Non-Markov Gaussian Term Structure Models: The Case of Inflation

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Abstract

Standard gaussian macro-finance term structure models impose the Markov property: the conditional mean is a function of the risk factors. Instead, we consider the family of gaussian models where yields are linear in the conditional mean (but not in the risk factors). To illustrate, if inflation is one of the factor, then (i) yields should span expected inflation but not inflation (ii) expected changes and policy surprises can have opposite effect on expected inflation, even contemporaneously, and (iii) expected inflation involves long lags of the risk factors.

We show that each of these features are important empirically. In particular, model-forecasts match survey forecast accuracy out-of-sample, spanning the expectations embodied in survey. In addition, allowing for the opposite effects of expected and unexpected changes yields a very different depiction of the short rate dynamics. Hence, the decomposition of nominal yields differ from that of the standard specification.

Keywords: Term Structure Models, Markov Dynamics, Inflation Risk Premium, Real Yields, Inflation Forecasts, Survey of Professional Forecasters

JEL Classification: E43, E47, G12

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1. Introduction

The Markov property is pervasive across affine macro-finance term structure models (MTSMs), where it implies that the conditional mean of the future risk factors $\mathcal{E}_t \equiv E_t[\mathcal{Z}_{t+1}]$ ($\mathcal{Z}_t \in \mathbb{R}^{N^Z}$) is a linear function of the current risk factors: $\mathcal{E}_t = \phi \mathcal{Z}_t$.\(^1\) If, in addition, the short rate is a linear function of $\mathcal{Z}_t$, then no-arbitrage requires that yields are linear: the standard model leaves no separate role for the conditional mean in the behavior of yields – $\mathcal{E}_t$ is a function of $\mathcal{Z}_t$. Instead, we consider models where $\mathcal{E}_{t+1}$ has the Markov property (but not $\mathcal{Z}_t$), where the short rate is linear in $\mathcal{E}_t$, and where yields are linear in $\mathcal{E}_t$. This family of models, named conditional mean MTSM (CM-MTSM), remains close to standard models, but offers key differences.

First, having freed $\mathcal{E}_t$ from its tight link with $\mathcal{Z}_t$, we switch their roles and let the conditional mean determine bond prices. This implies that yields span expectations about future $\mathcal{Z}_t$ (but not the current value). This prediction is consistent with a vast literature studying the predictive content of yields. In contrast, the standard model predicts that yields span the current value $\mathcal{Z}_t$. This is hardly supported in the data. For instance, Joslin et al. (2010) point out that the inflation rate is only weakly related to current yields and impose parametric restrictions limiting the information span of yields. Similarly, Duffee (2011) suggests that factors with opposite effects on expectations and on the risk premium can appear hidden in the measurement errors. We propose an alternative resolution where, in the example of inflation, bond prices are driven by expected inflation and yields should span inflation expectations.

Second, $\mathcal{E}_t$ mixes information from the innovations $u_t \equiv \mathcal{Z}_t - \mathcal{E}_{t-1}$ with the information content captured in the prior expectations $\mathcal{E}_{t-1}$. The weights on each component are equal if and only if $\mathcal{Z}_t$ has the Markov property, nesting standard MTSMs. In contrast, these weights differ in conditional mean models, allowing, in a direct and natural way, for the opposite effects of expectations and of policy innovations on the evolution of the economy (Phelps, 1967; Friedman, 1968).

Finally, the conditional mean $\mathcal{E}_t$ is ultimately a function of the entire history of $\mathcal{Z}_t$ – much like the conditional variance is a function of the history of squared returns in a GARCH model. In other words, the dynamics of $\mathcal{Z}_t$ involves distant lags. Therefore, CM-MTSM parsimoniously captures

\(^1\)The Markov property requires that the conditional distribution of $\mathcal{Z}_{t+1}$ is a function $\mathcal{Z}_t$, but this is equivalent to requiring that the conditional mean is a function of $\mathcal{Z}_t$ in the context of homoscedastic Gaussian MTSMs.
the well-known fact that the inflation dynamics involves several lags of inflation (Kim, 2007) and of other macro variables (Ang et al., 2006). Nonetheless, by the law of iterated expectations, it suffices to know \( E_t \) to forecast \( Z_{t+h} \) (\( E_t \) has the Markov property).

Formally, we develop a canonical form for the CM-MTSM family. The model identification and parameterization follow a path parallel to that of Joslin, Singleton, and Zhu (2011) and Joslin, Le, and Singleton (2011) (JSZ and JLS, respectively hereafter), but with one important distinction. In the standard MTSM, the state \( Z_t \) spans macro variables and yields by construction. In a CM-MTSM, there is a separation: the state \( Z_t \) spans the macro variables \( M_t \) while \( E_t \) spans the cross-section of yields. That distinction is possible only because \( Z_t \) does not have the Markov property. Nonetheless, the family of CM-MTSMs nests the canonical form of JSZ up to a matrix \( \Sigma^* \) controlling the weights put on \( u_t \) and \( E_{t-1} \) in updating expectations. A CM-MTSM has a VARMA(1,1) representation and a simple restriction on \( \Sigma^* \) leads to the standard VAR(1) case. The CM-MTSM family also nests the canonical dynamic term structure model of JSZ. The dimensions of \( \Sigma^* \) are \( N_M \times N_Z \), where \( N_M \leq N_Z \) is the number of macro variables: \( \Sigma^* \) drops from the parametrization when \( N_M = 0 \). Heuristically, the distinction between \( Z_t \) and \( E_t \) is observationally meaningless when excluding macro data.

Empirically, we develop a specification with two yield factors and two macro variables: the inflation rate and the unemployment rate (i.e., \( N_Z = 4 \) and \( N_M = 2 \)). We consider the case of Canada. The results highlight the three properties of CM-MTSMs listed above. First, yields span inflation expectations as measured by surveys of professional forecasters, delivering similar out-of-sample forecasting performance. In fact, the conditional mean models reduces forecast root mean squared errors (RMSE) by as much as 10–20% relative to the VAR(1). Second, inflation expectation updates use very different weights on the prior expectations \( E_{t-1} \) and on the innovations \( u_t \), relying mostly on information from the yield curve. In fact, the weights on expected yield changes and surprise changes have different signs. Finally, the inflation dynamics involve very long lags. As Kim (2007) points out, inflation expectations are persistent but the inflation shocks are mostly transitory. These results imply that CM-MTSMs can substantially improve the decomposition of nominal yields into a real yield, an inflation risk premium and expected inflation.

To help with the estimation, we use inflation forecasts from surveys, adding the likelihood of
survey data with the likelihood of yields and macro variables. Using surveys reduces sampling uncertainty and lessens the bias in the persistence parameters (Kim and Wright, 2005; Kim and Orphanides, 2012; Bauer et al., 2012; Jardet et al., 2013) and plays an essential role in our results. We also consider two economic restrictions to mitigate the effect of overparameterization. First, in the spirit of Duffee (2011), we restrict the parameter space to limit the variability of the conditional inflation Sharpe ratio.\footnote{See also Bauer and Diez de los Rios (2012) in the context of a multi-country term structure model.} We find that a very tight constraint on the inflation Sharpe ratio (as in Ang, Bekaert, and Wei 2008) is strongly rejected in the data. On the other hand, the magnitude and the variations of the inflation risk premium (and therefore of the real yield) shoot off the charts when the constraint is too loose relative to observed Sharpe ratios. Hence, this suggests that long-run risk structural models where macro variables are not Markovian and with a pricing kernel calibrated to reasonable Sharpe ratio may fit term structure data reasonably well Bansal and Shaliastovich (2013). Second, we follow Chernov and Mueller (2011) and imposes that the short real rate is not a function of expected inflation. This restriction ties up the response of the inflation risk premium to expected inflation and the response of the nominal short rate to expected inflation. This restrictions has has little effect on the fit of yields, but it delivers an alternative structural interpretation of short-horizon inflation risk premium. Finally, we consider the robustness of our results to the presence of time-varying inflation volatility. The denominator in the Sharpe ratio is constant in the benchmark model, which may alter the effect on the Sharpe ratio constraint. We find that our results are robust if the inflation variance follows a GARCH process.

Monfort and Pegoraro (2007) consider term structure models with regimes and where yield factors do not have the Markov property (but with finite lags). In practice, the combination of regimes with two lags of yield factors matches the predictability evidence. More recently, Chun (2011) also defines a term structure model where the conditional mean are the yield factors, but he does not define the dynamics of the corresponding macro variables. Joslin et al. (2013) consider an asymmetric formulation where the yield factors have the Markov property under the pricing measure, but can depend on several lags under the historical measure. Feunou and Meddahi (2007) analyzes more general departures from the Markov assumption.

Stock and Watson (2003) find little evidence of a predictive relationship from nominal yields
to future inflation and argue that this reflects “limitations of conventional models [...], not a fundamental absence of predictive relationships in the economy.” (p. 79). Ang et al. (2007) find that survey forecasts are difficult to match but they do not use survey at estimation. Faust and Wright (2011) discuss the importance of using survey data to forecast inflation with US data. We find that combining survey data with a CM-MTSM matches survey forecasts and improves upon other models. Pennacchi (1991) and Ang et al. (2008) combine economic restrictions with nominal yields in dynamic term structure models. Pennacchi (1991) considers an equilibrium model and uses surveys to pin down the evolution of expectations, but he does not measure the inflation risk premium. Ang et al. (2008) consider a no-arbitrage model with regimes and impose a zero one-period inflation risk premium to identify the level of real rates. Ragan (1995) is an early attempt to measure expected inflation from nominal yields in Canada. Garcia and Luger (2007) estimate an equilibrium-based model based on Canadian data where conditional expectations play a central role, but their focus is not on inflation forecasts.

The recent literature uses separate measurements for each element of Fisher’s decomposition. For instance, Chernov and Mueller (2011) combine nominal yields, surveys of inflation forecasts and data from inflation-indexed bonds. Similarly, Haubrich et al. (2012) use nominal yields, surveys of inflation forecasts and inflation swap data. Ajello et al. (2012) combine measures of core, food, and energy inflation series and obtain a decomposition of yields within a term structure model. Most existing results do not control for the excess variability of the risk premium or limit the Sharpe ratio to economically reasonable values. In addition, real bonds and inflation swaps are not available in many countries at the horizons of interest and, in any case, their use relies on maintained hypotheses about the degree of integration between different markets and about the magnitude and variability of the liquidity premium. These assumptions may not be supported in the data (e.g., Campbell et al. 2009; Fleckenstein et al. 2013).

Section 2 follows and introduces the model and discusses its properties in relation to existing specifications. Section 3 discusses the data, the estimation method and economic restrictions that can be used at the estimation step. Section 4 reports all the results and Section 5 concludes. All proofs are provided in the Appendix.

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3See also Chen et al. (2010); D’amico et al. (2010); Christensen et al. (2010).
2. A Macro-Finance Conditional Mean Model

2.1. Conditional Expectations as Yield Factors

We build the family of CM-MTSM using the conditional distribution of the latent factor \( Z_t \in \mathbb{R}^N \).

