < E[v | v \in [\omega_k, \omega_{k+1}]].$$

First, consider $b > 0$. If for any $\varepsilon > 0$, there exists and equilibrium such that $\omega_{k+1} - \omega_k < \varepsilon$, then for such equilibrium $E[v | v \in [\omega_k, \omega_{k+1}]] \leq \omega_k + \varepsilon$ which contradicts (16) for sufficiently small $\varepsilon$. Now, consider $b < 0$. If for any $\varepsilon > 0$, there exists and equilibrium such that $\omega_k - \omega_{k-1} < \varepsilon$, then for such equilibrium $E[v | v \in [\omega_{k-1}, \omega_k]] \geq \omega_k - \varepsilon$ which contradicts (15) for sufficiently small $\varepsilon$. q.e.d.

The fact that there exists $K$ such that there is an equilibrium with $K$ segments for any $1 \leq K \leq \overline{K}$, but not for $K > \overline{K}$ can be proven by the same argument as in the proof Theorem 1 in Crawford and Sobel (1982).

**Proof of Corollary 1.** Since there is an equilibrium partition for any $K \leq \overline{K}$, it is sufficient to show that there is an equilibrium with two intervals in the partition. For $K = 2$, we can use the same
argument as in the proof of Theorem 4 to show that (5) implies

$$\omega_1 + b - \mathbb{E}[v | v \in [\omega_0, \omega_2]] \leq 0. \hspace{1cm} (17)$$

Since in the equilibrium with $K = 2$, $\omega_0 = \underline{v}$ and $\omega_2 = \overline{v}$, and $\omega_1 \geq \underline{v}$, we get the desired conclusion.

It is easy to check that for $N = 2$, the inequality in (17) is an equality and an equilibrium with two segments exists whenever equation $\omega_1 + b - \mathbb{E}[v] = 0$ has a solution. Since $b > \mathbb{E}[v] - \underline{v}$, whenever $b \leq \mathbb{E}[v] - \underline{v}$, such solution exists by continuity which proves the sufficiency of condition in the corollary.

Proof of Theorem 2. In this proof, it is useful to introduce the following notations:

$$\Phi(\omega_{k-1}, \omega_k) \equiv \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{1}{n+1},$$

$$\Psi(\omega_{k-1}, \omega_k) \equiv \sum_{n=1}^{N-1} \binom{N-1}{n} F(\omega_{k-1}, \omega_k)^n F(\omega_{k-1})^{N-1-n} \frac{n}{n+1},$$

$$m(\omega_{k-1}, \omega_k) \equiv \mathbb{E}[v | v \in (\omega_{k-1}, \omega_k)].$$

Define function

$$H(\omega_{k-1}, \omega_k, \omega_{k+1}) \equiv \Psi(\omega_{k-1}, \omega_k)(\omega_k + b - m(\omega_{k-1}, \omega_k)) + \Phi(\omega_k, \omega_{k+1})(\omega_k + b - m(\omega_k, \omega_{k+1})). \hspace{1cm} (18)$$

It is easy to check that function $H$ coincides with the left-hand side of equation (5). By Theorem 1, any equilibrium of the second-price auction is outcome-equivalent to an equilibrium having the partition structure with thresholds $(\tilde{\omega}_k)_{k=1}^{K}$ solving the recursion

$$H(\tilde{\omega}_{k-1}, \tilde{\omega}_k, \tilde{\omega}_{k+1}) = 0 \hspace{1cm} (19)$$

with $\tilde{\omega}_0 = \underline{v}$ and $\tilde{\omega}_{K+1} = \overline{v}$. We show that if the NITS condition fails, then for any such solution, there exists a different solution to the recursion (19) with $K + 1$ partition cells. Since there are at most $\overline{K}$ partition cells, this implies that there exists an equilibrium satisfying NITS, and in particular, the most informative equilibrium satisfies NITS. We consider separately cases $b > 0$ and $b < 0$. Here, we consider only the case $b > 0$. Case $b < 0$ is analogous and for completeness, the Online Appendix provides the details.

If type $\underline{v}$ reveals herself to the bidder, then the bidder prefers to submit a losing bid. Suppose to contradiction that NITS fails and $\underline{v} + b < m(\underline{v}, \tilde{\omega}_1)$. We show by induction that for any $k \leq K+1$, there exists another solution $(\omega^k_j)_{j=1}^{K+1}$ to (19) such that $\omega^k_0 = \underline{v}$, $\omega^k_k > \tilde{\omega}_{k-1}$, and $\omega^k_{k+1} = \tilde{\omega}_k$. Theorem 2 follows from the claim applied to $k = K + 1$.

For $k = 1$, the failure of NITS implies that $H(\underline{v}, \underline{v}, \tilde{\omega}_1) = \Phi(\underline{v}, \tilde{\omega}_1)(\underline{v} + b - m(\underline{v}, \tilde{\omega}_1)) < 0$. At
the same time, \( H(\underline{v}, \overline{\omega}_1, \overline{\omega}_1) = \Psi(\underline{v}, \overline{\omega}_1)(\overline{\omega}_1 + b - m(\underline{v}, \overline{\omega}_1)) > 0 \), as \( \overline{\omega}_1 = m(\overline{\omega}_1, \overline{\omega}_1) \geq m(\underline{v}, \overline{\omega}_1) \). By continuity, there exists \( x \in (\underline{v}, \overline{\omega}_1) \) such that \( H(\underline{v}, x, \overline{\omega}_1) = 0 \) proving the claim for \( k = 1 \).

Suppose the statement is true for \( k \) and we next prove it for \( k + 1 \). Since \( \overline{\omega}_k \) solves (19), \( H(\overline{\omega}_{k-1}, \overline{\omega}_k, \overline{\omega}_{k+1}) = 0 \) or

\[
\Psi(\overline{\omega}_{k-1}, \overline{\omega}_k)(\overline{\omega}_k + b - m(\overline{\omega}_{k-1}, \overline{\omega}_k)) + \Phi(\overline{\omega}_k, \overline{\omega}_{k+1})(\overline{\omega}_k + b - m(\overline{\omega}_k, \overline{\omega}_{k+1})) = 0. \tag{20}
\]

Let \( \omega^k_k > \overline{\omega}_{k-1} \) and \( \omega^k_{k+1} = \overline{\omega}_k \) as in the inductive hypothesis and consider \( H(\omega^k_k, \overline{\omega}_k, \overline{\omega}_{k+1}) \):

\[
\Psi(\omega^k_k, \overline{\omega}_k)(\overline{\omega}_k + b - m(\omega^k_k, \overline{\omega}_k)) + \Phi(\overline{\omega}_k, \overline{\omega}_{k+1})(\overline{\omega}_k + b - m(\overline{\omega}_k, \overline{\omega}_{k+1}))
\]

which differs from (20) only in the first term. Since \( \omega^k_k > \overline{\omega}_{k-1} \), \( m(\omega^k_k, \overline{\omega}_k) > m(\overline{\omega}_{k-1}, \overline{\omega}_k) \). Moreover, the binomial distribution with probability of success \( \frac{F(\overline{\omega}_{k-1}, \overline{\omega}_k)}{F(\overline{\omega}_k)} \) first-order stochastic dominance the binomial distribution with probability of success \( \frac{F(\omega^k_k, \overline{\omega}_k)}{F(\overline{\omega}_k)} \). Hence,

\[
\frac{\Psi(\omega^k_k, \overline{\omega}_k)}{F(\overline{\omega}_k)^N} = \sum_{n=1}^{N-1} \binom{N-1}{n} \left( \frac{F(\omega^k_k, \overline{\omega}_k)}{F(\overline{\omega}_k)} \right)^n \left( \frac{F(\overline{\omega}_{k-1}, \overline{\omega}_k)}{F(\overline{\omega}_k)} \right)^{N-1-n} \frac{n}{n+1} < \sum_{n=1}^{N-1} \binom{N-1}{n} \left( \frac{F(\overline{\omega}_{k-1}, \overline{\omega}_k)}{F(\overline{\omega}_k)} \right)^n \left( \frac{F(\overline{\omega}_{k-1}, \overline{\omega}_k)}{F(\overline{\omega}_k)} \right)^{N-1-n} \frac{n}{n+1} = \frac{\Psi(\overline{\omega}_{k-1}, \overline{\omega}_k)}{F(\overline{\omega}_k)^N},
\]

as and \( \frac{n}{n+1} \) is increasing in \( n \). Therefore, \( H(\omega^k_k, \overline{\omega}_k, \overline{\omega}_{k+1}) < 0 \).

