SUBJECTIVE INDEPENDENCE AND CONCAVE EXPECTED UTILITY

EHUD LEHRER† AND ROEE TEPER‡

February 12, 2014

Abstract. When a potential hedge between alternatives does not reduce the exposure to uncertainty, we say that the decision maker considers these alternatives similar. We offer a novel approach and suggest that similarity is subjective and should be different across decision makers. Similarity can be recovered through a property of the individual’s preferences referred to as subjective codecomposable independence. This property characterizes a class of event-separable models and provides a behavioral foundation to Concave Expected Utility preferences. Our model accommodates recent paradoxes posed by Machina to Choquet expected utility, which employs an objective notion of similarity.

Keywords: Codecomposable independence, uncertainty aversion, concave expected utility, Choquet expected utility, Ellsberg paradox, Machina paradox.

JEL Classification: D81, D83

This paper contains parts of an online unpublished manuscript by Lehrer entitled “A Comment on an Example by Machina” (2007). The authors would like to thank Mark Machina for discussions which motivated us to think about this project. The authors also wish to thank Sophie Bade, Madhav Chandrasekher, Mehmet Ekmecki, Shiri Eron, Tzachi Gilboa, Faruk Gul, Juan Sebastian Lleras, Evan Piermont, Luca Rigotti, Marciano Siniscalchi and especially Simon Grant for their comments and remarks.

†Tel Aviv University, Tel Aviv 69978, Israel and INSEAD, Bd. de Constance, 77305 Fontainebleau Cedex, France. e-mail: lehrer@post.tau.ac.il. Lehrer acknowledges the support of the Israel Science Foundation, Grant #538/11.

‡Department of Economics, University of Pittsburgh. e-mail: rteper@pitt.edu.
1. Introduction

In the heart of decision theory, and economics in general, lies the Expected Utility model. It is well-known, however, that in environments of subjective uncertainty this model cannot capture agents' preferences for hedging against uncertainty. This phenomenon is referred to as uncertainty aversion. While the vast majority of the literature focuses on uncertainty aversion, an important issue has not received enough attention: what is the nature of alternatives between which an agent does not prefer to hedge, and more importantly, how would this change across different agents? What are the behavioral implications of that?

We offer a novel approach to valuation of hedging and suggest that the value of a hedge depends on a notion of subjective similarity. From a decision maker's perspective, hedging between similar alternatives does not reduce the amount of uncertainty she is exposed to. Hence, she reveals no preference for hedging between such alternatives. In contrast with existing literature, similarity is subjective and is in the ‘eyes of the beholder’; what one decision maker deems similar, might not seem similar to another. Our approach gives rise to a class of preferences represented by a subjective non-additive probability (capacity) capturing the decision maker’s belief, and a general integration scheme according to which expected utility is calculated. This class of preferences includes expected utility (Savage [23] and Anscombe and Aumann [1]) and Choquet expected utility (Schmeidler [24]) as special cases, both of which are based on objective similarity. We exemplify how subjective similarity leads to different predictions from existing models. Combining our approach with uncertainty aversion provides a behavioral foundation to a class of preferences termed Concave Expected Utility (CavEU) – a capacity-based model of uncertainty aversion that employs the concave integral (Lehrer [16]) for evaluating alternatives. This model can accommodate recent Ellsberg-like paradoxes emerging from Choquet preferences (Machina [21]). Lastly, our approach provides sufficient conditions for subjective and Choquet expected utility models; these conditions are weaker than the previous standard formulations.

1.1. Overview. Ellsberg’s urn example [9] raised conceptual issues with Savage’s [23] and Anscombe and Aumann’s [1] Subjective Expected Utility (SEU) theory. The example shows that the model cannot capture preferences for hedging exhibited in environments with subjective uncertainty. These difficulty stems from the sure-thing principle (in Savage) or the independence axiom (in Anscombe-Aumann). A solution was offered by Schmeidler [24] who presented Choquet Expected Utility (CEU) theory,
in which beliefs are represented by non-additive probabilities and expected utilities are calculated according to the Choquet integral [7].

Schmeidler assumed that independence is maintained between alternatives that are comonotonic. Two functions over a state space are comonotonic if both induce the same ordering when the states are ordered according to their associated outcomes. Mixing between comonotonic alternatives yields an alternative in the same comonotonicity class. Thus, it is implicit in Schmeidler’s approach that hedging between comonotonic acts should not reduce the amount of uncertainty the decision maker is exposed to, and no decision maker should have a strict preference for hedging between such alternatives.

We propose that valuation of hedging against uncertainty depends on a subjective notion of similarity. _Similarity_ of alternatives, as a perception of exposure to uncertainty, entails no strict preference for hedging. Thus, _similarity should be considered subjective_, while structural (objective) similarities need not dictate attitudes toward uncertainty. For instance, according to our approach the decision maker is let the liberty to have strict preferences for hedging between comonotonic acts.\(^1\) We offer an example for a subjective-independence axiom reflecting our approach.

Any act can be represented (or, decomposed) as a coin toss (a mixture) between betting on an event and a ‘complementary’ act. Typically, there are different possibilities to represent an act this way. The independence axiom we postulate requires that any act can be decomposed in at least one way for which the decision maker exhibits no strict preference for hedging.\(^2\) Note that according to this axiom, independence is required to hold for some decomposition, and not for all of them. This particular decomposition depends on the decision maker and typically differs from one decision maker to another. We therefore refer to this axiom as _subjective codecomposable independence_.

\(^1\)There are other weakening of independence in the literature. The issue of similarity as perceived by a decision maker and that, as such, should be subjective has not been discussed. For example, constant independence introduced by Gilboa and Schmeidler [13] postulates that the constants are similar to every act. This is an objective notion of similarity. Siniscalchi [26] introduced an independence axiom across acts that are ‘complementary’ in some sense, but given the underlying utils, this notion of similarity as perception of uncertainty is also objective. Castagnoli and Maccheroni [4] study independence within an abstract collection of sets of alternatives (see Section 6 for more details).

\(^2\)More formally, any act can be decomposed in such a way that the decision maker exhibits uncertainty neutrality among acts represented by the same bet and complementary act. For brevity, we use the former wording throughout the Introduction.
Our approach gives rise to a class of event-separable preferences including, among other, SEU and CEU. These preferences all admit a utility form expressed by a subjective non-additive probability (capacity), capturing the decision maker’s belief, and a general integration scheme according to which expected utility is calculated. Our theory suggests that the utility of an alternative is the expected value (with respect to the non-additive belief) of some decomposition of the alternative into bets. The decision maker wishes to simplify the task of evaluating an act, doing so by decomposing it to elementary alternatives, that is bets. This decomposition into basic ingredients alleviate the evaluation of and the comparison between alternatives. Unlike CEU, our model allows decompositions that do not have to follow a particular structure.

The subjective independence axiom allows us to identify the underlying belief with respect to which bets, and thus acts, are evaluated. One unresolved difficulty is the inability to identify the bet-decomposition structure supporting the evaluation scheme. In order to obtain more structure and some understanding as to how alternatives are being evaluated, we impose (in addition to subjective independence) the classical uncertainty aversion axiom and refer to such preferences as Concave Expected Utility (CavEU). Such preferences admit a representation that employs the concave integral (Lehrer [16]). Thus, subjective independence allows one to provide a behavioral foundation for this class of preference relations.