Its conditional mean,

\[
E_t \equiv E_t[Z_{t+1}],
\]

(1)

drives the one-period interest rate \( i_t \):

\[
i_t = \rho_0 + \rho_1' E_t.
\]

(2)

The dynamics of \( E_t \) under the risk-neutral measure is given by

\[
\Delta E_{t+1} = K_0^Q + K_1^Q E_t + \Sigma E_{t+1}^Q,
\]

(3)

where \( \epsilon_{t+1}^Q \) is a Gaussian white noise under \( Q \). The \( n \)-period nominal yield is defined by

\[
y_t^{(n)} = - \ln E_t^Q \left[ \exp - \left\{ \sum_{j=0}^{n-1} (\rho_0 + \rho_1' E_{t+j}) \right\} \right],
\]

(4)

with the following closed-form solution:

\[
y_t^{(n)} = a_n + b_n' E_t,
\]

(5)

where the coefficients \( a_n \) and \( b_n \) are given in Appendix A.1. Equations (1)-(3) are standard: they postulate a set of yield factors \( E_t \) (Equation (1)) driving the short rate (Equation (2)) and with Markovian dynamics under \( Q \) (Equation (3)) so that the solution for yields is affine (Equation (5)).

Note, however, that \( E_t \) plays two roles: it represents a set of risk factors driving yields as well as the conditional mean of \( Z_t \) under \( P \). This is an important distinction in the following.

2.2. Historical Dynamics

The change of measure, \( \xi_t \), is exponential-affine:

\[
\xi_{t+1} = \frac{\exp(\lambda_t \epsilon_{t+1}^Q)}{E_t[\exp(\lambda_t \epsilon_{t+1}^Q)]},
\]

(6)
with prices of risk $\lambda_t$ that are affine functions of the yield factors $\mathcal{E}_t$,

$$\lambda_t \equiv \tilde{\lambda}_0 + \tilde{\lambda}_1 \mathcal{E}_t. \quad (7)$$

From standard results, the combination of this pricing kernel with Gaussian innovations in Equation (3) implies that the dynamics under the historical measure $\mathbb{P}$:

$$\Delta \mathcal{E}_{t+1} = K_0^P + K_1^P \mathcal{E}_t + \Sigma \epsilon_{t+1}^P, \quad (8)$$

with parameters given by

$$K_0^P = K_0^Q - \Sigma \epsilon \tilde{\lambda}_0 \quad (9)$$
$$K_1^P = K_1^Q - \Sigma \epsilon \tilde{\lambda}_1. \quad (10)$$

### 2.3. Expectations and State Variables

The yield factors $\mathcal{E}_t$ span the conditional expectations of the future state $Z_{t+1}$ by construction. Therefore, it must be that $Z_{t+1} = \mathcal{E}_t + u_{t+1}$ for some zero-mean innovation $u_{t+1}$ that is unpredictable given time-$t$ information. To complete the dynamics of $Z_{t+1}$, we assume that $u_{t+1}$ corresponds to a rotation of the innovations in $\mathcal{E}_{t+1}$,

$$Z_{t+1} = \mathcal{E}_t + \Sigma \epsilon_{t+1}^P. \quad (11)$$

that is $u_{t+1} \equiv \Sigma \epsilon_{t+1}^P$ where $\Sigma$ is lower diagonal and positive-definite. Substituting $\epsilon_{t+1}^P = \Sigma^{-1}(Z_{t+1} - \mathcal{E}_t)$ in Equation (8), it is easy to check that $\Delta \mathcal{E}_{t+1} = K_0^P + K_1^P Z_t$, and that $Z_t$ is a standard Gaussian VAR(1) process if and only if:

$$\Sigma \epsilon \Sigma^{-1} = I_{N_x} + K_1^P. \quad (12)$$

However, $\mathcal{E}_t$ is not the conditional mean of $Z_{t+1}$ under the risk-neutral measure $Q$. This is given by

$$\mathcal{E}_{t+1}^Q = \mathcal{E}_t + \Sigma \Sigma^{-1} \left( K_0^Q - K_0^P \right) + \Sigma \Sigma^{-1} \left( K_1^Q - K_1^P \right) \mathcal{E}_t^P. \quad (13)$$
(see Appendix A.2.) Therefore, $E^Q_t$, is an affine transformation of $E_t$ and inherits the Markov property under $Q$, and the dynamics of $Z_{t+1}$ have the same form under $Q$ and $P$. The assumption that the same shocks drive $Z_t$ and $E_t$ should be uncontroversial. Most dynamic stochastic general equilibrium (DSGE) models imply that endogenous state variables and their expectations share the same set of structural shocks.\(^4\) Our approach is similar to Piazzesi and Schneider (2006), in the context of a term structure model with learning, and similar to the state dynamics in some long-run risk models.\(^5\) In addition, this approach is also closely related to the standard GARCH model where the squared innovations drive the conditional variance (we consider a GARCH model for the variance in Section 4.6).

So far, the dynamics for $Z_t$ is generic and requires identification assumptions. Proposition 1 below establishes a canonical form. This proposition is an adaptation of Proposition 1 in JSZ, but with risk factors given by the conditional mean $E_t$ instead of by $Z_t$.

**Proposition 1.** Every canonical CM-MTSM is observationally equivalent to a canonical CM-MTSM with $i_t = v \cdot E_t$, where $v$ is a vector of ones, and

\[
\Delta E_{t+1} = K^P_0 + K^P_1 E_t + \Sigma \epsilon^P_{t+1} \\
\Delta E_{t+1} = K^Q_0 + K^Q_1 E_t + \Sigma \epsilon^Q_{t+1},
\]

where $\epsilon^P_t$ and $\epsilon^Q_t$ are Gaussian white noise under $Q$ and $P$, respectively, and $K^Q_1$ is an ordered real Jordan form, $K^Q_{0,i} = \ell^Q_{\infty}$, $K^Q_{0,i} = 0$ for $i > 1$.

The proof follows the same steps as the proof for Proposition 1 in JSZ, but substituting $E_t$ for their generic $X_t$.

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\(^4\)Including cases where the linearized model has a VAR(1) representation. In addition, the state variables often do not have a finite-order autoregressive representation and, in that case, the solution to the DSGE has a recursive structure similar to Equation (8). See Fernandez-Villaverde et al. (2005), Ravenna (2007) and the references therein on the invertibility problem associated with finding a finite-order VAR representation in (linearized) DSGE models.

\(^5\)See, in particular, the NBER version of Bansal and Yaron (2004).
2.4. From Latent to Observable Factors

Observable variables may include macro variables $\mathcal{M}_t$ and yield factors $\mathcal{P}_t$. Take a number $0 \leq N_M \leq N_Z$ of macro variables $\mathcal{M}_t$ that are spanned by the state $Z_t$,

$$\mathcal{M}_t = \gamma_0 + \gamma_1 Z_t,$$

(16)

where $\gamma_0$ is a $N_M \times 1$ vector and $\gamma_1$ is a $N_M \times N_Z$ rectangular matrix. In addition, take $N_L = N_Z - N_M$ observed portfolios of yields $\mathcal{P}_t$ that are measured without errors,

$$\mathcal{P}_t = W y_t = a_W + b'_W \epsilon_t,$$

where $y_t$ stacks the cross-section of $N$ individual yields given by Equation (5), $W$ is a $N_L \times N$ matrix of weights, $a_W \equiv W [a_{n_1} \ldots a_{n_N}]'$, and $b_W = [b_{n_1} \ldots b_{n_N}] W'$. This framework is general and can accommodate different numbers of observable macro variables $\mathcal{M}_t$ and yield portfolios $\mathcal{P}_t$. Proposition 2 provides the joint dynamics of the observable.

**Proposition 2.** The $N_Z \times 1$ vector of observable $X_t$,

$$X_t \equiv \begin{bmatrix} \mathcal{M}_t \\ \mathcal{P}_t \end{bmatrix} = C + D_1 Z_t + D_2 \epsilon_t,$$

(17)

with $C$, $D_1$, and $D_2$ given by

$$C \equiv \begin{bmatrix} \gamma_0 \\ a_W \end{bmatrix}, \quad D_1 \equiv \begin{bmatrix} \gamma_1 \\ 0 \end{bmatrix}, \quad D_2 \equiv \begin{bmatrix} 0 \\ b'_W \end{bmatrix},$$

(18)

follows a conditional mean model analogous to Equations (11), 13, 14 and 15. This follows from applying the mapping in Equation (17) to the dynamics of $Z_t$. The mapping between parameters is provided in Appendix A.3.

We can then use the observed macro variables $\mathcal{M}_t$ and yield portfolios $\mathcal{P}_t$ to obtain an equivalent conditional mean model based on the observable $X_t$ instead of the latent $Z_t$. Combined with the model identification from Proposition 1, Proposition 2 gives rise to a canonical form similar to
that of JLZ but with one important distinction. In their Markovian set-up, the state variables \( Z_t \) necessarily span yields and macro variables, while in our case there is a separation. The state \( Z_t \) spans the macro variables \( M_t \) while \( E_t \) spans the cross-section of yields. This separation arises because \( Z_t \) is not Markovian – \( E_t \) is not a function of \( Z_t \). The following theorem formalizes the canonical form. Note that the parameterization introduces the matrix \( \Sigma^* \), with dimensions varying with the number of macro variables. This matrix plays a key role, as discussed in the following section.

**Theorem 1. Canonical Form**

Suppose that the \( N_L \) portfolios of yields \( P_t \) and the \( N_M \) macro variables \( M_t \) are observed without errors, with \( N_L + N_M = N_Z \). Then any canonical CM-MTSM is equivalent to a unique canonical CM-MTSM whose state variables are \( X'_t = [M_t \ P'_t] \). That is, the short rate is given by

\[
i_t = \rho_0 X + \rho'_1 X P_P X_{t},
\]

and the dynamics of \( E_P X_{t'} \) under \( Q \) and \( P \) are given by

\[
\Delta E_P X_{t+1} = K_0^Q X + K_1^Q X P_P X_{t} + \Sigma X e_Q X_{t+1}
\]

\[
\Delta E_P X_{t+1} = K_0^P X + K_1^P X P_P X_{t} + \Sigma X e_P X_{t+1},
\]

with:

\[
X_{t+1} = E_Q X_{t} + \Sigma X e_Q X_{t+1}
\]

\[
X_{t+1} = E_P X_{t} + \Sigma X e_P X_{t+1},
\]

under \( Q \) and \( P \), respectively, and where the link between \( E_Q X_{t} \) and \( E_P X_{t} \) is given by

\[
E_Q X_{t} = E_P X_{t} + \Sigma X \Sigma^{-1} X (K_0^Q X - K_0^P X) + \Sigma X \Sigma^{-1} X (K_1^Q X - K_1^P X) E_P X_{t}.
\]

The canonical form is parameterized by

\[
\Theta^X = \left( K_0^P X, K_1^P X, \lambda^Q, k^Q, \gamma_0, \gamma_1, \Sigma e_X, \Sigma^* \right),
\]
where $\Sigma^*$ is a $N_M \times N_Z$ matrix (see Appendix A.4 for details).