On the other hand, since \( \omega^k_{k+1} = \overline{\omega}_k \),

\[
H(\omega^k_k, \overline{\omega}_k, \omega^k_{k+1}) = \Psi(\omega^k_k, \overline{\omega}_k)(\overline{\omega}_k + b - m(\omega^k_k, \overline{\omega}_k)) > 0.
\]

By continuity, there exists \( x \in (\overline{\omega}_k, \omega^k_{k+1}) \) such that \( H(\omega^k_k, \omega^k_{k+1}, x) = 0 \). By continuity, we can find solution \((\omega^k_{j+1})_{j=1}^{K_{k+1}}\) to (19) with \( \omega^k_{k+1} > \overline{\omega}_k \) and \( \omega^k_{k+2} = \overline{\omega}_{k+1} \), which completes the proof of the inductive step.

\[ \square \]

**Proofs for Section 5**

By Lemma 1, the strategy of the advisor can be described by a function \( m(v) \) which specifies at what price the advisor sends message “quit” to the bidder. The following lemma shows that in the English auction, types of advisor either perfectly reveal themselves to the bidder or pool with neighboring types.

**Lemma 3.** Function \( m(v) \) is increasing on a subset of \([v, \overline{v}]\) of Lebesgue measure \( \overline{v} - v \).

**Proof of Lemma 3.** Suppose to contradiction that \( m(v) \) is strictly decreasing on a set of positive measure \( I \). Let \( v = \inf I \) and \( v' = \sup I \). Then \( m(v) > m(v') \) and \( G(v') - G(v) > 0 \). Let \( q \) and \( t \) be
the probability of winning and expected price paid conditional on using strategy \( m(v) \) and \( q' \) and \( t' \) be the probability of winning and expected price paid conditional on using strategy \( m(v') \). Then

\[
q_v - t \geq q' v - t',
\]

\[
q' v' - t' \geq q v' - t,
\]

implies that \( q' \geq q \). By quitting at price \( m(v) \) instead of \( m(v') \), the advisor increases the probability of winning by at least \( G(v') - G(v) > 0 \) and so, \( q > q' \) which is a contradiction. \( \square \)

**Lemma 4.** \( m \) is strictly increasing on \([v, v^*]\) and is constant almost everywhere on \([v^*, \overline{v}]\) where \( v^* \) satisfies (6) when \( v^* > v \) and \( v^* + b \geq E[v|v \geq v^*] \) when \( v^* = v \).

**Proof of Lemma 4.** Any equilibrium generates a partition \( \Pi \) of \([v, \overline{v}]\) satisfying for any \( \pi \in \Pi, v, v' \in \pi \iff m(v) = m(v') \). We say that types in \( \pi \in \Pi \) pool if \( m(v) \) is constant on \( \pi \), i.e. these types send message “quit” at the same price. We say that types in \([v', v'']\) separate, if \( m(v) \) is strictly increasing on \([v', v'']\), i.e. all these types send message “quit” at different prices. Define by \( \Pi^0 \) the closure of the set of all types that pool with some other type. Then \( \Pi^0 = [v, \overline{v}] \backslash \Pi^P \) is the set of all types that separate and denote by \( \partial \Pi^P \) the boundary of \( \Pi^P \).

Notice that the babbling equilibrium is an equilibrium of the English auction and it satisfies NITS if and only if \( E[v] \leq v + b \). So we focus on the case when there is a non-trivial information transmission in equilibrium, i.e. \( \Pi^0 \neq \emptyset \).

We first show that whenever an interval of types perfectly reveals their value to the bidder in the auction, then these types quit at the optimal time.

**Claim 3.** If \( m \) is strictly increasing on a subset \( S \) of \((v', v'')\) of (Lebesgue) measure \(|v'' - v'|\), then \( m(v) = v + b \) on \((v', v'')\).

**Proof:** There exists at most countable number of discontinuities of \( m \) on \( S \). Consider a type \( v \) at which \( m \) is continuous, i.e. there exist sequences \( v^-_j \to v - 0 \) and \( v^+_j \to v + 0 \) such that \( m(v^-_j) \to m(v) - 0 \) and \( m(v^+_j) \to m(v) + 0 \). We show that \( m(v) = v + b \). Suppose to contradiction that \( m(v) < v + b \). Choose \( j \) large so that \( m(v) < m(v^+_j) < v + b \). If type \( v \) sends “quit” at price \( m(v^+_j) \) instead of \( m(v) \), then she can additionally win against types in \([v, v^+_j]\) and pay at most \( m(v^+_j) < v + b \). Therefore, her utility is higher contradicting the rationality of type \( v \). Now, suppose to contradiction that \( m(v) > v + b \). Choose \( j \) large so that \( v + b < m(v^-_j) < m(v) \). If type \( v \) sends “quit” at price \( m(v) \) instead of \( m(v^-_j) \), then she additionally wins against types in \([v^-_j, v]\) and pays at least \( m(v^-_j) > v + b \). Therefore, she strictly gains from sending “quit” at price \( m(v^-_j) \) contradicting the rationality of type \( v \). Therefore, \( m(v) = v + b \) for all continuity points of \( S \). Since \( m(v) = v + b \) on a dense subset of \( S \), it is also true on the whole \( S \). Since \( S \) has measure \(|v'' - v'|\), set \( S \) is dense in \((v', v'')\) and so, \( m(v) = v + b \) on \((v', v'')\). \textbf{q.e.d.}

**Claim 4.** \( \Pi^0 = [v^*, \overline{v}] \) for some \( v^* \geq v \).
Proof: Consider \( v \in \partial \Pi^P \). Type \( v \) is indifferent between pooling with some interval of types \( \pi \ni v \) and separating. Indeed, since \( \pi \in \partial \Pi^P \) and \( \Pi^S \neq \emptyset \), there exists a sequence \( v_j \to v \) such that \( m(v_j) = v_j + b \) by Claim 1. Type \( v \) can mimic type \( v_j \) and for large \( j \) get utility arbitrarily close to her maximal utility. Therefore,

\[
m(v) = v + b = \mathbb{E}[v | v \in \pi].
\]

Suppose to contradiction to Claim 2 there exists a sequence \( v_j \to v \) such that \( v_j \in \Pi^P \) and \( v_j < v \). Then \( \mathbb{E}[v | v \in \pi] < v + b \) which is a contradiction. q.e.d.

Next, we show that all types in \( \Pi^P \) send "quit" at the same price.

Claim 5. \( \Pi^P = \pi \) for some \( \pi \in \Pi \).

Proof: Suppose to contradiction that there are two adjacent intervals of types \( \pi \) and \( \pi' \) such that types in \( \pi \) send "quit" at price \( m \) and types in \( \pi' \) send "quit" at price \( m' > m \). Consider type \( v \) that is at the boundary of \( \pi \) and \( \pi' \). By continuity, type \( v \) is indifferent between sending "quit" at price \( m \) and \( m' \). The benefit of quitting at \( m' \) rather than \( m \) is that type \( v \) wins against types in \( \pi \), but there is a risk that she will tie with types in \( \pi' \). The indifference of type \( v \) implies that \( m' > v + b \). But then consider a time when the running price reaches \( m' \). Type \( v \) is the lowest type. However, she gets a negative utility from pooling with types in \( \pi' \). This contradicts the NITS condition. q.e.d.