Among the class of uncertainty averse preferences, CavEU are more flexible than CEU preferences in the sense that, the acts among which uncertainty neutrality applies are subjectively determined and are not dictated by pre-specified structural similarity. Due to this flexibility, CavEU preferences are less vulnerable to ‘paradoxes’ such as those introduced by Machina [21] to CEU preferences. Machina suggests that non-separable models would be able to accommodate the type of behavior exhibited in his examples. This is interesting because, while there are indeed some non-separable models that accommodate the Machina examples, CavEU is an event-separable model just like CEU preferences.\(^3\)

1.2. Organization. The rest of the paper is organized as follows. The next section provides an informal discussion of Choquet Expected Utility, Concave Expected Utility

\(^3\)There are classic models of uncertainty aversion such as maxmin expected utility (Gilboa and Schmeidler [13]) that cannot accommodate the Machina examples (see Baillon et al. [3]). Cerreia et al. [5] develop a general class of uncertainty averse preferences (containing CavEU) capturing preferences for hedging. This model, for example, can accommodate the examples. For an additional discussion see footnote 9 in Machina [21] and also Dillenberger and Segal [8].
and the essential differences between the two approaches, and shows how CavEU can address Machina’s examples. The formal framework of choice under uncertainty is presented in Section 3.1. Subjective codecomposability and the emergence of a capacity are presented in Sections 3.2 and 3.3. Uncertainty aversion and the characterizations of CavEU preferences are formally discussed in Section 4. The relation of codecomposability to SEU and CEU is presented in Section 5. Section 6 concludes with a discussion of the related literature and additional aspects of the model we present. All the proofs are in the appendix.

2. Non-Additive Beliefs and the Machina Examples

This section provides an informal discussion and (partial) comparison between the classic Choquet Expected Utility model and an alternative theory considered in this paper. The aim of this section is to present and motivate some of the key concepts relevant to the point the current paper is trying to make. The formal decision theoretic foundation for the alternative theories appears in the following sections.

2.1. Choquet and Concave Expected Utility. The underlying domain of alternatives is the collection of (non-negative) random variables, or assets, over a state space $S$. A capacity $v$ assigns a number to each event in a monotonic fashion (with respect to inclusion). We interpret the capacity $v(E)$ as how likely the event $E$ is in the eyes of the decision maker. In other words, a capacity can be interpreted as the decision maker’s subjective belief. A finite collection $(a_i, E_i)_i$, where $a_i$ is a positive real number and $E_i$ is an event, is a decomposition of the alternative $g$ if $\sum_i a_i \mathbb{1}_{E_i} = g$. That is, $g$ can be decomposed to the collection of bets of the form $a_i \mathbb{1}_{E_i}$. Clearly, an alternative can be decomposed in several distinct ways. Similarly to the classical expected utility theory, the value of a decomposition $(a_i, E_i)_i$, with respect to a capacity $v$, is simply $\sum_i a_i v(E_i)$. When a capacity is a probability distribution over the states $S$, then the values of all decompositions of an alternative coincide and equal to its expected utility with respect to that probability. However, when the capacity is not a probability distribution (that is, not additive) then this fact is no longer true, and different decompositions obtain different values.

---

4We assume for the sake of simplicity that the vNM utility index was already identified.
5$\mathbb{1}_E$ is the characteristic function of the event $E$. 
Following Ellsberg’s urn example [9], Schmeidler [24] was the first to present an alternative to the classical subjective expected utility theory by incorporating non-additive beliefs. Schmeidler introduced a behavioral foundations for Choquet Expected Utility (CEU) theory. The Choquet expected utility (see, Choquet [7]) of $g$ takes the form

$$V_{CEU}(g) = \sum_i b_i \cdot v(E_i),$$

where $\sum_i b_i \cdot \mathbb{1}_{E_i}$ is the unique decomposition of $g$ such that $\{E_i\}_i$ form a chain (that is, $E_{i+1} \subseteq E_i$ for every $i$). We refer to such a decomposition as the Choquet decomposition. Preferences $\succeq$ over the domain discussed are CEU if they admit a representation as in Eq. (1).

As mentioned, every alternative has more than one decomposition. Resorting to the Choquet decomposition is only one of many decompositions of an alternative. Consider an alternative theory in which an asset is evaluated according to the maximum value over all of its decompositions. The decomposition in which the maximum is obtained is referred to as the optimal decomposition. Such valuation are denoted by $V_{CAV}$ and preferences that admit such a utility form, are referred to as Concave Expected Utility (CavEU). The term hints that such preferences always exhibit (weak) affinity for hedging.

To illustrate how CavEU may be different than CEU, consider the following example. Let the state space be $S = \{s_1, ..., s_4\}$ and define a capacity $v$ over the state space as follows: $v(s) = \frac{1}{12}$ for every state $s$, $v(\{s_1, s_2\}) = v(\{s_1, s_3\}) = v(\{s_2, s_3\}) = v(\{s_1, s_4\}) = \frac{1}{6}$, $v(\{s_2, s_4\}) = v(\{s_3, s_4\}) = \frac{3}{12}$, $v(\{s_1, s_2, s_3\}) = v(\{s_1, s_3, s_4\}) = v(\{s_2, s_3, s_4\}) = \frac{1}{3}$, $v(\{s_1, s_2, s_4\}) = \frac{2}{6}$ and $v(S) = 1$. Note that the contribution of the state $s_2$ to any event that contains neither $s_1$ nor $s_2$ is greater than the contribution of $s_1$. Formally, for any event $E$ that does not contain the states $s_1, s_2$, $v(E \cup \{s_1\}) \leq v(E \cup \{s_2\})$. Moreover, the inequality is strict when $E = \{s_4\}$. In this sense, if we interpret $v$ as how the decision maker perceives uncertainty, then $s_2$ is more likely than $s_1$.

Now, consider the assets $f = (0,1,2,3)$ and $g = (1,0,2,3)$. Note that $f$ and $g$ differ only in states $s_1$ and $s_2$. The asset $f$ returns the lower outcome in the less likely state and the higher outcome in the more likely one. It is the opposite case for the asset $g$; it returns the higher outcome in the less likely state. It is plausible then, that a decision maker perceiving uncertainty through $v$ would rank $f$ over $g$. Nevertheless, expected return according to CEU does not support that; $V_{CEU}$ of both $f$ and $g$ is $\frac{8}{12}$: $V_{CEU}(f) = v(\{s_2, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = v(\{s_1, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = V_{CEU}(g)$.
That is, the Choquet expected return, holding the belief \( v \), is the same for both \( f \) and \( g \) and the decision maker is indifferent between the two. However, \( \text{CavEU} \) ranks the asset \( f \) strictly higher than the asset \( g \): 

\[
\frac{8}{12} = v(\{s_1, s_3, s_4\}) + v(\{s_3, s_4\}) + v(\{s_4\}) = \text{CavEU}(g).
\]

The capacity \( v \) above can be presented as 

\[
v(E) = \min_i p_i(E)
\]

for every event \( E \), where 

\[
p_1 = (\frac{1}{12}, \frac{1}{6}, \frac{2}{3}), \quad p_2 = (\frac{1}{12}, \frac{1}{3}, \frac{1}{6}), \quad p_3 = (\frac{1}{12}, \frac{2}{3}, \frac{1}{6}) \text{ and } p_4 = (\frac{2}{3}, \frac{1}{12}, \frac{1}{6}).
\]

That is the capacity, as a modeling tool of perception of uncertainty, displays pessimism. In this case it is plausible to assume that the decision maker will exhibit uncertainty aversion. Nevertheless, \( \text{CEU} \) with respect to this capacity does not exhibit such aversion.\(^6\) On the other hand, the concave integral does. As will be formally shown, the example above is a generic one in the sense that the capacity representing any \( \text{CavEU} \) preferences can always be constructed as the minimum of additive beliefs over the state space. In contrast, \( \text{CEU} \) with respect to such capacities typically does not exhibit uncertainty aversion. Actually, when \( \text{CEU} \) preferences with respect to a capacity \( v \) do exhibit uncertainty aversion, then they coincide with \( \text{CavEU} \) with respect to \( v \) (see Proposition 2 and Theorem 2 below).

2.2. On Examples by Machina. In a recent paper, Machina [21] introduced two “paradoxes” for the Choquet Expected Utility model. He “exploits” tail-separability (as he refers to it) exhibited by such preferences and constructs several examples, in the spirit of Ellsberg, in which such preferences can not accommodate choices that may be considered natural. Tail-separability is a property of \( \text{CEU} \) theory stemming from the fact that alternatives are always evaluated according to their Choquet decomposition. \( \text{CavEU} \) preferences, however, are more flexible in the sense that event-separability is subjective; the optimal decomposition depends on the decision maker’s subjective similarity structure and is not pre-specified structurally as in Eq. (1). Hence, \( \text{CavEU} \) preferences are less vulnerable than \( \text{CEU} \) ones to such “paradoxes”.