2.5. Discussion

2.5.1. Forward-looking yields

Forward-looking yields are consistent with long-run risk equilibrium models (e.g., Bansal and Shaliastovich 2013; Hasseltoft 2012). Forward-looking rules for the short rate are discussed in Ang et al. (2007) in the context of affine term structure models. However, the distinction between contemporaneous rules (i.e., that depend on $X_t$) and forward-looking rules (i.e., that depend on $E_{X,t}$) is blurred in Markovian model, since $E_{X,t}$ is a function of $X_t$. Chun (2011) derives a model where expectations are measured directly from survey (without errors) and where expectations play the role of yield factors. However, the relationship with the corresponding macro variables is left unspecified.

2.5.2. Forward-looking risk premium

The change of measure $\xi_t$ and the prices of risk $\lambda_t$ are standard (Piazzesi, 2005) but with the difference that they are functions of $E_t$ and not $X_t$: the risk premium is forward-looking. The risk premium – the spread between $E_{X,t}^P$ and $E_{X,t}^Q$ – can be derived directly from Equation (22),

$$E_{X,t}^Q - E_{X,t}^P = \bar{\lambda}_0X + \bar{\lambda}_1X E_{X,t}^P.$$ 

Then, the risk premium in a CM-MTSM depends on the entire history of $X_t$ via the recursion for $E_{X,t}$ in Equation (8). Again, this distinction is blurred within Markovian models where each approach is equivalent, up to rotation of the parameters.

2.5.3. Dynamic term structure models

Our canonical form nest the dynamic term structure model in JSZ as well as the macro-finance term structure model in JLS. First, the observable variables are jointly Markovian in JLS. This case is nested whenever $\Sigma_{E_X} = \Sigma_X + K_{1X}^P \Sigma_X$. In this case, $X_t$ is Markovian under $P$ and, therefore, $E_{X,t}^P$ and $E_{t}^Q$ are affine functions of $X_t$ (see Theorem 1). In turns, this implies that $X_{t+1}$ follows a Gaussian VAR(1) under $Q$ as well, and that yields and macro variables are linear in $X_t$. 

Second, our canonical model nests the canonical Gaussian dynamic term structure model in JSZ. To see this, consider first the parameterization of $\Sigma_X$ in terms of $\Sigma^*$ and other model parameters, given in Appendix A.4:

$$\Sigma_X = D_2 D^{-1} \Sigma_{\mathcal{E}X} + \begin{pmatrix} \Sigma^* \\ 0 \end{pmatrix}_{(N_L \times N_Z)} \begin{pmatrix} N_L \times N_Z \end{pmatrix}$$

$$D \equiv D_1 + D_2 (K_1^P + I_{N_Z}),$$

where $D_1$ and $D_2$ are given in Equation (18). The JSZ specification uses only yield factors as observable variables, $\mathcal{X}_t \equiv \mathcal{P}_t$. In this case, $N_L = N_Z$, we have $\Sigma^* = 0$, as well as $D_1 = 0$ and $D_2 = b_W$, from Equation (17). Then, Equation (23) yields $\Sigma_{\mathcal{E}X} = \Sigma_X + K_1^P \Sigma_X$. This is a necessary and sufficient condition for which $\mathcal{P}_t$ follows a VAR(1) under under $\mathcal{Q}$ and $\mathcal{P}$, and it follows that yields are affine in $\mathcal{P}_t$, as in JSZ. Heuristically, yields can only reveal yield factors. Excluding other observable variables that could help distinguish between $\mathcal{E}_{X,t}$ and $\mathcal{X}_t$ implies that the role played by $\mathcal{E}_t$ as the conditional mean becomes observationally meaningless.

Finally, consider the opposite case, where we exclude yield factors but use only observable macro variables, $\mathcal{X}_t \equiv \mathcal{M}_t$. Then, $\Sigma^*$ is square (since $N_M = N_Z$), $\Sigma_X$ can be estimated separately from $\Sigma_{\mathcal{E}X}$, and $\mathcal{X}_t$ has an unrestricted conditional mean dynamics. Its $\mathcal{P}$-parameters can be estimated directly based on the macro variables. In addition, the cross-section of yields is driven entirely by the history of macro variables and the risk-neutral parameters can be estimated using $\mathcal{E}_{X,t}$, filtered from the recursion in Equation (8) as observable risk factors. We do not consider this case in the empirical application, but instead focus on the case where $\mathcal{X}_t$ mixes macro variables and yield portfolios.

2.5.4. Joint estimation versus two-step estimation

We cannot neatly separate the parameters affecting the conditional dynamics of $\mathcal{X}_t$ from those parameters governing the cross-section of yields (under the risk-neutral measure). First, the pa-

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6The definition in Equation (24) involves parameters from the generic representation. An alternative definition based on the parameters of canonical representation in Theorem 1 is given by

$$\left( I_{N_Z} \otimes D_1 + \left( I_{N_Z} + K_1^P \right) \otimes D_2 \right) \text{vec} \left( D^{-1} \right) = \text{vec} \left( I_{N_Z} \right),$$

which is well defined if the first term can be inverted (which is how we proceed, in practice).
parameters of the innovation covariance matrix, $\Sigma_{\mathcal{X}}$, are the same under each measure. This also arises in the standard case, but this matrix only affects the variance of the shocks and a two-step estimation remains convergent, albeit inefficient. However, a simple count of parameters in $\bar{\lambda}_0$ and $\bar{\lambda}_1$ shows that we cannot freely shift all other parameters in the dynamics for $\mathcal{E}_{\mathcal{X},t}$. Equation (23) shows that $\Sigma_X$ is not separately identified from $\Sigma_{\mathcal{E}_X}$ and, in addition, that the connection between the two depends on the parameters from the $\mathbb{Q}$-dynamics via the matrix $D$ and $D_2$ (Appendix A.4 details the parameterization from Theorem 1). Of course, one could neglect this connection and estimate the $\mathbb{P}$-dynamics separately in a first step, obtaining consistent estimates. However, the effects of efficiency loss are more severe in CM-MTSMs. The parameters of the $\mathbb{P}$-dynamics enter the likelihood of the cross-section of yields indirectly, since they determined the filtered estimates of $\hat{\mathcal{E}}_{\mathcal{X},t}$ (via the recursion in Equation 22). Hence, the asset pricing implications depend on the parameters of the $\mathbb{P}$-dynamics. This contrasts with the standard model, where the two dynamics can be estimated separately unless we impose additional restrictions on the prices of risk (Joslin et al., 2010).

2.5.5. **Real yields and fisher’s decomposition**

If the observable vector includes the inflation rate, then the real term structure is identified without further assumption – beyond the absence of arbitrage opportunity for real bonds. The real short rate, $r_t$, is given by the link between the nominal and the real stochastic discount factors, $SDF_{t+1}$ and $SDF^r_{t+1}$, respectively, given by $SDF_{t+1} \equiv e^{-i_t} = e^{-\pi_{t+1}} SDF^r_{t+1}$, which implies that

$$r_t \equiv -\log (E_t[SDF^r_{t+1}]) = i_t - E^{\mathbb{Q}}_t[\pi_{t+1}] - \frac{1}{2} \sigma_\pi^2,$$

(25)

where $\sigma_\pi^2 \equiv \text{Var}_t[\pi_{t+1}]$. This definition of the real short rate corresponds to Fisher’s, decomposition but where the expectation of future inflation is taken under the risk-neutral measure, $E^{\mathbb{Q}}_t[\pi_{t+1}]$. Using Equation (22), the real short rate can be written analogously to the nominal rate,

$$r_t = \rho_{0X}^r + \rho_{1X}^r \mathcal{E}_{\mathcal{X},t},$$

(26)
and it follows that the real yield curve is given by

\[ r_t^{(n)} = a_{r,n} + b_{r,n}E_{X,t}, \tag{27} \]

with coefficients \( \rho_{0,X}^r \) and \( \rho_{1,X}^r \) given in Appendix A.1, and the loadings \( a_{r,n} \) and \( b_{r,n} \) solving recursions analogous to \( a_n \) and \( b_n \). The generalized Fisher decomposition can be applied to any nominal yield with maturity \( n \),

\[ i_t^{(n)} \equiv c^{(n)} + r_t^{(n)} + E_{\pi,t}^{(n)} + irp_t^{(n)}, \tag{28} \]

where \( r_t^{(n)} \) is the real yield, \( irp_t^{(n)} \) is the \( n \)-period-ahead inflation risk premium, \( E_{\pi,t}^{(n)} \) is the \( n \)-period-ahead inflation expectation and \( c^{(n)} \) is a Jensen term (see Appendix A.5).

2.5.6. VARMA representation

The conditional mean representation was introduced by Fiorentini and Sentana (1998) in the context of time-series models, and differs significantly from the standard (VAR) representation used in the context of macro-finance models. Nonetheless, the processes for \( Z_t \) and \( X_t \) have equivalent representations within the broader family of VARMA processes. For instance, combining the equations for \( Z_t \) and \( E_t \) together (i.e., Equations (8) and (11)) yields the following equivalent unrestricted VARMA(1,1) process:

\[ Z_{t+1} = K_0^P + \phi^P Z_t - \theta^P \Sigma \xi_t^P + \Sigma \xi_{t+1}^P, \tag{29} \]

where

\[ \phi^P = I_{Nz} + K_1^P \quad \theta^P = \phi^P - \Sigma \xi \Sigma^{-1}, \]

and we can obtain similar representations for \( Z_t \) under \( Q \) as well as for \( X_t \) under each measure. Note that \( \theta_X^P = 0 \) if and only if Equation (12) holds. In addition, note that \( \theta_X^P \) in the VARMA representation for \( X_t \) cannot be estimated separately from the other parameters (see Equation 23).
2.5.7. Using the extended VAR representation

Any VARMA process has an equivalent VAR(1) representation, but with an extended state vector. Specifically, the extended vector \( \tilde{Z}_t' = [Z_t' \ \mathcal{E}_t'] \) follows the VAR(1) process:

\[
\Delta \tilde{Z}_t = K_{0Z}^Q + K_{1Z}^Q \tilde{Z}_{t-1} + \Sigma_{\tilde{Z}} \epsilon_t^Q,
\]

with several cross-equation restrictions given by

\[
K_{0Z}^Q = \begin{bmatrix} 0_{N_Z \times 1} \\ K_{0Z}^Q \end{bmatrix}, \quad K_{1Z}^Q = \begin{bmatrix} -I_{N_Z} & I_{N_Z} \\ 0_{N_Z \times N_Z} & K_{1Z}^Q \end{bmatrix}, \quad \Sigma_{\tilde{Z}} = \begin{bmatrix} \Sigma & 0_{N_Z \times N_Z} \\ \Sigma_{\mathcal{E}} & 0_{N \times N} \end{bmatrix}.
\]

Given this VAR(1) representation, it would be tempting to apply the canonical form of JLS to the extended vector \( \tilde{Z}_t \). Their canonical involves arbitrary invariant (affine) transformation of a latent VAR(1) process that achieves an observationally equivalent representation with a convenient parameterization. However, the argument applies only if the parameters \( K_{0Z}^Q, K_{1Z}^Q \) and \( \Sigma_{\tilde{Z}} \) are unrestricted. This is not the case here, and the cross-equation restrictions are the essence of our message. Those restrictions guarantee that the yield factors \( \mathcal{E}_t \) span the conditional expectations of \( Z_t \) and that the \( \mathcal{E}_t \) is a function of the history of \( Z_t \).