Finally, equation (6) follows from (21) and Claim 3.

Proof of Theorem 3. By Lemma 4 condition (6) is a necessary condition. For any \( v^* \) satisfying in addition (10), we construct an equilibrium in online strategies satisfying NITS for dynamic auctions. Then we show that if \( v^* \) fails (10), then it cannot be part of equilibrium.

Consider strategies described in the theorem. The optimality of the advisor and the bidder after she receives the message "quit" is verified in the text. Let us check the optimality of the bidder. Let \( N_p \) be the number of bidders remaining in the game at price \( p \) and \( v_p \) be the lowest type remaining in the game at price \( p \). The utility of the bidder from following the recommendation of the advisor starting from running price \( p \) is equal to

\[
V(N_p, v_p) = \frac{1}{(1 - F(v_p))^N_p F(v_p)^{N-1-N_p}} \left( \int_{v_p}^{v^*} (1 - F(s)) \left( \mathbb{E}[v | v > s] - s - b \right) dG_{N_p}(s) \right) + \sum_{n=1}^{N_p-1} \frac{1}{n} \left( \frac{N_p - 1}{n} \right) (1 - F(v^*))^{n+1} (F(v^*) - F(v_p))^{N_p-1-n} \frac{1}{n} (\mathbb{E}[v | v \geq v^*] - v^* - b).
\]

(22)
By the definition of \( v^* \), the last term is zero and so,

\[
V(N_p, v_p) = \frac{1}{(1 - F(v_p))^N_p F(v_p)^{N-1-N_p}} \left( \int_{v_p}^{v^*} (1 - F(s))(\mathbb{E}[v|v > s] - s - b) dG_{N_p}(s) \right). \tag{23}
\]

The bidder prefers to quit immediately at the first time \( V(N_p, v_p) \) becomes negative. Moreover, (10) implies for all \( v_p \leq v^* \), \( V(N_p, v_p) \geq 0 \) which proves the optimality of the bidder’s strategy.

**Proof of Corollary 4.** To show that \( v^* < \bar{v} \), notice that the left-hand side of (6) is greater than the right-hand side for \( v^* \) sufficiently close to \( \bar{v} \). Therefore, \( v_1^* < \bar{v} \) and for \( \tilde{v} \in (v_1^*, v_0^*) \), \( \mathbb{E}[v|v > \tilde{v}] - \tilde{v} - b < 0 \). This implies that \( v^* < \bar{v} \).

**Proof of Theorem 4.** We will show that for any equilibrium of the second-price auction, there is no \( \omega_k \in (v^*, \bar{v}) \). This implies that the partition generated by the second-price auction is cruder, and so the English auction is more efficient. Suppose to contradiction that there is \( \omega_k \in (v^*, \bar{v}) \) such that \( \omega_{k-1} \leq v^* \). Notice that in equation (5) \( \Lambda_{k+1} = 1/2, 1 - \Lambda_k \geq 1/2 \), \( \omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]] \geq 0 \) and \( \omega_k + b - \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \leq 0 \). Therefore, equation (5) implies

\[
G(\omega_{k-1}, \omega_k)(\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]]) + G(\omega_k, \omega_{k+1})(\omega_k + b - \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]]) \leq 0
\]
or

\[
\omega_k + b - \frac{G(\omega_{k-1}, \omega_k)}{G(\omega_{k-1}, \omega_{k+1})} \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]] - \frac{G(\omega_k, \omega_{k+1})}{G(\omega_{k-1}, \omega_{k+1})} \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \leq 0.
\]

Observe that\(^{28}\)

\[
\frac{G(\omega_{k}, \omega_{k+1})}{G(\omega_{k-1}, \omega_{k+1})} \geq \frac{F(\omega_{k}, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})}.
\]

Since \( \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \geq \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]], \)

\[
\omega_k + b - \frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})} \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]] - \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})} \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \leq 0
\]
or

\[
\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_{k+1}]] \leq 0.
\]

\(^{28}\)Indeed

\[
\frac{F^{N-1}(\omega_{k+1}) - F^{N-1}(\omega_k)}{F^{N-1}(\omega_{k+1}) - F^{N-1}(\omega_{k-1})} \geq \frac{F(\omega_{k+1}) - F(\omega_k)}{F(\omega_{k+1}) - F(\omega_{k-1})}
\]

if and only if

\[
\gamma F^{N-1}(\omega_{k+1}) + (1 - \gamma) F^{N-1}(\omega_{k-1}) \geq F^{N-1}(\omega_k)
\]

for \( \gamma \) satisfying \( \gamma F(\omega_{k+1}) + (1 - \gamma) F(\omega_{k-1}) = F(\omega_k) \) which holds by Jensen’s inequality.
Then
\[ \omega_k - b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_{k+1}]] \geq \omega_k - b - \mathbb{E}[v|v \in [v^*, \omega_{k+1}]] \]
\[ \geq \omega_k - b - \mathbb{E}[v|v \geq v^*] \]
\[ = \omega_k - v^* > 0, \]
which is a contradiction. \hfill \Box

**Proof of Theorem 5.** See the argument after the theorem in the main text. \hfill \Box

**Proof of Proposition 5.** The profit of the seller in the English auction is given by
\[ b + \int_{\bar{v}}^{v^*} \hat{v}dH(\hat{v}) + (1 - H(v^*))v^*, \tag{24} \]
where \( H \) is the distribution of the second order statistic. The derivative of (24) with respect to \( b \) equals 1 + (1 - \( H(v^*) \)) \( \frac{d}{db} v^* \). We can find \( \frac{d}{db} v^* \) by the implicit function theorem from (6),
\[ \frac{d}{db} v^* = -\left(1 - \frac{f(v^*)}{1-F(v^*)}b\right)^{-1}. \]
Since MRL is strictly decreasing, \( \frac{d}{db} v^* < 0 \) for \( b \in [0, \bar{b}] \). Thus, the derivative of the profit with respect to \( b \) equals
\[ \frac{H(v^*) - f(v^*)b}{1 - f(v^*)b}. \tag{25} \]
When \( b \) is close to 0, \( v^* \) is close to \( \bar{v} \). Using (6) and L'Hospital’s Rule, we get
\[ \lim_{v^* \to \bar{v}} \frac{f(v^*)}{1 - F(v^*)}b = \lim_{v^* \to \bar{v}} \frac{f(v^*) \int_{v^*}^{\bar{v}} (v - v^*)dF(v)}{(1 - F(v^*))^2} \]
\[ = f(\bar{v}) \lim_{v^* \to \bar{v}} \frac{1}{2f(v^*)} = \frac{1}{2}. \]
Since \( H(\bar{v}) = 1 \), expression (25) is positive for small \( b \). When \( b \) is approaches \( \bar{b} \), \( v^* \) approaches \( \bar{v} \) and so, \( \frac{f(v^*)}{1 - F(v^*)}b \to f(\bar{v})\bar{b} \) while \( H(v^*) \to 0 \). Therefore, expression (25) is negative for sufficiently large \( b \), which completes the proof of the first statement. To prove the second statement, notice that \( \lim_{N \to \infty} H(v^*) = 0 \). Therefore, there exists \( N(b) \) such that \( H(v^*) - \frac{f(v^*)}{1 - F(v^*)}b < 0 \) for all \( N > N(b) \). Since \( H(v^*) \) decreases monotonically in \( N \), there is \( \varepsilon(b) > 0 \) such that a decrease in \( b \) to \( b - \varepsilon(b) \) is beneficial for the seller for sufficiently large \( N \). The last statement follows from Corollary 1. \hfill \Box