2.2.1. The Reflection Example. An urn contains one hundred balls of four different colors \( a, b, c, \) and \( d \). All you know is that there are fifty balls that either \( a \) or \( b \), and fifty balls that are either \( c \) or \( d \). A decision maker (DM) chooses an act, then a ball is randomly drawn and a reward in utils\(^7\) is given to the DM according to the color

\(^6\)The capacity \( v \) as defined in the example is not a convex one. According to Schmeidler [24] \( \text{CEU} \) with respect to \( v \) does not adhere to uncertainty aversion.

\(^7\)Even though the analysis would go through if entries are monetary, we consider utils for brevity and simplicity.
of the ball and the chosen act. The following table summarizes the rewards related to four acts. We will later discuss two decision problems: \( f_1 \) vs. \( f_2 \), and \( f_3 \) vs. \( f_4 \).

Table 1. The Reflection Example

<table>
<thead>
<tr>
<th>Bet</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 )</td>
<td>400</td>
<td>800</td>
<td>400</td>
<td>0</td>
</tr>
<tr>
<td>( f_2 )</td>
<td>400</td>
<td>400</td>
<td>800</td>
<td>0</td>
</tr>
<tr>
<td>( f_3 )</td>
<td>0</td>
<td>800</td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td>( f_4 )</td>
<td>0</td>
<td>400</td>
<td>800</td>
<td>400</td>
</tr>
</tbody>
</table>

One can see that \( f_3 \) and \( f_4 \) are mirror images of \( f_2 \) and \( f_1 \) respectively. It is then plausible that, if a DM strictly prefers \( f_1 \) to \( f_2 \) (i.e. \( f_1 \succ f_2 \)) we would expect that she should strictly prefer \( f_4 \) to \( f_3 \) (i.e. \( f_4 \succ f_3 \)). This is inconsistent with \( CEU \) preferences, as explained in Machina [21].

Indeed, the Choquet decomposition of \( f_1 \) is \( 400 \cdot I_{abc} + 400 \cdot I_b \) and that of \( f_2 \) is \( 400 \cdot I_{abc} + 400 \cdot I_c \). Suppose that \( V_{CEU}(f_1) \succ V_{CEU}(f_2) \) as induced by some capacity \( v \). It implies that \( v(b) \succ v(c) \), which in turn is equivalent, by a similar calculation, to \( V_{CEU}(f_3) \succ V_{CEU}(f_4) \). To summarize, \( f_1 \succ f_2 \) and \( f_4 \succ f_3 \) together are inconsistent with \( CEU \) theory.

We show now that the reversal as discussed in Machina is possible under \( CavEU \) preferences. Consider the least monotonic capacity \( v \) such that \( v(a) = v(b) = v(c) = v(d) = 0, v(bc) = \frac{1}{100} \) and \( v(ab) = v(cd) = \frac{1}{2} \). The optimal decomposition of \( f_1 \) is \( 400 \cdot I_{ab} + 400 \cdot I_{bc} \). However the best decomposition of \( f_2 \) is \( 400 \cdot I_{ab} \). Thus, according to \( CavEU \) with respect to the described capacity, \( V_{CAV}(f_1) \succ V_{CAV}(f_2) \). The reason is that \( 400 \cdot v(ab) + 400 \cdot v(bc) > 400 \cdot v(ab) \). For a similar calculation \( V_{CAV}(f_4) \succ V_{CAV}(f_3) \). Indeed, an optimal decomposition for \( f_3 \) is \( 400 \cdot I_{cd} + 400 \cdot I_{bc} \). Hence, \( CavEU \) is consistent with \( f_1 \succ f_2 \) and \( f_4 \succ f_3 \).

\( ^{8} \)This capacity might be interpreted as that the DM believes that at least one of the balls must be \( b \) or \( c \).

\( ^{9} \)L’Hardion and Placido [20] find that the more common reversal is the opposite to the one discussed above, that is \( f_2 \succ f_1 \) while \( f_3 \succ f_4 \). Baillon, L’Haridon, and Placido [3] note and claim that \( CavEU \) preferences cannot accommodate the more common reversal. The algebra below shows that this is in fact inaccurate. If \( v(b) \succ v(c) = 0, v(bcd) \succ v(bc) + v(cd) \) and \( v(bd) = v(b) \) then \( V_{CAV}(f_3) = 400 \cdot v(bcd) + 400 \cdot v(b), V_{CAV}(f_4) = 400 \cdot v(bcd) + 400 \cdot v(c) \) and \( V_{CAV}(f_3) \succ V_{CAV}(f_4) \). On the other hand, if in addition \( v(abc) - v(bc) \succ v(ab) - v(b) \) and \( v(abc) \prec v(ac) + v(bc) \) then \( V_{CAV}(f_1) = 400 \cdot v(abc) + 400 \cdot v(b), V_{CAV}(f_2) = 400 \cdot v(ac) + 400 \cdot v(bc) \) and \( V_{CAV}(f_2) \succ V_{CAV}(f_1) \).
2.2.2. The 50-51 Example. An urn contains one hundred and one balls of four different colors $a, b, c$, and $d$. Fifty balls are either $a$ or $b$ and the other fifty one are either $c$ or $d$. The following table summarizes the rewards related to four acts. We will later discuss two decision problems: $f_5$ vs. $f_6$, and $f_7$ vs. $f_8$.

**Table 2. 50-51 Example**

<table>
<thead>
<tr>
<th>Bet</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_5$</td>
<td>800</td>
<td>800</td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td>$f_6$</td>
<td>800</td>
<td>400</td>
<td>800</td>
<td>400</td>
</tr>
<tr>
<td>$f_7$</td>
<td>1200</td>
<td>800</td>
<td>400</td>
<td>0</td>
</tr>
<tr>
<td>$f_8$</td>
<td>1200</td>
<td>400</td>
<td>800</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that $f_7$ and $f_8$ are obtained from $f_5$ and $f_6$ by increasing the rewards related to color $a$ (which are the highest possible) by 400 and by reducing the rewards related to color $d$ (which are the lowest possible) by 400. Machina argues that it is reasonable for uncertainty averse DM’s to prefer $f_5$ being to $f_6$ (denoted $f_5 \succ f_6$) and at the same time to prefer $f_8$ to $f_7$ ($f_8 \succ f_7$).

Why CEU cannot accommodate this kind of preferences? The Choquet decomposition of $f_5$ is $400 \cdot 1_{abcd} + 400 \cdot 1_{ab}$ and that of $f_6$ is $400 \cdot 1_{abcd} + 400 \cdot 1_{ac}$. Suppose that CEU with respect to some capacity $v$ induce $V_{CEU}(f_5) > V_{CEU}(f_6)$. It implies that $v(ab) > v(ac)$. On other other hand, the Choquet decomposition of $f_7$ is $400 \cdot 1_{abc} + 400 \cdot 1_{ab} + 400 \cdot 1_a$ and that of $f_8$ is $400 \cdot 1_{abc} + 400 \cdot 1_{ac} + 400 \cdot 1_a$. Hence, $V_{CEU}(f_8) > V_{CEU}(f_7)$ is equivalent to $v(ac) > v(ab)$. In short, $V_{CEU}(f_5) > V_{CEU}(f_6)$ cannot live together with $V_{CEU}(f_8) > V_{CEU}(f_7)$ under CEU. In other words, $f_5 \succ f_6$ and $f_8 \succ f_7$ together, are inconsistent with Choquet utility maximization theory.