First, the restrictions on \( K_{1Z}^Q \) imply that the Jordan form \( U K_{1Z}^Q U^{-1} \) (for an appropriate matrix \( U \), see J LZ) embodies several eigenvalue restrictions with no convenient expressions. Second, the desired rotation \( U \tilde{Z}_t U^{-1} \) is not arbitrary and depends on the cross-equation restrictions noted above. Third, the covariance matrix \( \Sigma_{\tilde{Z}} \) does not have full rank, implying that the covariance matrix of the transformed vector does not have full rank and, therefore, that the likelihood function is singular unless we can deduce the appropriate dimension reduction. But this depends on the initial matrix \( \Sigma_{\tilde{Z}} \), or else the transformations are not observationally equivalent. In addition, the standard essentially affine price of risk specification, \( \lambda_{\tilde{Z}_t} = \lambda_{\tilde{Z}_0} + \lambda_{\tilde{Z}_1} \tilde{Z}_t \), is not identified. Heuristically, not all the blocks of the matrix \( \lambda_{\tilde{Z}_1} \) are identified because there are only \( N_Z \) sources of risk.
3. Data and Estimation

This section details an empirical that highlights some of the key features and advantages of CM-MTSMs. The estimator is based on the joint likelihood of the observable $X_t$ and of the yield data. In addition, we introduce two economic restrictions that have been proposed in the literature to obtain better estimates of the risk premium. One restriction controls the variations of the Sharpe ratio associated with the inflation risk premium, and the other restriction creates a connection between the real short rate and the one-period inflation risk premium. In our implementation, we estimate the unconstrained model and variants based on different combinations of these restrictions. We also introduce data from surveys of professional forecasters to mitigate the effect of sampling uncertainty on the estimates of the persistence parameters. Results are provided in Section 4.

3.1. Data

The observable vector $X_t$ includes two macro variables ($N_M = 2$) and two yield portfolios ($N_L = 2$), so that the underlying model has four latent factors ($N_Z = 4$). Specifically, we use the monthly inflation rate $\pi_t \equiv \ln \frac{P_t}{P_{t-1}}$ and the unemployment rate $g_t$, so that $M_t \equiv [\pi_t, g_t]$, as well as the first two principal components (PCs) of yields, which we label level and slope, respectively, so that $P_t \equiv [l_t, s_t]$. The sampling frequency is monthly, from January 1992 to January 2013. We use yields on zero-coupon bonds with maturities of 3, 6, 9, 12 and 18 months, as well as 2, 3, 4, 5, 7, 8, 9, 10 years. Zero-coupon yields are available from the Bank of Canada web site. The inflation rate is computed from the seasonally adjusted all-items consumer price index in Canada (StatCan Table 326-0020), and we use the seasonally adjusted Canadian unemployment rate (StatCan Table 282-0089). Note that outstanding real bonds still have very long maturities in Canada and carry a time-varying liquidity premium. Moreover, inflation swap data are not available in Canada.

We also use inflation forecasts from surveys of professional forecasters by Consensus Economics (CE) to estimate the model. CE reports the median forecast across respondents. A survey is taken on the second week of each month but the inflation forecasts are updated only quarterly. On the last month of every quarter, the survey requests a forecast for the remaining quarters of the current calendar year, and for every quarter of the following calendar year. We use the first five quarters to obtain a balanced panel (longer-horizon forecasts are available only irregularly). Importantly,
inflation data are released at the end of each month with a one-month lag. For instance, in the second week of March, survey participants know the inflation rate up to the month of January.

3.2. Likelihood

3.2.1. State dynamics

The first component of the joint likelihood is the conditional likelihood of $X_t$:

$$
l(X_t|X_{t-1}; \Xi) = -\frac{1}{2} \left\{ \frac{N_z}{2} \ln(2\pi) + \ln(\det(\Sigma_X \Sigma_X')) + (X_t - \mathcal{E}_{X,t-1})'(\Sigma_X \Sigma_X')^{-1} (X_t - \mathcal{E}_{X,t-1}) \right\},
$$

(31)
as a function of parameter set $\Xi$, and where $\mathcal{E}_{X,t}$ is given by the recursion in Equation (8) with initial value $\mathcal{E}_{X,0} = E[X_t]$.

3.2.2. Yields

The second component of the joint likelihood is the conditional likelihood of yields observed with errors. The observable $X_t$ includes the first two principal components (PC) of yields measured without errors. We take the remaining PCs of yield as measured with errors, and stacked within a vector $P^e_t$:

$$
P^e_t \equiv W^e y_t = a^e + b^e \mathcal{E}_{X,t} + \eta^e_t,
$$

where $a^e \equiv W^e [a_{n_1} \ldots a_{n_{N^e}}]$, $b^e = [b_{n_1} \ldots b_{n_{N^e}}]W^e$, $W^e$ is the $N_J \times N^e$ matrix of loadings for the PCs that are measured with errors, and where $\eta^e_t$ stacks Gaussian measurement errors. The conditional log-likelihood of $P^e_t$ is then given by

$$
l(P^e_t|X_t; \Xi) = \sum_{n=1}^{N^e} \left\{ -\frac{1}{2} \left\{ \ln(2\pi \sigma_{e,n}^2) + \left( \frac{\eta^e_{t,n}}{\sigma_{e,n}} \right)^2 \right\} \right\},
$$

(32)

where $\sigma_{e,n}^2$ is the variance of the measurement errors.

3.2.3. Surveys of inflation forecasts

The survey asks for a forecast of the average of year-over-year inflation rates across all months in a given quarter. This is given by

$$
\pi^{CE}_{t,h} = 12 \sum_{j=h-11}^{h} \left( \frac{\pi_{t+j} + \pi_{t+j+1} + \pi_{t+j+2}}{3} \right),
$$

(33)
for $h = 3, 6, 9, 12$ and $15$ in our sample (i.e., the next five quarters). While the required forecast may be nominally intuitive, Equation (33) often involves inflation rates from the past and, in addition, the different monthly inflation rates across the forecast horizon do not all receive the same weight.

We compute a model-based forecast of Equation (33) at each date for each horizon. The model-based forecast may differ from the survey-based forecast by a Gaussian measurement error $\eta_{t,h}^{CE}$ with variance $\sigma_{CE,h}^2$. The log-likelihood of the survey data is then given by

$$l \left( \pi_t^{CE} | X_t; \Xi \right) = \sum_{h=1}^{5} \left( -\frac{1}{2} \ln \left( 2\pi \sigma_{CE,h}^2 \right) + \frac{\eta_{t,h}^{CE}}{\sigma_{CE,h}} \right)^2. \quad (34)$$

3.2.4. Combining likelihood

We combine the likelihoods together and write:

$$L(\Xi) = \sum_{t=1}^{T} \left( l \left( P_t^e | X_t; \Xi \right) + l \left( X_t | X_{t-1}; \Xi \right) + 1 \left( \pi_t^{CE} \right) l \left( \pi_t^{CE} | X_t; \Xi \right) \right), \quad (35)$$

where the indicator function $1 \left( \pi_t^{CE} \right)$ is equal to one if an update survey-based forecast is available at time $t$ (i.e., every three months). The estimator of $\Xi$ is based on the (log) likelihood in Equation (35).

Note that we fix the parameters controlling the unconditional means of the state variables to their respective sample averages and impose the usual stationarity conditions on $K_1^P$ and $K_1^Q$.

3.3. Economic Restrictions

3.3.1. Sharpe ratio restrictions

The inflation risk premium may be imprecisely estimated if the Sharpe ratios are left unrestricted. Indeed, $\bar{\lambda}_0^X$ and $\bar{\lambda}_1^X$ are estimated only imprecisely, often implying unreasonable Sharpe ratios, for a wide class of affine term structure models (Duffee, 2010). The one-period-ahead Sharpe ratio associated with inflation risk, $SR_{t,1}^\pi$,

$$SR_{t,1}^\pi \equiv \sigma_\pi^{-1} \left( E_t^Q [\pi_{t+1}] - E_t^P [\pi_{t+1}] \right) \quad (36)$$

is given by the first element of $\sigma_\pi^{-1} (\bar{\lambda}_0 + \bar{\lambda}_1 \mathcal{E}_{X,t})$. The first term in the numerator of $SR_{t,1}^\pi$ is the price that an investor is willing to pay to enter a contract that pays the realized inflation rate. The
second term in the numerator is the investor’s expectation of inflation. Hence, the numerator is the inflation premium – the premium to investors for entering a fair bet on inflation. The ratio $SR_{t,1}^\pi$ expresses this premium as a price per unit of risk, as measured by the volatility of an unexpected change in inflation.

We impose that $SR_{t,1}^\pi$ remains within a plausible range with very high probability. The population distribution of $SR_{t,1}^\pi$ is Gaussian with mean and variance,

\[
E[SR_{t,1}^\pi] = \sigma_\pi^{-1} (\bar{\lambda}_{0,\pi} + \bar{\lambda}'_{1,\pi} E[X, t])
\]

\[
Var[SR_{t,1}^\pi] = \sigma_\pi^{-2} \bar{\lambda}'_{1,\pi} Var(E[X,t]) \bar{\lambda}_{1,\pi}.
\] (37)

Formally, we impose that

\[
-\kappa \leq E[SR_{t,1}^\pi] - 1.96 Var[SR_{t,1}^\pi]^{-1/2} \leq E[SR_{t,1}^\pi] + 1.96 Var[SR_{t,1}^\pi]^{-1/2} \leq \kappa,
\] (38)

implying a probability of 5% that $SR_{t,1}^\pi$ is outside of the range $[-\kappa, \kappa]$ in population. To assess the effect of this restriction, we re-estimate the model while varying $\kappa$ from zero to infinity (the unconstrained case). Our approach is consistent with Duffee (2010), who constrains the sample mean of the Sharpe ratio (but also suggests restricting the population distribution directly), and with Chernov and Mueller (2011), who penalize the excessive variability of the term premium.

3.3.2. A restriction on the short real rate

Chernov and Mueller (2011) impose that the real short rate is not a function of inflation expectations. We consider the effect of this restriction within the family of CM-MTSMs. The restriction on the short rate implies that the first element of $\rho_{1X}^r$ is zero,

\[
e_{1}' \rho_{1X}^r = 0,
\] (39)

where $e_1$ is a vector of zeros but the first element is 1. Economically, the restriction creates a link between the policy response to expected inflation and the price of inflation risk:

\[
\rho_{1\pi} = 1 + \bar{\lambda}_{1,\pi},
\] (40)
where $\lambda_{1,\pi\pi}$ is the response of the inflation risk premium to expected inflation and $\rho_{1\pi}$ is the response of the nominal short rate to expected inflation. For instance, if the policy response is more than one-for-one (monetary policy is stabilizing with respect to inflation), then $\lambda_{1,\pi\pi} > 0$ and the inflation premium increases with higher inflation expectations. One interpretation is that the central bank is the marginal investor in the short-rate market and its policy actions affect the equilibrium by introducing a wedge between the compensation for inflation risk implicit in the nominal rate $i_t$ and the inflation risk premium.