**Proof of Proposition 6.** We consider the auction with contracts and show that if all bidders offer the delegation contracts with caps on the bids at \( v^* + b \), then this contract indeed maximizes each
bidders payoff. Given the strategies of other bidders, all bids above \( p^* \) lead to probability one of winning and the expected payment \( \mathbb{E}[\hat{v} + b] \). Bidding \( p^* \) leads to the probability of winning \( q^* \equiv \mathbb{E}[\frac{1}{M+1}] \) where \( M \) is the number of bidders with value above \( \hat{v} \) and the expected payment \( \mathbb{E}[\frac{1}{M+1}(\hat{v} + b)] \). Bidding \( p < p^* \) leads to the probability of winning \( G(p - b) \) and the expected payment \( \mathbb{E}[\hat{v} + b | \hat{v} \leq p - b] \). Since there is one-to-one mapping between bids and probabilities of allocation, we can equivalently formulate the problem of the bidder as follows. The bidder offers the contract to the advisor, in which to each report of type \( v \) allocation, we can equivalently formulate the problem of the bidder as follows. The bidder offers the contract to the advisor, in which to each report of type \( v \) by the advisor corresponds the probability of winning \( q \) and corresponding expected price \( t(q) \). Denote \( \overline{q} \equiv \lim_{p \rightarrow p^* \rightarrow 0} G(p - b) \). The preferences over \( q \)’s of the bidder and the advisor are

\[
\text{Bidder} : \quad qv - t(q), \quad (26)
\]

\[
\text{Advisor} : \quad q(v + b) - t(q). \quad (27)
\]

The bidder designs a contract that solves the program \( A \):

\[
\max_{q \in Q} \quad \int_{\mathbb{R}} (q(v) - t(q(v))) dF(v)
\]

s.t. \( v \in \arg \max_{v'} \{(v + b)q(v') - t(q(v'))\} \) for all \( v \),

where \( Q \equiv [\overline{q}, \overline{q}) \cup \{q^* - 1\} \). We first show some properties of function \( t(\cdot) \).

**Claim 6.** \( t(q) \) is strictly convex and twice differentiable on \([0, \overline{q}]\).

**Proof:** For any \( q < \overline{q} \), in the English auction type \( v = G^{-1}(q) \) of the advisor wins with probability \( q \). Consider any \( q, q' < \overline{q} \) and types \( v \) and \( v' \) that win in the English auction with probabilities \( q = G(v) \) and \( q' = G(v') \), respectively. Since in the English auction bidding \( v + b \) is strictly optimal for advisor of type \( v \), \( qv - t(q) > q'v - t(q') \). This implies that \( v > \frac{t(q') - t(q')}{q - q'} \) whenever \( q > q' \) and \( v < \frac{t(q) - t(q)}{q - q} \) whenever \( q < q' \), which in turn, implies the strict convexity of \( t \) on \([0, \overline{q}]\). Differentiability of \( t \) is implied by \( t(q) = q\mathbb{E}[\hat{v} + b | \hat{v} \leq G^{-1}(q)] \) for \( q < \overline{q} \). q.e.d.

Notice that function \( t \) cannot be extrapolated to a strictly convex function to the whole interval \([0, 1]\) because of the following claim.

**Claim 7.** Points \((\overline{q}, \lim_{q \rightarrow \overline{q}} t(q)), (q^*, t(q^*)), \) and \((1, t(1))\) lie on the same line with slope \( \lim_{q \rightarrow \overline{q}} t'(q) = v^* + b \).

**Proof:** In the English auction, the bidder with expected value \( \mathbb{E}[v|v > v^*] \) gets expected profit 0 from winning at price \( p^* \). Therefore, he is indifferent between \((\overline{q}, \lim_{q \rightarrow \overline{q}} t(q)), (q^*, t(q^*)), \) and \((1, t(1))\) and so, they lie on the same line (her indifference curve) with slope \( \mathbb{E}[v|v > v^*] = v^* + b \). Since \( t(q) \) is differentiable at \( q < \overline{q} \), for every \( v < v^* \), \( v + b = t'(q(v)) \) which implies that \( \lim_{q \rightarrow \overline{q}} t(q) = v^* + b \). q.e.d.
Denote by $q^A$ a solution to program $A$. We next show that the set $Q$ in program $A$ can be replaced by the union of $Q' \equiv [\overline{\omega}, \underline{\omega}] \cup \{q^*\}$ and $Q'' \equiv [\underline{\omega}, \overline{\omega}] \cup \{1\}$. Indeed, if the image of $q^A$ contains both $q^*$ and 1, then for all types $v > v^*$ of the advisor, and only for them, $q(v) = 1$. By Claim 7, the expected profit of the bidder conditional on $v > v^*$ is zero. This profit will not change, if all such types choose a report corresponding to the probability of winning $q^*$.

Suppose that $q^A \in Q'$ and the argument is symmetric if $q^A \in Q''$. We perturb function $t$ on $[\overline{\omega}, q^*]$ so that the perturbation $t_\varepsilon$ is strictly convex and twice differentiable. Consider the auxiliary program $A_{\varepsilon}$:

$$\max_{q \in [0,q^*], q \neq q^*} \int_0^\overline{\omega} (q(v) - t_\varepsilon(q(v))) dF(v)$$

s.t. $v \in \arg \max_{v'} \{(v + b)q(v') - t_\varepsilon(q(v'))\}$ for all $v$.

Claim 8. The delegation contract with the cap on the bids $v^* + b$ is a solution to program $A_{\varepsilon}$ for any $\varepsilon$.

**Proof:** We verify that conditions (c1), (c2), (c3') of Proposition 1 in Amador and Bagwell (2013) as well as convexity and differentiability assumptions are satisfied when we fix that other bidders follow their equilibrium strategies.

First, Claim 6 verifies the differentiability and convexity assumptions in in Amador and Bagwell (2013). Second, in program $B$, $\kappa$ in Amador and Bagwell (2013) is equal to 1. Let $q_f(v)$ be the optimal choice of $q$ by the advisor of type $v$. Then $F(v) - (v - t'(q_f(v)))f(v) = F(v) - b f(v)$ is nondecreasing if and only if $(\ln f(v))' \geq -\frac{b}{\varepsilon}$. By the assumption of the proposition, we get that (c1) holds. Third, $\tilde{v} - v^* - \int_0^\overline{\omega} (v - v^* - b) \frac{f(v)}{1 - F(v)} dv = \tilde{v} + b - \mathbb{E}[v | v \geq \tilde{v}] \leq 0$ for $\tilde{v} > v^*$ by the decreasing MRL, which verifies (c2). Finally, $v - t'(q_f(v)) = -b < 0$ which verifies (c3'). Therefore, the interval delegation is optimal and $v^*$ is the only candidate for cutoff. q.e.d.

Denote the solution to program $A_{\varepsilon}$ by $\hat{q}_A$. We show that $\hat{q}_A$ also solves program $A$. First, observe that $\hat{q}_A$ satisfies constraints of program $A$. Next, suppose to contradiction that $\hat{q}_A$ is not a solution and there is a $q^A$ that solves program $A$. For any $\varepsilon > 0$, we can find $\varepsilon$ small enough so that there exists $q^A_\varepsilon$ that is $\varepsilon$-close to $q^A$ in the $L_1$ norm and satisfies constraints of program $A_{\varepsilon}$. Since $q^A_\varepsilon$ is $\varepsilon$-close to $q^A$ in the $L_1$ norm, the maximized function in program $A_{\varepsilon}$ evaluated at $q^A$ is at most $\varepsilon$ away from the value of program $A$. But for small enough $\varepsilon$, this contradicts the optimality of $\hat{q}_A$ in program $A_{\varepsilon}$. q.e.d.