Consider now the least monotonic capacity that takes the following values: $v(ab) = \frac{50}{101}$, $v(ac) = .5$, $v(cd) = \frac{51}{101}$, $v(abc) = .75$ and $v(abd) = .76$. The optimal decomposition of $f_5$ is $400 \cdot 1_{abc} + 400 \cdot 1_{abd}$, and its value is $400 \cdot 0.75 + 400 \cdot 0.76 = 604$. The optimal decomposition of $f_6$ is $400 \cdot 1_{abcd} + 400 \cdot 1_{ac} = 602$. Thus, according to CavEU with respect to the specified capacity, $V_{CAV}(f_5) > V_{CAV}(f_6)$. On the other other hand, the optimal decomposition of $f_7$ is $400 \cdot 1_{abc} + 400 \cdot 1_{ab}$ and its value is $400 \cdot 0.75 + 400 \cdot \frac{50}{101}$. Furthermore, the optimal decomposition of $f_8$ is $400 \cdot 1_{abc} + 400 \cdot 1_{ac}$ and its value is $400 \cdot 0.75 + 400 \cdot 0.5$. Since $v(ab) < v(ac)$ we obtain that $V_{CAV}(f_8) > V_{CAV}(f_7)$. To conclude, under CavEU it is possible to obtain $f_5 \succ f_6$ and $f_8 \succ f_7$. 
3. Subjective Decomposability

3.1. Environment. Consider a decision making framework in which an object of choice is an act from the state space to utility outcomes. More formally, let $S$ be a finite non-empty set of states of nature. An act $f$ is a function from $S$ to $\mathbb{R}_+$. The collection of acts is denoted by $\mathcal{F}$ with typical elements being $f, g, h$. We interpret $f(s)$ as the payoff induced by act $f \in \mathcal{F}$ in state $s \in S$ and assume it is the utility exerted by the decision maker if $f$ is chosen and $s$ is the realized state. Note that we assume the decision maker’s vNM utility has already been identified.

Remark 1. One can also consider the restatement by Fishburn [11] of the classical Anscombe-Aumann [1] set-up. In that case, standard axioms imply that the vNM utility index can be identified and that the formulation of alternatives as utility acts, as we resort to here, is well defined. Such results have been established repeatedly and we rely on these results. Note, however, that the axioms presented throughout do not rely on the assumption that the utility has been identified, and are formulated in a manner so they could be applied directly to Anscombe-Aumann acts as well.

Abusing notation, for an act $f \in \mathcal{F}$ and a state $s \in S$, we denote by $f(s)$ the constant act that assigns the utility $f(s)$ to every state of nature. Utils (and constant acts) will be typically denoted by $a, b, c$. Mixtures (convex combinations) of acts are performed pointwise. That is, if $f, g \in \mathcal{F}$ and $\delta \in [0, 1]$, then $\delta f + (1 - \delta)g$ is the act in $\mathcal{F}$ that yields $\delta f(s) + (1 - \delta)g(s)$ utility for every $s \in S$. Mixture coefficients will be denoted by $\delta, \alpha$, etc.

In our framework, a decision maker is associated with a binary relation $\succeq$ over $\mathcal{F}$ representing his ranking. $\succ$ is the asymmetric part of the relation. That is, if $f \succ g$ but it is not true that $g \succeq f$. $\sim$ is the symmetric part, that is $f \sim g$ if $f \succeq g$ and $g \succeq f$. A binary relation $\succeq$ is complete if for every $f, g \in \mathcal{F}$, either $f \succeq g$ or $g \succeq f$. It is transitive if for $f, g, h \in \mathcal{F}$, $f \succeq g$ and $g \succeq h$ imply $f \succeq h$. As discussed in the Introduction, an important property of preferences is independence. It will be useful for later discussion to present the formal definition at this point. We say that a binary relation $\succeq$ satisfies the independence axiom over a (convex) collection of acts $\mathcal{F'} \subseteq \mathcal{F}$ if for every $f, g, h \in \mathcal{F'}$ and every $\delta \in (0, 1)$, $f \succeq g$ if and only if $\delta f + (1 - \delta)h \succeq \delta g + (1 - \delta)h$.

\textsuperscript{10}See Section 6.1 for a discussion on the structural assumptions on the utils.
3.2. **Codecomposable Independence.** A *bet* is an act that yields some utility \( b \in \mathbb{R}_+ \) over an event \( E \subseteq S \) and the utility 0 over the complement event. Such a bet will be denoted by \( b_E \). An act which is not a bet can always be represented as a convex combination, or a decomposition, of some bet and another act. That is, for \( f \in \mathcal{F} \) we can find a bet \( b_E \), an act \( f' \), and \( \delta \in [0, 1] \) such that \( f = \delta b_E + (1 - \delta)f' \). This is exemplified in Figure 1. As can be seen from Figure 2, there are many decompositions of this sort for an act. Pick one decomposition of \( f \), say, to the bet \( b_E \) and the
complementary act \(f'\) (as appears in Figure 1). In particular, \(f \in [b_E, f'] = \{ \alpha b_E + (1 - \alpha)f' : \alpha \in [0, 1] \}\). Now, as can be seen in Figures 1 and 3, every act \(g \in [b_E, f']\) can be decomposed, similarly to \(f\), to the bet \(b_E\) and the act \(f'\). A decision maker might (but not necessarily) perceive the acts \(f\) and \(g\), and any other act in \([b_E, f']\), to have a “similar” structure. If so, it is reasonable to assume, as in Schmeidler [24], that the decision maker will not have a strict preference for hedging (in other words, the decision maker will exhibit uncertainty neutrality when mixing) between such acts.

Note, however, that a different decision maker may not find \(g\) similar to \(f\). It is plausible that she entertains a different notion of similarity, and finds \(h \not\in [b_E, f']\) and \(f\) similar since they share a(nother) decomposition structure (see, for example, Figures 2 and 4). While one decision maker exhibits uncertainty neutrality when mixing between \(f\) and \(g\) due to his perception of similarity, another decision maker finds \(f\) and \(h\) similar, exhibiting uncertainty neutrality when mixing between \(f\) and \(h\) while exhibiting other uncertainty attitudes for mixtures between \(f\) and \(g\).

Our main axiom, \textit{Subjective Codecomposable Independence}, postulates that a similarity structure exists but is subjective: every act can be decomposed to some bet on an event and a complementary act such that, the decision maker exhibits uncertainty neutrality across all acts that can be decomposed to a bet on the same event and the complementary act.

Formally, for a bet \(b_E\) and an act \(f'\), let
\[
\text{cone}(b_E, f') = \{ \alpha b_E + \beta f' : \alpha, \beta \geq 0 \}
\]
be the the cone generated by \(b_E\) and \(f'\).

\textit{Subjective Codecomposable Independence}. For every non-bet act \(f\), there exist a bet \(b_E\) and \(f'\) such that \(f \in [b_E, f']\) and \(\geq\) satisfies independence over \(\text{cone}(b_E, f')\).

The next axiom can be considered as complementary to \textit{subjective codecomposable independence}, which does not have any bite when the act under consideration is a bet.

\textit{Worst-Outcome Bet Independence}. For every two bets \(b_E\) and \(d_G\), \(b_E \succeq d_G\) if and only if \(\alpha b_E \succeq \alpha d_G\) for every \(\alpha \in (0, 1)\).

\textit{Worst-outcome bet independence} allows us to directly compare different bets on different events, and states that whenever a bet \(b_E\) is preferred to a bet \(d_F\), then mixing both with the worst bet 0 does not reverse the preference.\(^{11}\)

\(^{11}\)A stronger version which compares any two acts and their mixtures with the worst outcome, referred to as worst-outcome independence, can be found in Chateauneuf and Faro [6].
3.3. A Capacity Emerges. To explore the implications of subjective codecomposable independence we need to present some notations and definitions. A capacity \( v \) over \( S \) is a function \( v : 2^S \to [0, 1] \) satisfying: (i) \( v(\emptyset) = 0 \) and \( v(S) = 1 \); and (ii) \( K \subseteq T \subseteq S \) implies \( v(K) \leq v(T) \).