4. Results

4.1. Forecasting Inflation

An accurate decomposition of yields requires accurate forecasts of future inflation. The natural criteria are the out-of-sample RMSEs from these forecasts. This section compares the out-of-sample RMSEs from a battery of alternative models:

(i) Random walk models

- $RW1$: \[ E_t \left( \frac{\sum_{j=1}^{h} \pi_{t+j}}{h} \right) = \pi_t \]
- $RW2$: \[ E_t \left( \frac{\sum_{j=1}^{h} \pi_{t+j}}{h} \right) = \frac{1}{12} \sum_{j=0}^{11} \pi_{t-j} \]

(ii) Stationary models

- $AR_\pi$: \[ \Delta \varepsilon_{\pi,t} = K_{0\pi} + K_{1\pi} \pi_t \]
- $CM_\pi$: \[ \Delta \varepsilon_{\pi,t} = K_{0\pi} + K_{1\pi} \varepsilon_{\pi,t-1} + \Sigma \varepsilon_{\pi,t} \epsilon_{\pi,t} \]
- $S-CM_\pi$: \[ \Delta \varepsilon_{\pi,t} = K_{0\pi} + K_{1\pi} \varepsilon_{\pi,t-1} + \Sigma \varepsilon_{\pi,t} \epsilon_{\pi,t} \]

(iii) Inflation and Unemployment

- $VAR_M$: \[ \Delta \varepsilon_{M,t} = K_{0M} + K_{1M} M_t \]
- $CM_M$: \[ \Delta \varepsilon_{M,t} = K_{0M} + K_{1M} \varepsilon_{M,t-1} + \Sigma \varepsilon_{M,t} \epsilon_{M,t} \]

(iv) Inflation, Unemployment, Level and Slope

- $VAR_{M,\pi}$ and $VAR_X$: \[ \Delta \varepsilon_{X,t} = K_{0X} + K_{1X} X_t \]
\[ \Delta \mathcal{E}_{X,t} = K_{0M} + K_{1M} \mathcal{E}_{X,t-1} + \Sigma \mathcal{E}_{X} \mathcal{E}_{X,t} \]

With the exception of three univariate cases, RW1, RW2 and \( S - CM_{\pi} \), every model is nested in Equation (35). The labels RW1 and RW2 correspond to simple random walk models; the labels \( AR_{\pi} \) and \( CM_{\pi} \) to univariate autoregressive and conditional means for inflation; and the stochastic conditional mean model \( S - CM_{\pi} \) allows for an additional shock \( \epsilon_{\pi,t} \) to the conditional mean \( \mathcal{E}_{\pi,t} \). We also estimate several multivariate models. The labels \( VAR_{M} \) and \( CM_{M} \) correspond to bivariate models combining the inflation rate and the unemployment rate, \( \mathcal{M}' = (\pi_t, g_t) \); the labels \( VAR_{M,P} \) and \( CM_{M,P} \) correspond to trivariate models combining the macro variables with the first yield factor (the level factor); and the labels \( VAR_{X} \) and \( CM_{X} \) correspond to models combining the macro variables with both yield factors (the level and slope factors).

We estimate each model with data from December 1986 to December 1991. Then, we compute inflation forecasts for horizons up to two years ahead, keeping track of the forecast errors against the realized values of inflation. The estimation window is lengthened by one month, new forecasts and forecast errors are produced, and the exercise is repeated until we reach the end of the sample. Figure 1 shows the time series of expected inflation at horizons of 3 months, 1 year, 2 years and 5 years. The level of expected inflation is strongly pro-cyclical and has declined from 3% early in the sample to around 2% starting around 1994. Figure 1 also shows that the slope of expected inflation changes sign over time. Short-run inflation expectations stand below long-run inflation expectations when unemployment is high, and vice-versa when unemployment is low. Table II displays the ratios of the resulting forecast RMSEs for each model relative to the forecast RMSE from the \( AR_{\pi} \) specification. Overall, the conditional mean models offer improvement relative to the AR1 model, but their Markovian counterparts do not, reaching as much as 20% when using macro variables and two yield factors. We explore these results in details in the following sections 4.1.1. The added-value of survey’s information

Using CE survey forecasts at estimation is essential to mitigate over-fitting the inflation data. Table I displays the ratio of monthly forecast RMSEs obtained when using survey data to the RMSEs from the same model estimated without surveys. Perhaps unsurprisingly, the information from surveys improves quarterly inflation forecasts (Panel A). For instance, RMSEs typically decrease by 5–10% at the 1-quarter horizon and by as much as 15–30% at longer horizons. More
importantly, using quarterly surveys delivers substantial RMSE improvements for monthly forecasts (Panel B). For instance, including surveys improves forecast RMSEs of the $VAR_X$ model by 22% and those the $CM_X$ model by 27%. Also, surveys are particularly useful at long horizons, improving the 3-month forecast RMSEs from the $CM_X$ model by only 5%. Figure 2A shows the RMSE ratios. The importance of survey data is readily apparent. Estimates that neglect survey data suffer from substantial bias and small-sample sampling errors (Kim, 2007; Kim and Orphanides, 2012), while adding surveys to the measurement equations effectively lengthens the sample.

4.1.2. From quarterly surveys to monthly forecasts

Combining macro and yield data is key to obtain accurate monthly forecasts. Panel A of Table II displays the ratio of forecast RMSEs relative to the $AR_\pi$ model for quarterly forecasts, when survey are available. The quarterly results suggest that the $S - CM_\pi$ model is preferable at horizons up to two quarters ahead and that the $CM_X$ model is preferable at longer horizons. However, Panel B shows that the performance of the stochastic mean model does not carry to the monthly frequency. The $CM_{M,P}$ and $CM_X$ models, which use yield factors, outperform other models, delivering 5–15% RMSE improvements at horizons between six months and one year. The $CM_X$ model eventually dominates, with improvements of 20% or more at horizons beyond one year. Figure 2C shows the RMSE ratios from the conditional mean models. The importance of using yield data is readily apparent. Multivariate conditional mean models use the information from the shape of the term structure to update inflation forecasts in the absence of updated survey data.

4.1.3. The added-value of conditional mean models

Table III reports the forecast RMSEs from each $CM$ model relative to the RMSEs obtained from the corresponding $VAR$ model. Panels A and B provide results for monthly and quarterly forecasts. The results are: the benefits of using conditional mean models increase as we expand the span of $X_t$. In fact, the $CM_X$ model stands out again, improving over the corresponding $VAR$ model by 6% and 17% at the 1-year and 2-year horizons, respectively. Figure 2B shows the RMSE ratios and illustrates the significant improvements obtained from conditional models. These results contrast with the common observation that more parsimonious models offer better out-of-sample results. Survey data play a key role in obtaining these results.
4.1.4. Can survey forecasts be improved?

CM models can match the accuracy of quarterly CE inflation forecasts. Table IV reports the ratio of each model’s forecast RMSEs relative to the RMSEs obtained when using the survey inflation forecast directly. Figure 2D shows these ratios. No model systematically improves upon the accuracy of survey forecasts, but most models perform reasonably well. The worst performers are the $AR_\pi$ and $VAR_{M}$ model with deteriorations up to 8–9%. On the other hand, a conditional mean model using yield factors and estimated with survey data matches or improves survey RMSEs. In addition, Figure 3 compares the time series of the model-implied forecasts with the forecasts from surveys, and with the realized values of inflation, at horizons of 1, 2, 3 and 4 quarters. One-quarter and two-quarter-ahead model-forecasts are very close to CE forecasts. Model-implied and CE forecasts are still close to each other at longer horizons, but the forecast errors increase, as expected. This contrasts with results in Ang et al. (2007) and Faust and Wright (2011), who conclude that no model can match the accuracy of survey forecasts.

4.2. Why Conditional Mean Models for Inflation Forecasts?

How does a conditional mean model offer more accurate forecasts? To answer this question, consider a VAR(1) where the conditional mean of $X_{t+1}$ is a rotation of the current state,

$$
\text{VAR: } \mathcal{E}_{X,t} = K_{0X} + (K_{1X} + I_{N_{z}})X_t
= K_{0X} + (K_{1X} + I_{N_{z}})\mathcal{E}_{X,t-1} + K_{1X}(X_t - \mathcal{E}_{X,t-1})
= K_{0X} + (K_{1X} + I_{N_{z}})\mathcal{E}_{X,t-1} + K_{1X}u_{X,t},
$$

with $u_{X,t} \equiv (X_t - \mathcal{E}_{X,t-1})$, and contrast this with the case of the conditional mean model,

$$
\text{CM: } \mathcal{E}_{X,t} = K_{0X} + (K_{1X} + I_{N_{z}})\mathcal{E}_{X,t-1} + \Sigma \mathcal{E}_{X} \mathcal{e}_{t}^p
= K_{0X} + (K_{1X} + I_{N_{z}})\mathcal{E}_{X,t-1} + \Sigma \mathcal{E}_{X} \Sigma_{X}^{-1}(X_t - \mathcal{E}_{X,t-1})
= K_{0X} + (K_{1X} + I_{N_{z}})\mathcal{E}_{X,t-1} + \Sigma \mathcal{E}_{X} \Sigma_{X}^{-1}u_{X,t},
$$

where the weights on the prior expectations, $(K_{1X} + I_{N_{z}})$, can differ from the weights on the innovations, $\Sigma \mathcal{E}_{X} \Sigma_{X}^{-1}$. The estimates for the dynamics of inflation expectations $\mathcal{E}_{\pi,t} = E_t[\pi_{t+1}]$
illustrate the importance of separating these channels:

\[
E_{\pi,t} = 1.03 \times E_{\pi,t-1} - 0.005 \times u_{\pi,t} - 0.07 \times E_{g,t-1} + 0.35 \times u_{g,t} \\
+ 0.003 \times E_{l,t-1} - 0.03 \times u_{l,t} - 0.04 \times E_{s,t-1} + 0.26 \times u_{s,t}.
\] (43)

Equation (43) details the estimates for the first line of \(E_{X,t}\). Table V reports parameter estimates of the matrices \(K_{1X} - I_{N_Z}\) and \(\Sigma_X \Sigma_X^{-1}\). The differences between the performance of VAR(1) and conditional mean model can be explained as follows.

First, while expected inflation is persistent, \(\hat{K}_{1X,\pi} = 0.03\) (but the matrix \(\hat{K}_{1X} + I_{N_Z}\) has no unit root), inflation innovations are transitory and have close to no effect on the expectation update. Also, inflation innovations have no significant effect on \(E_{l,t}\) and \(E_{s,t}\), while its effect on \(E_{g,t}\) is small (see the first column of \(\Sigma_X \Sigma_X^{-1}\) in Table V). Hence, the intuition from Kim (2007) that inflation combines a persistent conditional mean component with transitory noise (i.e., an ARMA(1,1) process) carries over in our multivariate context.

Second, since the coefficient on inflation innovation is almost zero, updates of expected inflation are driven by innovations in the other variables. Multiplying coefficients by the corresponding standard deviation shows that slope and unemployment innovations are the most significant economically. The conditional mean model uses the span of \(u_{X,t}\) to capture predictable variations in expected inflation.