Proof of Theorem 4. We will show that for any equilibrium of the second-price auction, there is no $\omega_k \in (v^*, \overline{\omega})$. This implies that the partition generated by the second-price auction is cruder, and so the English auction is more efficient. Suppose to contradiction that there is $\omega_k \in (v^*, \overline{\omega})$ such that $\omega_k \leq v^*$. Notice that in equation (5) $\Lambda_{k+1} \leq 1 - \Lambda_k \geq \frac{1}{2}$, $\omega_k + b - \mathbb{E}[v | v \in [\omega_{k-1}, \omega_k]] \geq 0$

\footnote{For example, we can consider perturbation $t_\varepsilon(q) = t(q) + \varepsilon \max\{0, (q - \overline{\omega})^3\}$.}
and } \omega_k + b - \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \leq 0. \text{ Therefore, equation (5) implies }

G(\omega_{k-1}, \omega_k)(\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]]) + G(\omega_k, \omega_{k+1})(\omega_k + b - \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]]) \leq 0

or

\omega_k + b - \frac{G(\omega_{k-1}, \omega_k)}{G(\omega_{k-1}, \omega_{k+1})} \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]] - \frac{G(\omega_k, \omega_{k+1})}{G(\omega_{k-1}, \omega_{k+1})} \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \leq 0.

Observe that

\frac{G(\omega_k, \omega_{k+1})}{G(\omega_{k-1}, \omega_{k+1})} \geq \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})}.

Since } \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \geq \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]],

\omega_k + b - \frac{F(\omega_{k-1}, \omega_k)}{F(\omega_{k-1}, \omega_{k+1})} \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k]] - \frac{F(\omega_k, \omega_{k+1})}{F(\omega_{k-1}, \omega_{k+1})} \mathbb{E}[v|v \in [\omega_k, \omega_{k+1}]] \leq 0

or

\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_{k+1}]] \leq 0.

Then

\omega_k - b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_{k+1}]] \geq \omega_k - b - \mathbb{E}[v|v \in [v^*, \omega_{k+1}]]

\geq \omega_k - b - \mathbb{E}[v|v \geq v^*]

= \omega_k - v^* > 0,

which is a contradiction. \qed

Proofs for Section 6

Proof of Theorem 6. We first show that strategies described in Theorem 6 constitute an equilibrium of the Dutch auction.

Indeed, the left-hand side of equation (12) is greater than } v \text{ and bounded from above by } \mathbb{E}[v]. The right-hand side of equation (12) is less than } v \text{ for small } v^* \text{ and is greater than } \mathbb{E}[v] \text{ for sufficiently large } v^*. \text{ By continuity, equation (12) has a solution.}

To prove that conjectured strategies constitute an equilibrium, we need to show that the advisor sends the message “stop” at the optimal time given that bidder follows her recommendation, and

\^30\text{Indeed } \frac{F^{N-1}(\omega_{k+1}) - F^{N-1}(\omega_k)}{F^{N-1}(\omega_{k+1}) - F^{N-1}(\omega_{k-1})} \geq \frac{F(\omega_{k+1}) - F(\omega_k)}{F(\omega_{k+1}) - F(\omega_{k-1})}

if and only if

\gamma F^{N-1}(\omega_{k+1}) + (1 - \gamma) F^{N-1}(\omega_{k-1}) \geq F^{N-1}(\omega_k)

for } \gamma \text{ satisfying } \gamma F(\omega_{k+1}) + (1 - \gamma) F(\omega_{k-1}) = F(\omega_k) \text{ which holds by Jensen’s inequality.}
that the bidder prefers to follow recommendations of the advisor.

**Optimality of the advisor**  First, we show that strategy \( \sigma(\cdot) \) is optimal for the advisor. The advisor of type \( v \) solves the following problem

\[
\max_{\sigma} (v + b - \sigma) G(\sigma^{-1}(\sigma)),
\]

for which the first-order condition is

\[
g(v)(v + b) = (G(v)\sigma(v))'
\]

with the initial condition \( \sigma(v^*) = v^* + b \). From (29),

\[
\sigma(v) = \frac{G(v^*)}{G(v)}(v^* + b) + \frac{1}{G(v)} \int_{v^*}^{v} g(\hat{v})(\hat{v} + b) d\hat{v} =
\]

\[
G(v^*) \left( \mathbb{E}[v | v < v^*] \right) + \frac{G(v) - G(v^*)}{G(v)} \left( \mathbb{E}[\hat{v} | \hat{v} \in [v^*, v]] + b \right) =
\]

\[
b \frac{G(v) - G(v^*)}{G(v)} + \mathbb{E}[\hat{v} | \hat{v} < v] - \left( \mathbb{E}[\hat{v} | \hat{v} < v^*] - \mathbb{E}[v | v < v^*] \right) \frac{G(v^*)}{G(v)} =
\]

\[
b + \mathbb{E}[\hat{v} | \hat{v} < v] + (v^* - \mathbb{E}[\hat{v} | \hat{v} < v^*]) \frac{G(v^*)}{G(v)}.
\]

The equilibrium bid is equal to expectation of \( \max\{v^*, \hat{v}\} + b \) conditional on \( \hat{v} < v \). Given (30), the utility of the advisor from winning the auction equals

\[
v - \mathbb{E}[\hat{v} | \hat{v} < v] - (v^* - \mathbb{E}[\hat{v} | \hat{v} < v^*]) \frac{G(v^*)}{G(v)} =
\]

\[
\frac{1}{G(v)} \left( G(v)v - G(v^*)v^* - \mathbb{E}[\hat{v} : \hat{v} \in [v^*, v]] \right) =
\]

\[
\int_{v^*}^{v} G(\omega) \frac{G(v)}{G(\omega)} d\omega > 0.
\]

Hence, if the bidder follows her strategy, then it is optimal for the advisor to follow her strategy.

**Optimality of the bidder**  By the single-crossing property of payoffs, when the bidder knows \( v \), the bidder prefers to stop the auction earlier. Hence, having received the message “stop” from the advisor, the bidder prefers to stop immediately. It remains to check that the bidder does not want to quit the auction before she gets a recommendation from the advisor. By (30), if the bidder
quits at time $t$, then her payoff equals
\[
E[v|v < v_p] - \sigma(v_p) = E[v|v < v_p] - b - E[\hat{v}\hat{v} < v_p] - (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}. \tag{32}
\]

On the other hand, if the bidder follows her equilibrium strategy, then her expected utility is given by
\[
E[(v - \sigma(v))1\{v > \hat{v}\}|\hat{v}, v < v_p] = 
\frac{1}{F(v_p)^2} \int_{v^*}^{v_p} \left( v - b - E[\hat{v}\hat{v} < v] - (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v)} \right) F(v) dF(v), \tag{33}
\]
where the first equality is by the fact that $E[(v - \sigma(v))1\{v > \hat{v}\}|v < v^*, \hat{v} < v_p] = 0$, the second equality is by (30). We need to show that (32) is less than (33). We evaluate the difference
\[
\int_{v^*}^{v_p} \left( v - b - E[\hat{v}\hat{v} < v] - (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v)} \right) F(v) dF(v) - 
F^2(v_p) \left( E[v|v < v_p] - b - E[\hat{v}\hat{v} < v_p] - (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)} \right). \tag{34}
\]
The derivative of (34) divided by $f(v_p)F(v_p)$ is equal to
\[
v_p - b - E[\hat{v}\hat{v} < v_p] - (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)} = 
2 \left( E[v|v < v_p] - b - E[\hat{v}\hat{v} < v_p] - (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)} \right) - 
F(v_p) \left( \frac{1}{F(v_p)} (v_t - E[v|v < v_p]) - \frac{(N-1)F^{N-2}(v_p)}{G(v_p)} (v_p - E[\hat{v}\hat{v} < v_p] - (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}) \right) = 
v_p - 2E[v|v < v_p] + b + E[\hat{v}\hat{v} < v_p] + (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)} - 
\left( v_p - E[v|v < v_p] - (N-1)(v_p - E[\hat{v}\hat{v} < v_p]) + (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}(N-1) \right) = 
(N-1)(v_p - E[\hat{v}\hat{v} < v_p]) + (E[\hat{v}\hat{v} < v_p] - E[v|v < v_p]) + b - (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)}(N-2) = 
v_p - E[v|v < v_p] + b + (N-2) \left( v_p - E[\hat{v}\hat{v} < v_p] - (v^* - E[\hat{v}\hat{v} < v^*]) \frac{G(v^*)}{G(v_p)} \right) \tag{35}\]
In (35) $v_p - E[v|v < v_p] + b > 0$ by the fact that $v^*$ is the unique solution to (12). The remaining term in (35) is positive by (31). Hence, the derivative of (34) is positive. Therefore, since at $v_p = v^*$, the expression (34) is equal to zero by (12), for $\geq v^*$ the expression (34) is non-negative.
This proves that the bidder prefers to follow recommendations of the advisor rather than stop the auction earlier.