**Definition 1.** We say that a binary relation \( \succeq \) over all acts \( \mathcal{F} \) admits an event-separable representation if there exist a homogeneous functional \( V : \mathcal{F} \to \mathbb{R} \) and a capacity \( v : 2^S \to [0, 1] \) such that:

1. \( V \) represents \( \succeq \), that is \( V(f) \geq V(g) \Leftrightarrow f \succeq g \) for every \( f, g \in \mathcal{F} \);
2. \( V(b_E) = b \cdot v(E) \) for every bet \( b_E \); and
3. for every act \( f \in \mathcal{F} \),
   \[
   V(f) = \sum a_E v(E) \text{ for some } \sum a_E \mathbb{1}_E = f,
   \]
   where \( a_E \geq 0 \).

A collection \( \{(a_E, E) : E \subseteq S, a_E \geq 0\} \) is a decomposition of \( f \) if \( \sum a_E \mathbb{1}_E = f \). Given a capacity \( v \) over events, the value of such a decomposition is \( \sum a_E v(E) \). Thus, a binary relation admits an event-separable representation if an act is ranked according to the value, with respect to the capacity, of one of its decompositions into bets. Choquet Expected Utility is one example, out of many, for such preferences. As discussed in Eq. 1, the decomposition according to which an act is evaluated is the Choquet decomposition of the act to a chain.

In addition to the axioms above, which rise from the new approach we present, we will postulate throughout the following standard assumptions:

**Weak Order.** \( \succeq \) is complete and transitive.

**Monotonicity.** For every \( f, g \in \mathcal{F} \), \( f(s) > g(s) \) for all \( s \in S \) implies \( f \succ g \).

**Continuity.** For every \( f \in \mathcal{F} \) the sets \( \{g \in \mathcal{F} : g \succeq f\} \) and \( \{g \in \mathcal{F} : g \preceq f\} \) are closed.

**Theorem 1.** Let \( \succeq \) be a binary relation over \( \mathcal{F} \) satisfying weak order, monotonicity, continuity, worst-outcome bet independence, and subjective codecomposable independence. Then, \( \succeq \) admits an event-separable representation. Moreover, if both \( (V, v) \) and \( (V', v') \) represent \( \succeq \), then \( V' = V \) and \( v' = v \).

Theorem 1 states that given standard assumptions and subjective codecomposable independence, a binary relation admits an event-separable representation. The axioms
are sufficient to identify a unique non-additive belief and to state that alternatives are ranked according to the value of one of their decompositions.\textsuperscript{12}

Another way to think about the representation (or, the axioms) is the following. In the eyes of the decision maker, every act is equivalent to a lottery over different bets on different events. A bet is being evaluated according to the (non-additive) belief; similar to SEU theory, the act itself is being evaluated as the expected value, with respect to the lottery, of all such bets. Unlike SEU and CEU, different decision makers may consider different decompositions (as lotteries over bets) for the same alternative.

Note that \textit{subjective codecomposable independence} is a weak assumption; it is not possible to determine exactly what is the decomposition according to which an alternative is ranked. A question is whether making stronger behavioral assumptions can help identify the integration mechanisms, and whether such integration mechanisms are natural and interesting while being different than Choquet? We investigate this direction in Section 4 below. We impose the classical uncertainty aversion axiom and show that the only decision model that is based on a capacity, as in Theorem 1, is the \textit{Concave Expected Utility} model discussed in Section 2 in the context of the Machina examples.

3.4. \textbf{Proof Sketch: Theorem 1.} The detailed proof of Theorem 1 appears in the Appendix. We here present an intuitive sketch. \textit{Subjective codecomposable independence} guarantees that every act $f$ is contained in a cone generated by some bet and another act, where an affine function over this cone represents the preferences. Due to \textit{worst-outcome bet independence} every bet can be directly compared to the constants, which guarantees that all these affine functions (one for every cone) can be calibrated with the one defined on the constant acts and thereby can be merged together in a consistent way. This yields a functional $V$ over all acts that represents the preferences and is affine over the subjective cones. In particular, part 1 in Definition 1 is satisfied.

Now, due to \textit{worst-outcome bet independence}, $V$ is homogeneous over bets. That is, $V(\alpha b_E) = \alpha V(b_E)$. In particular $V(0) = 0$. Thus, normalizing $V(\mathbb{1}_S) = 1$ and defining $v(E) = V(\mathbb{1}_E)$, we obtain that $v$ is a capacity due to \textit{monotonicity}. This implies that part 2 of Definition 1 is satisfied.

\textsuperscript{12}Note that, unlike the customary representation result, Theorem 1 provides only sufficient conditions for the utility representation. However, assuming that $V$ satisfies standard properties as continuity and monotonicity, and in addition a proper additivity property in the line of our main axiom, it is possible to show that the axioms we present are also necessary. We believe that the insightful part of the result is the one presented.
In case \( f \) is a non-bet act the main idea is the following. *Subjective codecomposable independence* assures us that \( f \) is a mixture (convex combination) of some bet \( b_{E_f} \) and a ‘residual’ act \( g \) such that \( V \) is affine over cone\((b_{E_f}, g)\). In particular, \( V(f) = V(\alpha b_{E_f} + (1 - \alpha)g) = \alpha V(b_{E_f}) + (1 - \alpha)V(g) \), and since \( V \) is homogeneous over bets, we get that \( V(f) = \alpha V(\mathbb{1}_{E_f}) + (1 - \alpha)V(g) \). By repeating this argument for \( g \) and substitute \( V(g) \) into \( V(f) \), one obtains a summation of bets and a smaller residual, the one obtained from decomposing \( g \). This procedure is reiterated over and over again. However, and this is the delicate part of the proof, the sequence of residuals obtained from this process might not vanish. That is, it might converge to an act \( h \) which is not a bet and satisfies \( V(h) > 0 \).

When decomposing \( f \) such that \( V(f) = \alpha bV(\mathbb{1}_{E_f}) + (1 - \alpha)V(g) \), we show that it is possible to choose the decomposition in a way that \( a_f = \alpha b \) is maximal. Then, when decomposing the residual, say \( g \), it is impossible that \( g \) is decomposed in an affine fashion to a bet on \( E_f \). This will contradict the maximality of \( a_f \). Since the state space is finite, there are finite number of events. This means that there must be a finite step in this process of decomposing the residuals. From part 2 of Definition 1, which was established above, the proof is complete.

4. Uncertainty Aversion

Since Schmeidler [24] and Gilboa and Schmeidler [13] uncertainty aversion has been one of the most studied phenomenon in the theory of decision making. Unlike Schmeidler [24] who focused on comonotonic-independence, following our discussion regarding subjective similarity, we here wish to impose the weaker *subjective codecomposable independence*, and add structure by postulating uncertainty aversion.

*Uncertainty Aversion.* For every \( f, g \in \mathcal{F} \), if \( f \sim g \) then \( \delta f + (1 - \delta)g \succeq g \) for every \( \delta \in [0, 1] \).

Before we state the result, we make a formal definition of the concave integral (Lehrer [16]) with respect to capacities. The concave integral of an act \( f : S \to \mathbb{R}_+ \) with respect to a capacity \( v : 2^S \to [0, 1] \) is defined by

\[
\int_{\text{Cav}} f dv = \max \left\{ \sum a_E v(E) : \sum a_E \mathbb{1}_E = f, a_E > 0 \right\}.
\]

The integral considers all possible decompositions of an act and evaluates it according to the decomposition with the maximal value (with respect to the capacity). We refer
to preferences $\succeq$ over all acts $\mathcal{F}$ as CavEU if there exist a capacity $v$, such that for all $f, g \in \mathcal{F}$

$$f \succeq g \iff \int_{\mathcal{Cav}} f \, dv \geq \int_{\mathcal{Cav}} g \, dv.$$  

We know from Theorem 1 that along with the standard axioms, subjective codecomposable independence implies that preferences admit an event-separable representation. It turns out that from this family of preferences, there is only one class adhering to uncertainty aversion. This is the class of CavEU preferences.

**Theorem 2.** Let $\succeq$ be a binary relation over $\mathcal{F}$. Then the following are equivalent:
1. $\succeq$ satisfies weak order, monotonicity, continuity, worst-outcome bet independence, subjective codecomposable independence and uncertainty aversion; and
2. $\succeq$ is CavEU.