Third, the estimated weights on the prior expectations and estimated weights on the innovations have opposite signs. Expected inflation increases with a higher expected level of the yield curve, but decreases with a surprise increase in the level. Coefficients of the slope factor exhibit a similar pattern with opposite signs (an increase of \(s_t\) lifts short-maturity yields relative to longer maturities, and corresponds to a tighter stance of monetary policy). Expected inflation also decreases with a higher expected unemployment rate but increases with a surprise increase in the unemployment rate, presumably reflecting anticipated policy actions.

Whether the signs of these relationships are reasonable depends on the correlation structure between the innovations. Panel (B) reports the standard deviations (on the diagonals) and the correlations (off the diagonals) implied from the estimate of the covariance matrix \(\Sigma_X \Sigma_X^{-1}\) and
Σε_ξΣ′ε_ξ. Inflation innovations are very noisy and show little correlation with other innovations but, again, expected inflation exhibits significant correlations with other components of the conditional mean vector. Importantly, changes in expected inflation are negatively correlated with changes in the expected level of the yield curve, and negatively correlated with increases of short-maturity yields.

4.3. Fitting the Cross-Section of Yields and Surveys

The CM_ξ model also provides a better fit of the cross-section of yield. Table VI compares the RMSEs of measurement errors from the VAR_ξ and CM_ξ models. Table VI also reports RMSEs for the CM_ξ-0 model, which are discussed in the next section. Panel A shows that survey RMSEs are very close between the VAR_ξ and CM_ξ models, and range between 25 and 50 basis points (bps). This reflects the extent of the disagreement between inflation forecasts based on surveys and those based on the model.

Panel B compares the RMSEs of yield measurement errors. Each model fits the first two PCs of yields by construction. The RMSEs for third PCs is smaller in the CM_ξ model relative to the VAR_ξ model, while the RMSEs are close for higher-order PCs. This translates into a better fit of yields, since the third PC explains a larger share of variations. Panel C shows that the RMSEs range between 5 and 18 bps in the CM_ξ model. RMSEs from the VAR_ξ model are larger at shorter maturities, as much as 10 bps lower for the 3-month yield, decline to less than 5 bps for maturities up to three years, and are always between 0 and 3 bps for longer maturities, up to 10 years.

4.4. What Sharpe Ratio?

This section assesses the effect of the Sharpe ratio constraint in Equation (38). First, Figure 4 shows how much each component of the joint likelihood of the CM model gains between the case with a constant Sharpe ratio and the unconstrained case (κ = inf). The yield likelihood quickly gains 100 points as we reach sensible values of κ – around 0.2. The yield likelihood then crawls up, eventually reaching a gain of 258 points in the unconstrained case. This increase occurs in an area of the parameter space where the Sharpe ratio is unreasonably large, suggesting that the unconstrained
model is overparameterized and overfits the data (Duffee, 2010). Note that the log-likelihoods of
the CE survey data and that of the state variables do not change as we vary the constraint.

Do these likelihood improvements translate into lower yield RMSEs? The $CM_X=0$ label in
Table VI corresponds to the $CM$ model $\kappa = 0$. Panel B shows that moving from the restricted
model to the unrestricted model leads to a marginally better fit of higher-order components at the
cost of a small deterioration in the fit of $PC3$. Table C shows that these small changes leave the
yield RMSEs essentially unchanged. Hence, some of the likelihood improvements in Figure 4 can
be attributed to an improved fitting of what amount to be largely idiosyncratic yield variations
captured by higher-order components – overfitting. Furthermore, Figure 5 compares the inflation
risk premium implied from the unrestricted $CM_X$ model (Panel A) and a version of the model
with $\kappa = 0.2$ (Panel B). The excess variability of the inflation risk premium stands out in the
unconstrained case: the scale of the first panel is five times larger! The 2-year inflation risk premium
varies between -20% and 5% annually, and the 5-year inflation risk premium varies between -5%
and 15% annually! Contrast these results with the constrained case, where the 2-year and 5-year
inflation risk premium remain close to their sample average, around 1%, and range between -0.5%
and 3%. The stark contrast between the constrained and the unconstrained case remains for any
economically reasonable value of the constraint parameter.

What are the implications from this excess variability of the inflation risk premiums? Results
(unreported) show that the conditional mean of future inflation is essentially unchanged when
loosening the constraint (survey forecasts pins down the model’s inflation forecasts). With yield
RMSEs remaining constant, the excess inflation risk premium variability must be mirrored by an
excess variability of the real yields. Figure 7 compares the impact of $E_{X,t}$ on real and nominal
yields as we relax the constraint. Panel A reports the product of each of $X_t$, component’s standard
deviations with the corresponding 2-year real yield loading, showing very large increases in the
estimated impact as we relax the Sharpe ratio constraint. In contrast, Panel B shows no change in
the impacts on the 2-year nominal yield.

Relaxing the Sharpe ratio also affects parameter estimates in the short-rate equation. Figure 6
shows the evolution of the elements of $\hat{\rho}_{X1}$ as we relax the constraint. Strikingly, we find that re-
stricting the Sharpe ratio leads to reasonable estimates of the short-rate coefficients. Panel A and
Panel B reports the estimated response coefficient for expected inflation and expected unemployment. When the Sharpe ratio is restricted within an economically reasonable interval, the estimates range between 2 and 2.5 for expected inflation, and between -0.7 and -0.8 for expected unemployment. However, the estimates increase steadily in magnitude as we relax the constraint, reaching close to 3.5 and -1.2 in the unrestricted case for expected inflation and expected unemployment, respectively.

4.5. A Constraint on the Short Rate?

This section assesses the effect of excluding expected inflation (and inflation) from the real short rate (see Section 3.3.2). Recall that the short-rate constraint ties up $\rho_{1\pi}$ and $\bar{\lambda}_{1,\pi\pi}$ (see Equation (40)). The evidence based on the Sharpe ratio constraint shows that the matrix of prices of risk $\bar{\lambda}_1$ is overparameterized, suggesting that the effect of the short-rate constraint depends on the variability in the inflation Sharpe ratio. Figure 9A shows the likelihood gains from relaxing the Sharpe ratio constraint when inflation is excluded from the real short rate. The gains are large initially, as we move from $\kappa = 0$ to $\kappa = 0.03$, which is still very low, but non-existent past this low level of risk premium volatility (note that we changed the scale, with $\kappa$ varying from 0 to only 0.2 for the purpose of clarity). There is little interaction with the Sharpe ratio constraint for reasonable values of $\kappa$, say around 0.20. Panels B and C show the changes in the estimated coefficients for expected inflation and unemployment in the nominal short rate. Again, there are rapid changes for very low values of $\kappa$, but the estimates quickly settle down.

Of course, the constraint is not entirely innocuous. Reintroducing inflation in the real short rate increases the likelihood by a little more than 100 points when $\kappa = 0.2$, but this gain in the likelihood of yields is similar to the gain from freeing the Sharpe ratio constraint, which, as saw in the previous section, does not translate into economically significant differences either. What role can the constraint on the real rate play? Excluding inflation from the real rate is motivated by results from results based on DSGE models (see Chernov and Mueller 2011) and can be seen as a structural restriction identifying the one-month inflation risk premium with the response of the central bank to inflation expectations. In that sense, this assumption yields a different interpretation of nominal yield data, with very little effect on the fit of yields.
4.6. Nominal Yield Decompositions

4.6.1. Benchmark decomposition

The generalized Fisher equation derived in Section 2.5,

\[ i_t^{(n)} = c_t^{(n)} + \pi_t^{(n)} + r_t^{(n)} + irp_t^{(n)}, \]

(see Equation (28)), involves a real yield, the expected inflation and the inflation risk premium. Table VII reports sample summary statistics for each component for maturities of 3 and 6 months, as well as 1, 2, 5 and 10 years, computed from the CMZ model with \( \kappa = 0.2 \). The average nominal yield curve is upward sloping – the slope between 3-month and 10-year yields averages close to 1.3% in our sample – but exhibits significant variability throughout the cycle: the average volatility slopes downward from 3.28% to 2.44%. Real yields are the most important contributors to the variations and persistence of nominal yields. But the positive slope of nominal yields is due to the slope of the inflation risk premium. Real yields are downward sloping – the average slope is -0.7% – suggesting that investors perceived real bonds as hedges. In addition, the average expected inflation is mostly flat around above 2%. In contrast, the inflation risk premium average ranges from almost zero at the shortest maturity to 1.6% at the 10-year maturity.

Figure 8 shows the time series of the real yield, of the expected inflation and of the inflation risk premium for maturities of three months and two years. The real yields move in tandem across maturities and, by and large, the business cycle variations in the level of the nominal yield curve are attributed to the underlying real curve. However, the inflation risk premium is negatively correlated with the real yields. Therefore, real yields appear more procyclical than nominal yields once we adjust for the inflation risk premium. Importantly, this correlation is strongest at longer maturity. Hence, the slope of the inflation risk premium exhibits cyclical variations and drives the slope of the nominal curve. The short-horizon inflation risk premium has been negative for some time after the year 2000. Exposures to inflation risk are perceived as hedges at those horizons. David and Veronesi (2013) provide empirical evidence that inflation shocks can be good news and bad news in the context of a general equilibrium with learning. This effect may also arise with Epstein-Zin preferences: inflation may signal higher consumption growth, leading to upward revisions of the
continuation utility value.

4.6.2. Time-varying inflation volatility

We establish the robustness of our results to conditional inflation volatility. There is ample evidence that the volatility of inflation innovations is time-varying. For instance, the autocorrelation of squared inflation innovations is high and significant. The benchmark model implements a restriction on the inflation Sharpe ratio where the ratio’s denominator is constant. However, the inflation risk premium can increase without affecting the inflation Sharpe ratio if the inflation volatility increases.

We develop a simple no-arbitrage GARCH extension of the benchmark model. Starting from the canonical representation in Theorem 1, we let the matrix $\Sigma_t^*$ be time-varying, implying that the conditional variance of $X_t$ is also time-varying, $\Sigma_{X,t}$. Importantly, the dynamics for $\mathcal{E}_{X,t}$ remains homoscedastic under $\mathbb{P}$ and $\mathbb{Q}$, implying that nominal yields remain affine and that the estimation can proceed as described in Section 3.2 with only minor changes to the conditional likelihood of $X_t$. To complete the model, we assume that the first element of $\Sigma_t^*$ follows a univariate Threshold-GARCH (Zakoian, 1994):

$$
\sigma_{\pi,t+1}^2 = (1 - \alpha - \beta)\sigma_{\pi,0}^2 + \alpha \left( u_{\pi,t+1}^2 I[\pi_{t+1} < \bar{\pi}] + \sigma_{\pi,t}^2 I[\pi_{t+1} > \bar{\pi}] \right) + \beta \sigma_{\pi,t}^2,
$$

and we keep every other element of $\Sigma_t^*$ constant. Hence, the variance of inflation and its covariances with the other state variables vary through time. We use a threshold mainly to exclude an outlier in the squared inflation residuals associated with a tax change.