**Proof of Theorem 7.** We want to show that there is no partition \((\omega_k)_{k=1}^K\) induced by the equilibrium of the second-price auction such that \(\omega_k \in [v, v^*]\). Since \(v^*\) is the unique solution to (12) and \(v + b - \mathbb{E}[v|v \leq v] = b < 0, \omega_k + b - \mathbb{E}[v|v \leq \omega_k] < 0\). Therefore,

\[
\omega_k + b - \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)] \leq \omega_k + b - \mathbb{E}[v|v < \omega_k] < 0,
\]

which contradicts the fact that the first term in (5) should be positive. \(\square\)

**B Online Appendix (Not for Publication)**

**B.1 Exponential Example**

This subsection solves an example with two bidders \((N = 2)\) and exponential distribution of values with parameter \(\lambda\). We first describe equilibria in the second-price auction.

**Proposition 7.** Suppose \(b > 0\). The following strategies constitute the most informative equilibrium of the second-price auction. There exists a sequence \((\omega_k)_{k=0}^K\) with \(\omega_0 = 0\) and \(K < \infty\) such that

- for all \(k = 1, \ldots, K < \infty\), the advisor with type \(v \in [\omega_{k-1}, \omega_k)\) sends message \(m_k \equiv \mathbb{E}[v|v \in [\omega_{k-1}, \omega_k)]\);
- the bidder submits a bid equal to the message received.

For \(b < \frac{1}{\lambda}\), \((\omega_k)_{k=0}^K\) are given by the following recursion

\[
\frac{\omega_{k-1}e^{-\lambda\omega_{k-1}} - \omega_{k+1}e^{-\lambda\omega_{k+1}}}{e^{-\lambda\omega_{k-1}} - e^{-\lambda\omega_{k+1}}} = \omega_k + b - \frac{1}{\lambda} \tag{36}
\]

with the terminal condition \(\omega_{K+1} = \infty\) where \(K\) is the maximal length of recursion possible so that \(\omega_1 > 0\). For \(b \geq \frac{1}{\lambda}\), there is only a babbling equilibrium, i.e. \(\omega_1 = \infty\).

The following strategies constitute an equilibrium of the second-price auction when \(F\) is exponential, \(N = 2\), and \(b < 0\). Let \(x \in [-b, \frac{1}{\lambda} - b]\) be the solution to equation \(xe^{2x+1}2x+1 = \frac{1}{\lambda} - b\) and \(\omega_k = kx\) for \(k = 1, 2, \ldots\). For all \(k\), the advisor of type \(v \in [\omega_{k-1}, \omega_k)\) sends message \(m_k = \mathbb{E}[v|\omega_{k-1}, \omega_k)]\) and the bidder submits a bid equal to the message received.

\[31\]In fact, by varying \(K\) in the statement of Proposition 7, we can get any equilibrium of the second-price auction.
Proof. Theorem 1 shows generally that advisor’s strategy in the static auction takes a partition form as described in the proposition. Here, we simply derive this strategy. Given the exponential assumption, we can compute messages \( m_k \) explicitly as functions of thresholds \( \omega_{k-1} \) and \( \omega_k \):

\[
m_k = E[v|v \in [\omega_{k-1}, \omega_k]] = \frac{1}{\lambda} + \frac{\omega_{k-1}e^{-\lambda\omega_{k-1}} - \omega_k e^{-\lambda\omega_k}}{e^{-\lambda\omega_{k-1}} - e^{-\lambda\omega_k}}. \tag{37}
\]

Threshold types \( \omega_k \) of advisor must be indifferent between sending messages \( m_k \) and \( m_{k+1} \). This gives us

\[
\frac{1}{2} P(v \in [\omega_{k-1}, \omega_k])(\omega_k + b - m_k) = -\frac{1}{2} P(v \in [\omega_k, \omega_{k+1}]) (\omega_k + b - m_{k+1}), \tag{38}
\]

which implies

\[
-\left(e^{-\lambda\omega_{k-1}} - e^{-\lambda\omega_k}\right) (\omega_k + b - m_k) = \left(e^{-\lambda\omega_k} - e^{-\lambda\omega_{k+1}}\right) (\omega_k + b - m_{k+1}).
\]

Combining this equation with (37) yields the recursive equation (36), where the initial value condition is \( \omega_0 = 0 \) and \( \omega_k \) must be an increasing sequence. When \( b > \frac{1}{\lambda} \), there is no solution to this recursion, which implies that the unique equilibrium is babbling: \( m = E[v] = \frac{1}{\lambda} \forall v \). Consider \( b \leq \frac{1}{\lambda} \). Denoting by \( x_k \equiv \omega_k - \omega_{k-1} \), we can rewrite the recursion (36) in terms of \( x_{k+1} \) and \( x_k \) as

\[
x_{k+1} + x_k = \left(\frac{1}{\lambda} - b - x_k\right) \left(e^{\lambda(x_k+x_{k+1})} - 1\right). \tag{39}
\]

The requirement that \( \omega_k \) is increasing translates into \( x_k \) being positive. Define function \( \psi \) to be \( x_{k+1} = \psi(x_k) \), such that \( x_k \) and \( x_{k+1} \) satisfy recursion (39). The following auxilliary claim will be helpful for proving the proposition:

Claim 9. \( \psi \) is well defined. For \( x \in (0, \frac{1}{\lambda} - b] \), it holds \( \psi'(x) \geq 1 \), \( \lim_{x \to 0} \psi(x) > 0 \), and \( \lim_{x \to \frac{1}{\lambda} - b} \psi(x) = \infty \).