We know, due to Theorem 1, that the belief (or, capacity) representing the preferences is unique. However, the Theorem allows for very general preferences and there is not much more that can be said about such beliefs. Since now we have restricted attention to preferences that adhere to uncertainty aversion, it is possible to identify a particular structure for the decision maker’s beliefs.

**Proposition 1.** $v$ can represent a CavEU preference relation (in the sense of Definition 1) if and only if $v$ can be written as a minimum of finitely many measures over $S$ (that is, $v = \min_i \mu_i$).

The proposition states that CavEU preferences can be represented by a belief which, as a modeling tool of perception of uncertainty, displays pessimism. It is reasonable to assume that a decision maker entertaining such beliefs will exhibit uncertainty aversion. Nevertheless, other integration mechanisms, such as the Choquet integral, with respect to pessimism exhibiting capacities do not display such aversion.\(^{13}\)

5. **Codecomposable Independence and Expected Utility Models**

It is interesting to see the links between the codecomposable independence approach to existing models. Clearly, both SEU and CEU are particular classes of preferences\(^{13}\)

\(^{13}\)For CEU preferences, for example, it holds as such capacities are typically not convex; see Schmeidler [24].
admitting an event-separable representation. It turns out that providing stronger versions of our independence axiom yields exactly SEU and CEU. Note that both versions resort to objective similarity considerations.

For an act \( f \) and a utility level \( a \in \mathbb{R}_+ \), let \( E'_a = \{ s \in S : f(s) \geq a \} \) be the event in which \( f \) performs better than \( a \). We refer to such an event as a cumulative event for \( f \). When considering a cumulative event for an act \( f \), we may ignore the utility level at times and write \( E^f \). A stronger codecomposable independence axiom can be formulated taking into account decomposition of acts to all bets over (respectively) cumulative events.

**Cumulative Codecomposable Independence.** For every act \( f \), bet \( b_{E^f} \) and \( f' \) such that \( f \in [b_{E^f}, f') \), \( \succeq \) satisfies independence over \( \text{cone}(b_{E^f}, f') \).

The axiom postulates that if \( f, g, h \in \mathcal{F} \) can all be expressed as a linear and positive combination of a bet \( b_{E^f} \) and an act \( f' \), then independence involving \( f, g, h \) holds. In terms of similarity, like Schmeidler’s approach, this again is an objective similarity assumption. Note that, in this case, \( f, g \) and \( h \) are comonotonic, and hence this axiom is weaker than Schmeidler’s comonotonic-independence. Resulting from such a strengthening of subjective codecomposable independence is the following proposition. Note that worst-outcome bet independence is implied by cumulative codecomposable independence.

**Proposition 2.** The following two statements are equivalent:
1. \( \succeq \) satisfies weak order, continuity, monotonicity, and cumulative codecomposable independence;
2. \( \succeq \) admits a CEU representation.

Lastly, we explore a further strengthening of our approach by postulating that for every decomposition of \( f \) to a bet \( b_E \) and a complimentary act \( f' \), the preference relation satisfies independence over \( [b_E, f'] \).

**Codecomposable Independence.** For every bet \( b_E \) and act \( f' \), \( \succeq \) satisfies independence over \( \text{cone}(b_E, f') \).

Assuming codecomposable independence along with the axioms specified above allows us to formulate the following result.

**Proposition 3.** The following two statements are equivalent:
1. $\succeq$ satisfies weak order, continuity, monotonicity, worst-outcome bet independence, and codecomposable independence;

2. $\succeq$ admits an SEU representation.

Proposition 3 states that given the standard axioms, codecomposable independence allows us to identify a subjective probability with respect to which the decision maker calculates the expected utility of the different alternatives and ranks them accordingly. Note that worst-outcome bet independence, is, again, not needed as it is implied by codecomposable independence.

It should be noted that recently Borah and Kops [2] suggest that the independence axiom can be substantially weakened while still maintain a subjective expected utility representation. Roughly speaking, they show that it is enough to require independence across acts that are similar in the sense that they differ only in one state.

6. Final Discussion

6.1. On the Assumptions on Utils. We assume that the vN–M utility is non-negative. It is possible to assume the utility is negative while strengthening the axiomatic structure by postulating $c$-independence (instead of the weaker worst-outcome bet independence). This will imply that the functional form is translation covariant and any non-negative utility act can then be analyzed by translating it to a non-negative one.

We have also assumed throughout that the utility is unbounded. Subjective codecomposable independence is stronger when the utility is bounded. It might be impossible to obtain some decompositions for an act since the complementary act to a particular bet may require levels of utility that are not specified (or identified) by the decision maker's preferences. That implies more structure on the capacity or the integration mechanism. To exemplify, assume that the vN-M utility is bounded by $[0, 1]$ and consider the utility act $f = (0.5, 1)$ over the state space $\{s_1, s_2\}$. Considering the bet $\mathbb{1}_{s_1}$ then $f$ can be decomposed as $f = 0.5 \cdot \mathbb{1}_{\{s_1\}} + 0.5 \cdot 2 \cdot \mathbb{1}_{\{s_2\}}$, however the complementary act, in this case $2 \cdot \mathbb{1}_{\{s_2\}}$, is not in the domain of the vN-M utility. Consider the least monotonic capacity $v$ over the state space $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ that satisfies $v(s_1, s_4, s_5) = v(s_2, s_5, s_6) = v(s_3, s_4, s_6) = \frac{2}{3}$ and $v(S) = 1$. Note that $v(E) \leq \frac{2}{3}$ whenever $E \neq S$. It is easy to verify that $v$ is totally balanced. Consider the act $f = (0.5, 0.5, 0.5, 1, 1, 1)$. It can be decomposed as $f = \frac{1}{2} \mathbb{1}_{\{s_1,s_4,s_5\}} + \frac{1}{2} \mathbb{1}_{\{s_2,s_5,s_6\}} + \frac{1}{2} \mathbb{1}_{\{s_3,s_4,s_6\}}$, which implies that $\int f C_{\text{av}} dv \geq 3 \cdot \frac{1}{2} \cdot \frac{2}{3} = 1$. 
Let the utility be bounded in $[0, 1]$ and assume, adhering to subjective codecomposable independence, that $\int^{Cav} f dv = \sum a_E v(E)$ for some decomposition $f = \sum a_E \mathbb{1}_E$, where $\sum a_E = 1$. Thus, $\sum a_E v(E) = \sum_{E \neq S} a_E v(E) + a_S v(S) \leq \frac{2}{3} \sum_{E \neq S} a_E + a_S = \frac{2}{3} (1 - a_S) + a_S \leq \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} = \frac{5}{6} < \int^{Cav} f dv$ (the second inequality is due to that $a_S$ cannot exceed $\frac{1}{2}$ because $f = \sum a_E \mathbb{1}_E$). This is a contradiction. Hence, it cannot be that the capacity $v$ represents preferences that adhere to subjective codecomposable independence and uncertainty aversion when the utility is bounded. This illustrates how subjective codecomposable independence entails more structure on preferences when utility is bounded relative to unbounded utility. Note that the capacity specified above is non-convex. Indeed, when the concave integral is taken with respect to a convex capacity, the resulting preferences must satisfy subjective codecomposable independence and uncertainty aversion. It appears, however, that convexity is not necessary. In a separate note we show (see Lehrer and Teper [18]) that a weaker property than convexity, termed the Sandwich property, is necessary and sufficient for the representation of CavEU preferences when utility is bounded.

6.2. How Does It Fit in the Lit? Castagnoli and Maccheroni [4] notice that many of the weakening of independence (e.g., Schmeidler [24] and Gilboa and Schmeidler [13]) take a form of independence within a collection of cones with a particular structure. They study the implication of independence within an abstract class of cones and show that preferences can be represented with an abstract function which is linear within every cone.