Figure 10 compares the models in two key dimensions. Panel A compares measures of 2-year expected inflation. Allowing for time-varying inflation volatility produces a measure of expected inflation that is very close to that from the benchmark model. Panel B compares the 2-year real yields implied by each model. Again, allowing for conditional inflation volatility produces estimates of real yields that are remarkably close to the benchmark case. Overall, we conclude that our results are robust to time-varying inflation volatility.
5. Conclusion

We introduce the class of CM-MTSMs where the conditional mean \( \mathcal{E}_t \) drives the term structure of yields \( y_t^{(n)} = a_n + b_n \mathcal{E}_t \) as well as the dynamics of the state variables, \( \mathcal{E}_t \equiv E_t[Z_{t+1}] \). We depart from the Markov assumption: \( \mathcal{E}_t \) potentially depends on the entire history of \( Z_t \). This parsimonious extension gives different roles to \( Z_t \) and \( \mathcal{E}_t \) and provides a better fit to the dynamics of expected inflation and of nominal yields. Including inflation within a CM-MTSM as a showcase, the results illustrate the distinctive properties of CM-MTSMs (i) yields span expected inflation as measured by survey forecasts (ii) expected changes and surprises in the level and the slope have opposite effects inflation expectations, (iii) the inflation dynamics involves long lags of macro and term structure data. A simpler Markovian VAR(1) model does not capture these features. Importantly, it is essential to include survey data to overcome the effect of sampling variability on the estimates of persistence parameters. Finally, we find that a using sensible restriction on the distribution of the Sharpe ratios is essential to deliver plausible estimates of the inflation risk premiums and of the real yields.

The dynamics in CM-MTSMs are consistent with several general equilibrium specifications where the observable macro variables are not Markovian (e.g., Bansal and Yaron (2004); Bansal and Shaliastovich (2013)). In addition, our results obtain while keeping the variability of the Sharpe ratio in line with plausible calibrations used in endowment economies. Hence, our results bode well for the ability of models where the pricing kernel is obtained from principle to capture the underlying economic structure. We leave important extensions for future research several. First, we focus on short horizons, typically less than two years. Kozicki and Tinsley (2001, 2006) highlight the role of shifting endpoints in the evolution of long-horizon inflation rates in US data. In Canada, Amano and Murchison (2006) show the importance of a shifting endpoint for inflation in the early 1990s during the transition toward a 2% inflation-targeting regime. Second, we focus on inflation forecasts, but conditional forecasts of future inflation rates are not independent of forecasts of other macro variables. Extending our approach to include additional variables, such as unemployment, could help identify the dynamic relationships between economic activity, inflation and risk as perceived by bond investors. The framework proposed here can be extended in these directions.
A. Appendix

A.1. Yield Coefficient Recursions

The price of a nominal zero-coupon bond with maturity \( n \) is given by

\[
D_t(n) = \exp \left( A_n + B_n' \Sigma E_t \right)
\]

with coefficients given by the recursions

\[
B_{n+1} = B_n + \frac{1}{2} B_n' \Sigma E_t' \Sigma E_t B_n - \rho_0,
\]

\[
A_{n+1} = A_n + K_0^Q + B_n' \Sigma E_t' \Sigma E_t B_n - \rho_0.
\]

(44)

for \( n > 0 \) and with initial conditions \( A_0 = 0 \) and \( B_0 = 0 \). The nominal yield coefficients are given by \( a_n = -\frac{A_n}{n} \) and \( b_n = -\frac{B_n}{n} \). From Equations (22) and (25), it follows that the real rate coefficients in Equation (26) are given by

\[
\rho_{0,R} = \rho_0 - \frac{1}{2} \sigma^2 - \epsilon_0 \Sigma X \Sigma^{-1} \left( K_0^0 - K_0^R \right)
\]

\[
\rho_{1,R} = \rho_1 - \epsilon_1 - \left( \Sigma X \Sigma^{-1} \left( K_1^0 - K_1^R \right) \right)' e_1.
\]

(45)

A.2. Risk Premium

From Equations (8), (11) and (15), it follows that

\[
E_t Q_t \equiv E_t Q_t [Z_{t+1}]
\]

\[
= E_t + \Sigma E_t^t [\epsilon_{t+1}]
\]

\[
= E_t + \Sigma E_t^{-1} E_t \left( \Delta E_{t+1} - K_0^R - K_1^R E_t \right)
\]

\[
= E_t + \Sigma E_t^{-1} \left( K_0^Q - K_1^Q E_t - K_0^R - K_1^R E_t \right)
\]

\[
= E_t + \Sigma E_t^{-1} \left( K_0^Q - K_0^R \right) + \Sigma E_t^{-1} \left( K_1^Q - K_1^R \right) E_t^R.
\]

(46)

A.3. Proposition 2

The \( N_Z \times 1 \) vector of observable \( X_t \) defined in Equations (17)–(18) has the same dynamics as the process for \( Z_t \). Define \( D \) as

\[
D = D_1 + D_2 \left( K_1^R + I_{N_Z} \right).
\]

(47)

We require that \( D \) is invertible. Then, using the mapping between \( X_t \) and \( Z_t \) given in Equation (17), the parameters for the dynamics of \( X_t \) are given by

\[
K_{0,X}^R = DK_0^R - DK_1^R D^{-1} \left( C + D_3 K_0^R \right)
\]

\[
K_{1,X}^R = DK_1^R D^{-1}
\]

(48)

\[
K_{0,X}^Q = DK_0^Q - DK_1^Q D^{-1} \left( C + D_3 K_0^Q \right)
\]

\[
K_{1,X}^Q = DK_1^Q D^{-1}
\]

(49)

\[
\Sigma E_X = D \Sigma E
\]

\[
\Sigma X = D_1 \Sigma + D_2 \Sigma E.
\]

(50)

A.4. Theorem 1

The canonical form for \( Z_t \) in Proposition (1) is exactly identified. We verify that the parameterization for \( X_t \) in Theorem (1) is also identified. First, we redefine \( D \) in Equation (47) to substitute out the matrix \( K_1^R \),
which belongs to the dynamics of $\mathcal{Z}_t$. Given $D \equiv D_1 + D_2 \left(K_1^P + I_{N_2}\right)$ and using $K_{1X}^P = DK_1^P D^{-1}$ from Proposition (2), it follows that

\[
D = D_1 + D_2 \left(K_1^P + I_{N_2}\right) = D_1 + D_2 + D_2 D^{-1} K_{1X}^P D,
\]

where $D_1$ and $D_2$ are defined in Equation (18). Rearranging:

\[
I_N = (D_1 + D_2) D^{-1} + D_2 D^{-1} K_{1X}^P
\]

\[
\text{vec}(I_N) = \text{vec} \left((D_1 + D_2) D^{-1}\right) + \text{vec} \left(D_2 D^{-1} K_{1X}^P\right)
\]

\[
\text{vec}(I_N) = (I_N \otimes (D_1 + D_2)) \text{vec} (D^{-1}) + \left(K_{1X}^P \otimes D_2\right) \text{vec} (D^{-1}),
\]

and, therefore, $D$ is defined implicitly by

\[
(I_N \otimes D_1 + (I_N + K_{1X}^P) \otimes D_2) \text{vec} (D^{-1}) = \text{vec} (I_N).
\]

We require that

\[
(I_{N_2} \otimes D_1 + (I_{N_2} + K_{1X}^P) \otimes D_2)
\]

is invertible so that $D$ is well-defined. Second, we explicitly provide the mapping between the parameters of $\mathcal{Z}_t$ and the parameters of $\mathcal{X}_t$. The mapping between the $\mathbb{P}$ parameters is given by

\[
K_0^P = (D - K_{1X}^P D_2)^{-1} \left(K_{0X}^P + K_{1X}^P C\right)
\]

\[
K_1^P = D^{-1} K_{1X}^P D
\]

\[
\Sigma_\varepsilon = D^{-1} \Sigma_\varepsilon X
\]

\[
D_1 \Sigma = \Sigma_X - D_2 D^{-1} \Sigma_\varepsilon X,
\]

and the mapping for the $\mathbb{Q}$ parameters is given by

\[
K_{0X}^Q = DK_0^Q - DJ \left(\lambda^Q\right) (D_1 + D_2)^{-1} (C + D_2 D^{-1} K_{0X}^P)
\]

\[
K_{1X}^Q = DJ \left(\lambda^Q\right) D^{-1}.
\]

(See Proposition (1).) Importantly, both sides of the last equation in (54) have the same rank as $D_1$, which cannot be inverted since its lower block is zero. In other words, we cannot identify $\Sigma_X$ and $\Sigma_\varepsilon X$ separately. To complete the parameterization in the theorem, rewrite

\[
D_1 \Sigma = \begin{bmatrix} \Sigma^* \\ 0 \end{bmatrix},
\]

defining $\Sigma^*$ implicitly. Then, $\Sigma_X$ is given by the last equation in (54) in terms of the other parameters,

\[
\Sigma_X = D_2 D^{-1} \Sigma_\varepsilon X + \begin{bmatrix} \Sigma^* \\ 0 \end{bmatrix}.
\]

Turning to the short rate, from Equation (17)–(18) and Proposition (1), it follows that the coefficients in Equation (19) are given by

\[
\rho_{0X} = -\rho' \left(C + D_2 K_0^Q\right)
\]

\[
\rho'_{1X} = \rho' D^{-1}.
\]

The dynamics of $\mathcal{X}_t$ in Equations (20)–(21) follow from Proposition (2). Finally, the spread between $\mathcal{E}_{X,t}^P$ and $\mathcal{E}_{X,t}^Q$ in Equation (22) follows from Proposition (2) and Equation (13).
A.5. **Generalized Fisher Decomposition**

The generalized Fisher decomposition can be applied to any nominal yield with maturity \( n \),

\[
i_t^{(n)} = c_t^{(n)} + r_t^{(n)} + \pi_t^{(n)} + irp_t^{(n)}, \tag{59}
\]

where \( r_t^{(n)} \) is the real yield, \( irp_t^{(n)} \) is the \( n \)-period-ahead inflation risk premium:

\[
irp_t^{(n)} = \frac{1}{n} \left( E_t^Q \left[ \sum_{j=1}^{n} \pi_{t+j} \right] - E_t \left[ \sum_{j=1}^{n} \pi_{t+j} \right] \right),
\]

where \( E_t^{\pi^{(n)}} \) is the \( n \)-period-ahead inflation expectation:

\[
E_t^{\pi^{(n)}} = E_t^P \left[ \frac{\sum_{j=1}^{n} \pi_{t+j}}{n} \right],
\]

and where \( c^{(n)} \) is a Jensen term:

\[
c^{(n)} = \frac{1}{2} \sigma^2 + \frac{1}{2n} \sum_{j=0}^{n-1} j^2 \left( b'_r \Sigma_m \Sigma'_m b_{r,j} - b'_j \Sigma_m \Sigma'_m b_j \right).
\]
References


Figure 1. Term structure of expected inflation. Monthly measures of expected average inflation, over the horizons 3 months, 1 year, 2 years and 5 years, from January 1992 to December 2012.
Figure 2. Out-of-sample forecast RMSEs. Panel A plots the ratio of forecast RMSEs for models estimated using surveys to the RMSES of the same model estimated without surveys. Panel B plots the ratio of inflation forecast RMSEs from CM models over that of the corresponding VAR models (using survey data in every case). Panel C plots the ratio of forecast RMSEs for models estimated based on an expanding information set (i.e., number of variables included in $\mathcal{X}_t$) to the RMSES of an AR1 (using survey data in every case). Panel D plots the ratio of quarterly inflation forecast RMSEs over that of survey forecast RMSEs where the models are estimated using surveys. Survey forecasts are available only quarterly and forecast errors are sampled quarterly to compute RMSEs.
Figure 3. Inflation and expected inflation. Quarterly inflation, survey forecasts and model-implied inflation forecasts, from Q1-1992 to Q4-2012, at horizons of 1, 2, 3 and 4 quarters ahead.
Figure 4. Likelihood gain and the Sharpe ratio constraint. Components of the $CM_X$ model log-likelihood across values of the inflation risk premium Sharpe ratio constraint parameter $\kappa$ between 0 and $\infty$.