Proof: First, we show that \( \varphi \) is well-defined. Since \( x_k > 0 \), we can rewrite (39) as

\[
\frac{x_{k+1} + x_k}{e^{\lambda(x_{k+1}+x_k)} - 1} = \frac{1}{\lambda} - b - x_k.
\]

Let \( h(x) \) be the function implicitly defined by the solution to the equation

\[
\frac{h}{e^{\lambda h} - 1} = \frac{1}{\lambda} - b - x. \tag{40}
\]

The left-hand side of (40) is decreasing in \( h \) and it takes values in \( (0, \frac{1}{\lambda}] \) for \( h \geq 0 \). Therefore, the solution \( h(x) \) always exists whenever \( x \in (0, \frac{1}{\lambda} - b] \). Next, we show that \( \psi'(x) \geq 1 \) for \( x \in (0, \frac{1}{\lambda} - b] \).
The derivative of $h$ is given by

$$h'(x) = \frac{(e^{\lambda h(x)} - 1)^2}{e^{\lambda h(x)}(e^{-\lambda h(x)} - 1 + \lambda h(x))} \geq 0,$$

which follows from $e^{-z} > 1 - z$ for $z > 0$. Then $\psi(x) = h(x) - x$ and

$$\psi'(x) = \frac{e^{\lambda h(x)} - 1 - \lambda h(x)}{e^{-\lambda h(x)} - 1 + \lambda h(x)}.$$

We have $\psi'(x) \geq 1$ if and only if $e^{\lambda h} - e^{-\lambda h} - 2\lambda h \geq 0$ for $h \geq 0$. This is implied by the fact that $e^{\lambda h} - e^{-\lambda h} - 2\lambda h$ is increasing in $h$ and equals zero at $0$.\footnote{Its derivative is $\lambda e^{\lambda h} + \lambda e^{-\lambda h} - 2\lambda = \lambda e^{-\lambda h}(e^{\lambda h} - 1)^2 \geq 0$.} Finaly, from (39) it follows that

$$\lim_{x \to 0} (1 - \lambda b) \frac{e^{\lambda \psi(x)} - 1}{\lambda \psi(x)} = 1$$

and so, $\lim_{x \to 0} \psi(x) > 0$. Also from (39) it follows that $\psi(x) \to \infty$ as $x \to \frac{1}{\lambda} - b$. \textbf{q.e.d.}

By Claim 9, $\psi(\cdot)$ is strictly above the diagonal line for $b > 0$, as depicted in Figure 4a. It is must be that $x_{K+1} = \infty$ for some $K$, since otherwise $x_k$ eventually becomes negative). Then, we can construct any equilibrium working backwards: $x_K = \frac{1}{\lambda} - b$, $x_{K-1} = \psi^{-1}(x_K)$ and so on until we reach $x_1$. The most informative equilibrium corresponds to $K$ such that $\psi^{-1}(x_1) \leq 0$. Since $\lim_{x \to 0} \psi(x) = 0$, $K$ is finite. From sequence $(x_k)^{K}_{k=1}$, we reconstruct threshold types $\omega_k = \omega_{k-1} + x_k$ and $\omega_0 = 0$.

We can use the analysis of the case $b > 0$ to analyze the case $b < 0$. Indeed, the derivation of the recursion (39) does not depend on the value of $b$. Recall the function $\psi$ which is defined implicitly as a value of $x_{k+1} = \psi(x_k)$ that satisfies (39) for given $x_k$. When $b < 0$, there is a fixed point of $\psi$ that gives the PBEM described in Proposition 7 (see Figure 4b).

Since $\overline{\omega}_w = \infty$, it is not clear in what sense the PBEM in Proposition 7 satisfies the NITS condition. However, we can construct a sequence of equilibria with $\overline{\sigma} < \infty$ that satisfies NITS and converges to the equilibrium in Proposition 7 as $\overline{\sigma} \to \infty$. Indeed, fix an integer $K$. Let $x_{K+1} = 0$ and recursively define $x_k = \psi^{-1}(x_{k+1})$. It is easy to verify that these strategies constitute an equilibrium when $F$ is the exponential distribution with parameter $\lambda$ truncated at $\overline{\sigma} = \sum^{K}_{k=1} x_k \to \infty$ as $K \to \infty$. Since $x_{K+1} = 0$, type $\overline{\sigma}$ perfectly reveals herself and so, this equilibrium satisfies NITS. Moreover, for any $\epsilon > 0$ there exists $K$ such that for any $K$, $x > x_k > x - \epsilon$ for all but $K$ indexes $k$. This way, even though the equilibrium in Proposition 7 cannot be verified to satisfy the NITS condition, it is a limit of equilibria satisfying NITS.

\textbf{Proposition 8.} Suppose $b < 0$. There exists an equilibrium of the Dutch auction described by a tuple $\{v^*, \sigma(\cdot)\}$ as follows. Types of advisor $v \leq v^*$ send message “stop” when $p = v^* + b$. Any
type of advisor $v \geq v^*$ sends message “stop” at time $t$ when $p = \sigma(v)$. The bidder follows the recommendation of advisor when the running price is above $v^* + b$ and stops the auction if the running price is $v^* + b$. Threshold $v^*$ is the solution to

$$v^* \left( 1 - e^{-\lambda v^*} \right) = 1 - b$$

and bidding strategy $\sigma(\cdot)$ is given by

$$\sigma(v) = E[\max\{v^*, \hat{v}\} + b|\hat{v} < v], \text{ for } v \geq v^*. \quad (42)$$

Proof. First, observe that (41) is the equation $E[v|v < v^*] = v^* + b$ for the exponential distribution. The left-hand side of (41) is a strictly increasing function\(^{33}\) which is $\frac{1}{\lambda}$ at $v^* = 0$ and converges to infinity as $v^* \to \infty$, while the right-hand side is greater than $\frac{1}{\lambda}$. Hence, there is a unique solution to (41).

We first verify that the advisor does not have incentives to deviate from her strategy. As a preliminary step, we derive the equilibrium of the first-price auction where bids are submitted directly by advisors and the lowest participating bidder has type $v^*$ and simply bids her value

\(^{33}\)Indeed, its derivative is equal to

$$\frac{e^{-\lambda v}}{(1 - e^{-\lambda v})^2} (e^{\lambda v} - (1 + \lambda v)) > 0$$
\( v^* + b \). The advisor with type \( v \) solves the following problem

\[
\max_{\sigma} (v + b - \sigma) F(\sigma^{-1}(\sigma)), \tag{43}
\]

for which the first-order condition is

\[
f(v)(v + b) = (F(v)\sigma(v))' \tag{44}
\]

with the initial condition \( \sigma(v^*) = \mathbb{E}[v|v < v^*] = v^* + b \). From (44),

\[
\sigma(v) = \sigma(v^*) \frac{F(v^*)}{F(v)} + \frac{1}{F(v)} \int_{v^*}^{v} f(\hat{v})(\hat{v} + b)d\hat{v} = \mathbb{E}[\hat{v}|\hat{v} < v] + \frac{F(v) - F(v^*)}{F(v)}b, \tag{45}
\]

which gives equation (42). Let \( p^* = \sigma(v^*) \). The utility of the advisor from winning the auction is

\[
v - \mathbb{E}[\hat{v}|\hat{v} < v] + b \frac{F(v^*)}{F(v)} \geq v + b - \mathbb{E}[\hat{v}|\hat{v} < v].
\]

Since \( v^* \) solves (41), the advisor gets a positive utility from the auction for \( v > v^* \).

If the bidder follows the recommendation of the advisor, then the strategy to stop when \( p = \sigma(v) \) is optimal for the advisor when \( v > v^* \), as it is an equilibrium strategy in the Dutch auction where the advisor decides when to stop. For \( v \leq v^* \), the advisor gets utility \( \frac{1}{N}(v - v^*) \leq 0 \) if she follows the strategy and \( v + b - \sigma(v_p) \) if she stop at a price above \( p^* \). Since

\[
v + b - \sigma(v_p) \leq v + b - \sigma(v^*) = v - v^* \leq \frac{1}{N}(v - v^*),
\]

sending the message “stop” at price \( p^* \) is optimal for the advisor.

Notice that the mixed derivative in \( b \) and \( \sigma \) of the maximized function (43) is positive. Hence, if the bidder submits the bid, then she chooses a higher bid. Therefore, it is optimal for her to stop when she gets the message from the advisor with type \( v > v^* \).