In terms of models of choice under uncertainty, the most related ones are confidence preferences presented by Chateauneuf and Faro [6], maxmin expected utility (MEU) that were introduced by Gärdenfors and Sahlin [12] and axiomatized by Gilboa and Schmeidler [13] and, of course, CEU preferences.

CavEU is clearly a particular case of confidence preferences, but requires more structure since not every confidence preferences satisfy the decomposability property. To see that, consider MEU preferences, which are a particular case of confidence preferences. Not every MEU preference relation can be represented as a (concave) integral; MEU satisfies translation covariance (due to the c-independence axiom) while it is clear from subjective codecomposable independence that it does not have to be satisfied by CavEU. The subclass of CavEU preferences that do admit an MEU representation

\[14\] In particular, CavEU are those confidence preferences where the support of the confidence function has finitely many extreme points that constitute the set of measures dominating the capacity.
(or equivalently, satisfy \(c\)-independence) are those that can be represented with a capacity having a large core (see, Lehrer [16]). In other words, a representation in the style of Eq. (3) of preferences over acts ranged to the entire real line (i.e., utils could be negative and positive) can be carried out by a capacity having a large core. As for the axiomatization of such preferences, worst-outcome bet independence appearing in Theorem 2 would have to be strengthen to \(c\)-independence.

This brings us to CEU preferences. Lovasz [19] (pp. 246-249) and Schmeidler [24] show that the Choquet integral is a concave one if and only if the capacity is convex. Hence we have that when the capacity is not convex CavEU and CEU differ. In addition, due to Lehrer [16] and Teper and Lehrer [17], CEU and CavEU coincide if and only if the capacity representing the preferences is convex (and in this case it is also MEU). The latter point emphasizes, as seen in Section 2, that within the class of uncertainty averse preferences, the class of CavEU preferences is more general than that of CEU.

References


\(\text{The definition of large core is due to Sharkey [25].}\)
Appendix: Proofs

Proof of Theorem 1.

For an act \( f \in \mathcal{F} \), let \( c(f)_S \sim f \) be a constant act that is indifferent to \( f \). Due to monotonicity, continuity and weak order, \( c(f) \) exists and is unique. Define the real valued function \( V : \mathcal{F} \to \mathbb{R} \) by \( V(f) = c(f) \). Clearly, \( V \) represents the preferences. That is, for every \( f, g \in \mathcal{F} \), \( f \succeq g \) if and only if \( V(f) \geq V(g) \). Note that \( V(\alpha \mathbb{1}_S) = \alpha \) for every \( \alpha \geq 0 \). Now,

\begin{equation}
V(a_E) = V(ac(\mathbb{1}_E)_S) = ac(\mathbb{1}_E) = aV(\mathbb{1}_E)
\end{equation}

for every \( a \geq 0 \) and \( E \subseteq S \), where the left equality is due to worst-outcome bet independence (applied to the bets \( c(\mathbb{1}_E)_S \) and \( \mathbb{1}_E \)). This homogeneity property implies that \( V(0) = 0 \). Define \( \nu : 2^S \to [0,1] \) by \( \nu(E) = V(\mathbb{1}_E) \). \( \nu \) is a capacity. Indeed,
\( v(\emptyset) = V(0) = 0, v(S) = V(1_S) = 1 \) and \( v \) is monotonic since \( V \) is monotonic.\(^{16}\) Now, if \( f = a_E \) is a bet, then the claim in the theorem is immediate due to Eq. (4).

Pick a non-bet act \( f \in \mathcal{F} \). The main idea of the proof lies behind the following 4 claims.

**Claim 1.** There is a bet \( b_E \) and an act \( f' \) such that \( f \in \text{cone}(b_E, f') \) and \( V \) is affine over \( \text{cone}(b_E, f') \). In particular, \( V \) is homogeneous, that is, \( V(\alpha f) = \alpha V(f) \) for every \( \alpha \geq 0 \).

**Proof.** Indeed, pick a non-bet act \( f \in \mathcal{F} \). *Subjective codecomposable independence* implies that there exist an event \( E_f \subseteq S \) and an act \( f' \) such that \( f \in [b_{E_f}, f'] \) (i.e., \( f = (1- \delta)b_{E_f} + \delta f' \) for some \( \delta \in (0, 1) \)) and \( \geq \) respects independence over \( \text{cone}(b_{E_f}, f') \). Note that since \( f \) is not a bet, \( f' \) and \( b_{E_f} \) are algebraically independent. Since \( \geq \) respects independence over \( \text{cone}(b_{E_f}, f') \) it can be represented (over \( \text{cone}(b_{E_f}, f') \)) by a unique affine function that agrees with \( V \) on \( 0, b_{E_f} \) and \( f' \). In other words, for every \( g = a_1b_{E_f} + a_2f' \in \text{cone}(b_{E_f}, f') \), \( V(g) = a_1V(b_{E_f}) + a_2V(f') \) represents \( \geq \). In particular, \( V(\alpha f) = \alpha V(f) \) for every \( \alpha \geq 0 \). \( \square \)

The following claim is an immediate corollary of Claim 1.

**Claim 2.** There exists an event \( E_f \), an act \( f' \in \mathcal{F} \), \( a > 0 \) and \( \delta \in [0, 1] \) such that \( f = a_{E_f} + \delta f' \) and \( V(f) = V(a_{E_f}) + \delta V(f') \).

**Proof.** There exist an event \( E_f \subseteq S \) and an act \( f' \) such that \( f \in [b_{E_f}, f'] \) (i.e., \( f = (1- \delta)b_{E_f} + \delta f' \) for some \( \delta \in (0, 1) \)) and \( V(f) = (1- \delta)V(b_{E_f}) + \delta V(f') \). Now, letting \( a = (1- \delta)b \) we have that \( (1- \delta)V(b_{E_f}) = aV(1_{E_f}) = V(a_{E_f}) \). \( \square \)

**Claim 3.** There is a maximal \( a \geq 0 \) such that \( f = \delta g + a_{E_f} + \sum_{E \neq E_f} b_E \) and \( V(f) = \delta V(g) + V(a_{E_f}) + \sum_{E \neq E_f} V(b_E) \).

**Proof.** Enumerate all events but \( E_f \) and consider now

\[
\mathcal{E}_f = \left\{ (b_{E_f}, b_{E_1}, \ldots, b_{E_{2^{|S|-1}}}, g, \delta) : g \in \mathcal{F}, \delta \in [0, 1], f = \delta g + \sum_E b_E, V(f) = V(g) + \sum_E V(b_E) \right\}.
\]

This set is not empty due to Claim 2, closed\(^{17}\) since \( V \) is continuous, and bounded since \( f \) is. Hence \( \mathcal{E}_f \) is a non-empty compact set. Let

\[
a^* = \arg\max \left\{ a : (a_{E_f}, b_{E_1}, \ldots, b_{E_{2^{|S|-1}}}, g, \delta) \in \mathcal{E}_f \right\}.
\]

\(^{16}\)Note that due to *monotonicity* and *continuity*, \( f(s) \geq g(s) \) for every \( s \) implies \( f \succeq g \).

\(^{17}\)In the product topology over \( \mathbb{R}^{2^{|S|-1}} \times \mathcal{F} \times [0, 1] \).
Since $\mathcal{E}_f$ is compact, $a^*f$ is well defined. □

From Claim 3 there is a residual act $h$ and a collection of bets $\{b_E\}_{E \neq E_f}$ such that $f = \delta h + a^*_E + \sum_{E \neq E_f} b_E$ and
\begin{equation}
V(f) = \delta V(h) + V(a^*_E) + \sum_{E \neq E_f} V(b_E).
\end{equation}

Now, if the residual act $h$ is a bet, then from Eq. (4) the theorem is proved. Assume $h$ is not a bet. The following claim is a twist on Claim 2.

Claim 4. There exist an event $E_h \neq E_f$, an act $f' \in \mathcal{F}$, $a > 0$ and $\delta \in [0, 1]$ such that $h = \delta a_{E_h} + \delta f'$ and $V(h) = V(a_{E_h}) + \delta V(f')$.