Figure 5. The inflation risk premiums and the inflation Sharpe ratio constraint. Panel A shows the 2-year and 5-year inflation risk premium from the $CM_X$ model estimated with no restriction on the distribution of the inflation rate Sharpe ratio. Panel B shows the same inflation risk premium, but from the $CM_X$ estimated with the inflation rate Sharpe ratio constraint parameter set to $\kappa = 0.2$. 
Figure 6. Nominal short rate coefficients. Estimates of each element of $\rho_{1X}$ from the $CM_X$ model but varying the parameters in the Sharpe ratio constraint between 0 and $\infty$ (i.e., no constraints). Panel A shows coefficients on inflation expectations $E_{\pi,t}$, Panel B shows coefficients on unemployment expectations $E_{g,t}$, Panel C and D show the coefficients on expectations of the yield portfolio $E_{P_1,t}$ and $E_{P_2,t}$. 
Figure 7. Impacts on the 2-year yield. Estimated impact of one standard deviation change of each element of $E_{X,t}$ on the 2-year real yield (Panel A) and on the 2-year nominal yield (Panel B) across values of the Sharpe ratio constraint parameter in the $CM_X$ model between 0 and $\infty$. The estimated impact equals the product of the standard deviation and of the 2-year yield loadings, for each element of $E_{X,t}$.

Figure 8. Nominal yield decomposition. Decomposition of the 1-year and 5-year nominal yields, from January 1992 to December 2012, into its components: the real yield, the expected inflation and the inflation risk premium.
Figure 9. A Constraint on the short rate. Panel A shows the likelihood gains as we relax the Sharpe ratio constraint, but where we impose the constraint on the short-rate. Panel B shows the expected inflation coefficient in the nominal short-rate equation with and without the short rate constraint across different Sharpe ratio constraint parameter $\kappa$. Panel C shows estimates of the expected unemployment coefficient.
Figure 10. Decomposition with time-varying inflation volatility. Compare the 2-year inflation expectations (Panel A) and the 2-year real yields (Panel B) from the benchmark $CM$ model and from the extended $CM$ model with time-varying inflation volatility. Both models are estimated $\kappa = 0.2$ in the inflation rate Sharpe ratio constraint.

Table I. The added value of inflation surveys in out-of-sample forecasting.

Out-of-sample inflation forecast RMSEs from each model relative to the RMSEs from the same model estimated without survey data. Estimation uses survey data in every case. Forecast horizons in months. Macro and yield data 01/1986-12/2012. Survey data Q1-1992/Q4-2012.

Panel A– Quarterly forecasts

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Panel B– Monthly forecasts

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Table II. Forecasting inflation out-of-sample


Panel A– Quarterly forecasts

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Panel B– Monthly forecasts

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<th>VAR_{M, P_{1}}</th>
<th>CM_{M, P_{1}}</th>
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Table III. The added value of conditional mean models in out-of-sample forecasting.

Out-of-sample inflation forecast RMSEs from each CM model relative to the RMSEs from the corresponding VAR model. Estimation is based on the joint likelihood of the model’s state variables (e.g., $X_t$) and of survey data in each case. Forecast horizons in months. Macro and yield data 01/1986-09/2011. Survey data Q1-1992/Q2-2011. Panel A and B reports RMSE ratios from monthly and quarterly forecasts.

<table>
<thead>
<tr>
<th>Horizon</th>
<th>$CM_{π}$</th>
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<th>$CM_{M,P_1}$</th>
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Panel B– Quarterly forecasts

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<td>0.97</td>
<td>1.02</td>
</tr>
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<tr>
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<td>0.96</td>
<td>0.97</td>
<td>0.92</td>
<td>0.97</td>
</tr>
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<td>0.96</td>
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<td>0.93</td>
<td>0.95</td>
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<td>0.92</td>
<td>0.95</td>
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Table IV. Can model forecasts improves survey forecast?


<table>
<thead>
<tr>
<th>Horizon</th>
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<th>$S - CM_{π}$</th>
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<th>$CM_{M,P_1}$</th>
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<td>1.00</td>
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<td>0.87</td>
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<tr>
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<td>0.99</td>
</tr>
<tr>
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<td>1.09</td>
<td>1.04</td>
<td>1.03</td>
<td>1.09</td>
<td>1.01</td>
<td>1.05</td>
<td>1.00</td>
<td>1.04</td>
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</table>
Table V. Parameter estimates

Parameter estimates for the no-arbitrage conditional mean model. Panel A reports parameters for the historical dynamics, $K_{1X} + I_N^Z$ and $\Sigma_{\varepsilon_X} \Sigma_{X}^{-1}$, respectively. Panel B reports the correlation of innovations to $X_t$ and innovations to $\mathcal{E}_X$, implicit in the matrices $\Sigma_X \Sigma_X^\prime$ and $\Sigma_{\varepsilon_X} \Sigma_{\varepsilon_X}^\prime$, respectively. Diagonal elements correspond to the standard deviations of the corresponding innovations in annualized basis points. Panel C reports parameters for the risk-neutral dynamics, $K_{1X}^\varphi + I_N^Z$. Estimation uses survey data and $\kappa = 0.2$. Asymptotic standard errors in parentheses when applicable. Macro and yield data 01/1984-12/2012. Survey data Q1-1992/Q4-2012.

Panel A—Historical dynamics

$K_{1X} + I_N^Z = \begin{pmatrix}
1,03 & -0,07 & 0,003 & -0,04 \\
(0,15) & (0,06) & (0,003) & (0,04) \\
0,36 & 0,82 & 0,003 & -0,10 \\
(0,25) & (0,11) & (0,005) & (0,07) \\
-1,28 & 0,68 & 0,969 & 0,36 \\
(1,16) & (0,51) & (0,026) & (0,31) \\
-0,30 & 0,02 & 0,000 & 0,99 \\
(0,21) & (0,10) & (0,005) & (0,06) \\
\end{pmatrix}$

$\Sigma_{\varepsilon_X} \Sigma_{X}^{-1} = \begin{pmatrix}
-0,005 & 0,35 & -0,03 & 0,26 \\
(0,002) & (0,07) & (0,01) & (0,06) \\
-0,011 & 0,73 & -0,04 & -0,04 \\
(0,002) & (0,06) & (0,01) & (0,03) \\
0,000 & -0,01 & 0,99 & -0,05 \\
(0,002) & (0,08) & (0,02) & (0,09) \\
0,002 & -0,10 & 0,01 & 0,90 \\
(0,001) & (0,03) & (0,01) & (0,03) \\
\end{pmatrix}$

Panel B—Standard deviations and correlations

$\Sigma_X \Sigma_X^\prime \rightarrow \begin{pmatrix}
4,01 & 0,02 & 0,08 & 0,01 \\
(0,16) & (0,08) & (0,08) & (0,08) \\
0,18 & -0,07 & 0,04 \\
(0,01) & (0,07) & (0,07) \\
1,09 & -0,04 \\
(0,05) & (0,06) \\
0,44 \\
(0,02) \\
\end{pmatrix}$

$\Sigma_{\varepsilon_X} \Sigma_{\varepsilon_X}^\prime \rightarrow \begin{pmatrix}
0,14 & 0,47 & -0,31 & 0,80 \\
(0,03) & (0,10) & (0,10) & (0,06) \\
0,15 & -0,34 & -0,14 \\
(0,01) & (0,08) & (0,11) \\
1,08 & -0,03 \\
(0,12) & (0,07) \\
0,39 \\
(0,03) \\
\end{pmatrix}$

Panel C—Risk-neutral dynamics

$K_{1X}^\varphi + I_N^Z = \begin{pmatrix}
0,78 & 0,08 & -0,001 & 0,03 \\
(0,14) & (0,06) & (0,002) & (0,04) \\
0,24 & 0,89 & -0,001 & -0,10 \\
(0,40) & (0,17) & (0,004) & (0,11) \\
-0,60 & 0,21 & 0,99 & 0,02 \\
(0,30) & (0,13) & (0,01) & (0,08) \\
-0,68 & 0,26 & -0,003 & 1,12 \\
(0,39) & (0,17) & (0,007) & (0,10) \\
\end{pmatrix}$
Table VI. Measurement errors

Standard deviation of measurement errors in annualized percentage for PCs, yields and CE inflation forecasts, respectively, with asymptotic standard errors in parenthesis. CE forecasts cover quarterly horizons from 1 to 5 quarters ahead. Yield combinations measured with errors, $Y^*_n,t$, are the principal components of yields (except the first two measured without errors). Estimation uses survey data and $\kappa = 0.2$. Monthly macro and yield data 01/1984-12/2012. Quarterly survey data Q1-1992/Q4-2012.

Panel A– Survey measurement errors

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<td>(0.002)</td>
<td>(0.002)</td>
<td>(0.004)</td>
<td>(0.003)</td>
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<td>0.59</td>
<td>0.52</td>
<td>0.32</td>
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<tr>
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<td>(0.003)</td>
<td>(0.004)</td>
<td>(0.003)</td>
<td>(0.007)</td>
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Panel B– PCA measurement errors

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<td>(0.01)</td>
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<td>(0.01)</td>
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Panel C– Yield measurement errors

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<td>0.09</td>
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Sample summary statistics of the nominal yield $y_t^{(n)}$, the real yield $r_t^{(n)}$, the expected inflation $E_{π_t}^{(n)}$, and the inflation risk premium, $irp_t^{(n)}$, at different maturities, $n$ computed from the model. Estimation uses survey data and $κ = 0.2$.

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<th>Panel B– 6 Months</th>
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<td>$r_t^{(n)}$</td>
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<tr>
<td>$E_{π_t}^{(n)}$</td>
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<tr>
<td>$irp_t^{(n)}$</td>
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<td>$ρ(12)$</td>
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<td>$E_{π_t}^{(n)}$</td>
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<td>$irp_t^{(n)}$</td>
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<th>Panel F– 10 Years</th>
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<td>$y_t^{(n)}$</td>
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<tr>
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<tr>
<td>$irp_t^{(n)}$</td>
<td>0.81</td>
</tr>
<tr>
<td>$ρ(1)$</td>
<td>0.98</td>
</tr>
<tr>
<td>$ρ(12)$</td>
<td>0.85</td>
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</tbody>
</table>