To finish the proof, we show that the bidder does not want to stop the auction earlier. Let \( v_p \equiv \sigma^{-1}(p) \) for all \( p > p^* \). Denote by \( \hat{v} \) the value of the opponent bidder. The expected utility of the bidder at time \( t \) from following the recommendation of the advisor is

\[
\mathbb{E}[\mathbb{E}[v|v < v_p] - \sigma(v_p)] = \mathbb{E}[\mathbb{E}[v|v < v_p] - \sigma(v_p) - \frac{F(v_p) - F(v^*)}{F(v^*)}] F(v_p),
\]

where we used the fact that at stage \( p^* \), the bidder gets utility zero from winning. We need to compare this utility with the utility that the bidder gets if she quits before the advisor’s message

\[
\mathbb{E}[v|v < v_p] - \sigma(v_p) = -b \frac{F(v_p) - F(v^*)}{F(v^*)},
\]
which boils down to showing that

$$E[(v - \sigma(v))1\{v > \hat{v}\}|v^* < v < v_p; \hat{v} < v_p] + b$$

is non-negative. Using (45) and $E[\hat{v}|\hat{v} < v] = \frac{1}{\lambda} \left(1 - \frac{v F(v)}{F(v_p)}\right)$ for the exponential distribution, we can re-write (46) as follows

$$\int_{v^*}^{v_p} \left(-b - \frac{1}{\lambda} + \frac{v + b F(v^*)}{F(v)}\right) \frac{F(v)}{F(v_p)} \frac{dF(v)}{F(v_p) F(v) - F(v^*)} + b$$

or rearranging terms

$$\int_{v^*}^{v_p} \left(\frac{F(v)}{F(v_p)} \left(-b - \frac{1}{\lambda} + v + b F(v^*)\right) dF(v) + b F(v_p) \left(F(v_p) - F(v^*)\right)\right).$$

We will show that (47) is increasing in $v_p$. Since (47) is zero at $v_p = v^*$, this would imply that (47) is non-negative for all $v_p > v^*$. The derivative of (47) is equal to

$$f(v_p) \left(F(v_p) \left(-b - \frac{1}{\lambda} + v_p + b F(v^*) + b (2 F(v_p) - F(v^*))\right)\right) = f(v_p) F(v_p) \left(b - \frac{1}{\lambda} + \frac{v_p}{F(v_p)}\right) > 0$$

where the inequality follows from the fact that $v^*$ is the unique solution to (41).

To explore these effects quantitatively let $\lambda = \frac{1}{5}$. The left panel of Figure 5 depicts the efficiency of the auction formats captured by the expected value of the winner. Because of the perfect information transmission, the English auction is efficient if $b \leq \frac{1}{\lambda}$ and so its efficiency does not vary with $b$ on this interval. The gap in efficiency between the English auction and the second-price auction increases as $b$ increases up to $\frac{1}{\lambda}$. This happens because communication in the second-price auction becomes less and less informative as the bias increases. In the right panel of Figure 5, we depict the revenue of the seller for different auction formats for $\lambda = \frac{1}{5}$. The gap in efficiency and revenue between two auctions increases with the size of the bias for $b < \frac{1}{\lambda}$. Notice the discontinuity with respect to $b$. If $b$ is greater than $\frac{1}{\lambda}$, then all equilibria are babbling and the seller gets revenue $\frac{1}{\lambda}$. That is, the seller benefits from having the bias only when this bias is not too large.

**B.2 Proof of Theorem 2 (Case $b < 0$)**

If type $\overline{v}$ reveals herself to the bidder, then the bidder prefers to submit a bid that is guaranteed to win. Type $\overline{v}$ does not want to reveal herself if and only if $\overline{v} + b - m(\overline{w}_K, \overline{v}) \leq 0$. Suppose to contradiction that NITS fails and $\overline{v} + b > m(\overline{w}_K, \overline{v})$. We show by induction that for any $k \leq K + 1$, there exists another solution $(\omega_{1j}^k)_{j=1}^{K_j}$ to (19) such that $\omega_{1j+1}^k = \overline{v}, \omega_{K_j-k}^k < \overline{w}_K, k + 1, and \omega_{K_j-k-1}^k = \overline{w}_K$. Theorem 2 follows from the claim applied to $k = K$. 

56
(a) $b > 0$: the English auction (solid line) and the second-price auction (dashed line).

(b) $b < 0$: the Dutch auction (solid line) and the second-price auction (dashed line).

*Figure 5: Efficiency and revenue comparison. Bias $b$ is plotted on the horizontal axis.*
For $k = 1$, the failure of NITS implies that $H(\tilde{\omega}_K, \bar{\nu}, \bar{\nu}) = \Psi(\tilde{\omega}_K, \bar{\nu})(\bar{\nu} + b - m(\tilde{\omega}_K, \bar{\nu})) > 0$. At the same time, $H(\tilde{\omega}_K, \tilde{\omega}_K, \bar{\nu}) = \Phi(\tilde{\omega}_K, \bar{\nu})(\tilde{\omega}_K + b - m(\tilde{\omega}_K, \bar{\nu})) > 0$, as $\tilde{\omega}_K$ satisfies (19). By continuity, there exists $x \in (\tilde{\omega}_K, \bar{\nu})$ such that $H(\tilde{\omega}_K, x, \bar{\nu}) = 0$ proving the claim for $k = 1$.

Suppose the statement is true for $k$ and we next prove it for $k + 1$. Since $\tilde{\omega}_k$ solves (19), $H(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1}) = 0$ or

$$
\Psi(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}))+
\Phi(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})) = 0. \quad (48)
$$

Let $\omega_{K-k}^k < \tilde{\omega}_{K-k+1}$ and $\omega_{K-j-k-1}^k = \tilde{\omega}_{K-k}$ as in the inductive hypothesis and consider $H(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}, \omega_{K-j-k}^k)$:

$$
\Psi(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k})(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}))+
\Phi(\tilde{\omega}_{K-k}, \omega_{K-j-k}^k)(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k}, \omega_{K-j-k}^k)),
$$

which differs from (48) only in the second term. Since $\omega_{K-j-k}^k < \tilde{\omega}_{K-k+1}$, $m(\tilde{\omega}_{K-k}, \omega_{K-j-k}^k) < m(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})$. Moreover,

$$
\Phi(\tilde{\omega}_{K-k}, \omega_{K-j-k}^k) = \sum_{n=1}^{N-1} \binom{N-1}{n} F(\tilde{\omega}_{K-k}, \omega_{K-j-k}^k)^n F(\tilde{\omega}_{K-k})^{N-1-n} \frac{1}{n+1} + G(\omega_{K-j-k}^k, \tilde{\omega}_{K-k}) \cdot 0
$$

$$
< \sum_{n=1}^{N-1} \binom{N-1}{n} F(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1})^n F(\tilde{\omega}_{K-k})^{N-1-n} \frac{1}{n+1}
$$

$$
= \Phi(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k+1}).
$$

Hence, $H(\tilde{\omega}_{K-k-1}, \tilde{\omega}_{K-k}, \omega_{K-j-k+1}^k) > 0$. On the other hand, since $\omega_{K-j-k-1}^k = \tilde{\omega}_{K-k}$,

$$
H(\tilde{\omega}_{K-k}, \tilde{\omega}_{K-k}, \omega_{K-j-k+1}^k) = \Phi(\tilde{\omega}_{K-k}, \omega_{K-j-k+1}^k)(\tilde{\omega}_{K-k} + b - m(\tilde{\omega}_{K-k}, \omega_{K-j-k+1}^k)) < 0.
$$

Therefore, there exists $x \in (\tilde{\omega}_K, \tilde{\omega}_{K+1})$ such that $H(x, \tilde{\omega}_{K-k}, \omega_{K-j-k+1}^k) = 0$. By continuity, we can find solution $(\omega_{K-j-k+1}^{k+1})_{j=1}^{K+1}$ to (19) with $\omega_{K-j-k-1}^k < \tilde{\omega}_{K-k}$ and $\omega_{K-j-k+2}^k = \tilde{\omega}_{K-k}$, which completes the proof of the inductive step.

References


59