Proof. The proof is similar to that of Claim 2, but we need to show that $E_h \neq E_f$. Assuming $E_h = E_f$ contradicts the maximality of $a^*f$. Indeed, if $E_h = E_f$ then $f = \delta f' + a_{E_f} + \sum_{E \neq E_f} b_E$ and $V(f) = \delta V(f') + V(a_{E_f}) + V(a^*_E) + \sum_{E \neq E_f} V(b_E)$, and $a + a^*f > a^*f$. □

Claim 4 suggests that it is not possible to apply subjective codecomposable independence to decompose $h$ while resorting to $E_f$. The same arguments apply when we decompose $h$ according to Claim 3; if $E_f$ appears in such a decomposition of $h$, and we substitute the decomposition of $V(h)$ with that of $V(f)$ appearing in Eq. (5), we again obtain a contradiction to the maximality of $a^*f$.

Following this argument, decompose $h$ as $V(h) = \gamma V(k) + V(a^*_{E_h}) + \sum_{E \neq E_h, E_f} V(d_E)$, and substitute its decomposition with that of $f$ appearing in Eq. (5) to obtain
\begin{equation}
V(f) = \delta \gamma V(k) + V(\delta a^*_{E_h}) + \sum_{E \neq E_h, E_f} V(\delta d_E) + V(a^*_E) + \sum_{E \neq E_f} V(b_E).
\end{equation}

Again, if $k$ is a bet then the theorem is proved. Otherwise, repeat the procedure we did for $h$. We decompose $k$ as in Claim 2, where following similar arguments as before, $E_h \neq E_h, E_f$, and so does every event in the decomposition of $k$ as in Claim 3, which we substitute with $V(k)$ in the decomposition of $f$. We repeat this procedure for residual acts of each step. If at any point the residual act is a bet, then the theorem is proved. Otherwise, it must be that this procedure ends after at most $2^{|S| - 1}$ steps. Indeed, since at each step the associated event (as in Claim 2 for $f$) according to which we decompose the residual act from the prior step, must be different from all events resorted to in previous steps, at the $2^{|S| - 1}$th step the residual act must be a bet. Otherwise we can decompose it using subjective codecomposable independence, which contradicts the fact that we already exhausted all possible event.
By Claim 1 $V$ is homogeneous. To prove uniqueness, suppose that there are two homogeneous $V, V' : F \to \mathbb{R}$ that represent $\succeq$ and satisfy $V(1_S) = V'(1_S) = 1$. Fix $f \in F$. Without loss of generality $V(f) \leq V'(f)$. Let $b \in \mathbb{R}_+$ such that $V(f) \leq V(b_S) = b = V'(b_S) \leq V'(f)$. Since both represent $\succeq$, $f \preceq b_S \preceq f$. Thus, $V(f) = b = V'(f)$. Since this is true for every $f$, $V = V'$ and by definition $v = v'$.

Proof of Theorem 2. The concave integral satisfies subjective codecomposable independence due to Proposition 5 in Even and Lehrer [10] (and it is immediate that the rest of the axioms are implied by integral).

Following Theorem 1, we have that $\succeq$ is represented by a homogeneous and continuous $V$ such that $V(1_E) \geq v(E)$. Ambiguity aversion implies that $V$ is a concave functional. By Lemma 1 in Lehrer [16] we have that $V(\cdot) \geq \int \text{Cav}(\cdot)dv$. However, for every $f \in F$ concavity of $V$ implies that $V(f) \leq \sum \alpha E V(1_E)$ for all decompositions of $f$, implying that $V(f) \leq \int \text{Cav} fdv$. Therefore $V(\cdot) = \int \text{Cav}(\cdot)dv$.

Proof of Proposition 1. Due to Lemma 1 in Lehrer and Teper [17], without loss of generality we can assume that $v$ is totally balanced. Thus, the proposition is proved due to Theorem 1 in Kalai and Zemel [14].

Proof of Proposition 2. It is clear that the axioms are satisfied by the CEU preferences. As for the inverse direction, we show first that cumulative codecomposable independence implies worst-outcome bet independence. Assume cumulative codecomposable independence and consider the two following bets: $b_E$ and $d_G$. Without loss of generality $b_E \succeq d_G$. Due to continuity there is $c \geq 0$ such that $b_E \sim c_{E \cup G}$. In particular, $b_E \succeq c_{E \cup G} \succeq d_G$. We show that for every $\alpha \geq 0$, $\alpha b_E \succeq \alpha c_{E \cup G} \succeq \alpha d_G$, implying $\alpha b_E \succeq \alpha d_G$ and worst-outcome bet independence.

Consider the act $f = 1_E + 1_{E \cup G}$. By cumulative codecomposable independence, $f \in [1_{E \cup G}, 1_E]$, and $\succeq$ satisfies independence over cone$(1_{E \cup G}, 1_E)$. In particular, $b_E \succeq c_{E \cup G}$ implies that for every $\alpha \geq 0$, $\alpha b_E \succeq \alpha c_{E \cup G}$. We can now apply the same reasoning to $f = 1_G + 1_{E \cup G}$ and conclude that since $c_{E \cup G} \succeq d_G$, then $\alpha c_{E \cup G} \succeq \alpha d_G$ for every $\alpha \geq 0$. Thus, worst-outcome bet independence is satisfied.

It remains to show that given cumulative codecomposable independence (which obviously implies subjective codecomposable independence), the decomposition of any act obtained in the proof of Theorem 1 is the Choquet one. To see that, pick an

\footnote{Note that the additivity property presented in Proposition 5 in Even and Lehrer [10] is not satisfied by every functional form as in Definition 1 and is not a characteristic property of such preferences. It is strictly stronger than subjective codecomposable independence.}
act $f \in \mathcal{F}$ and, let $a_1 = \max\{f(s) : s \in S\}$ and $E_1 = \{s \in S : f(s) = a_1\}$. Also denote $a_2 = \max\{f(s) : s \in E_1^c\}$. Let $f'$ be the act defined by $f'(s) = f(s)$ whenever $s \in E_1^c$ and $a_2$ otherwise (that is, $f'$ coincides with $f$ over the complement of $E_1$, and over $E_1$ it is defined as the second highest value $f$ attains). Now, $f = f' + (a_1 - a_2)\mathbb{1}_{E_1} = \frac{a_2}{a_1}(a_1 f') + \frac{a_1 - a_2}{a_1}(a_1 E_1)$. Note that $E_1$ is cumulative to $f$, hence by cumulative codecomposable independence we have that $V(f) = \frac{a_2}{a_1}V\left(\frac{a_1}{a_2}f'\right) + \frac{a_1 - a_2}{a_1}V(a_1 E_1) = V(f') + (a_1 - a_2)V(\mathbb{1}_{E_1})$.

Proof of Proposition 3. It is clear that the axioms are satisfied by the EU preferences. As for the other implication, codecomposable independence implies cumulative codecomposable independence, which by the previous proof, implies worst-outcome bet independence. Thus, all that is needed to show is that given codecomposable independence the capacity obtained in the proof of Theorem 1 is additive, hence a probability.

Pick any event $E \subset S$ and state $s \in S \setminus E$ and consider an act of the form $f = 2\mathbb{1}_{\{s\}} + \mathbb{1}_{E}$. On one hand, from the proof of Proposition 2 we know that $V(f) = v(E \cup \{s\}) + v(\{s\})$. On the other hand, we can write $f = \frac{1}{2}(4\mathbb{1}_{\{s\}}) + \frac{1}{2}(2\mathbb{1}_{E})$ and due to codecomposable independence we have that $V(f) = \frac{1}{2}(4v(\{s\})) + \frac{1}{2}(2v(E)) = 2v(\{s\}) + v(E)$. Thus, $v(\{s\}) + v(E \cup \{s\}) = 2v(\{s\}) + v(E)$, implying that $v(\{s\}) + v(E) = v(E \cup \{s\})$. Since $E$ is an arbitrary event, we get that $v(F) = \sum_{s \in F} v(s)$ for any event $F \subset S$, implying that $v$ is a probability over $S$. □