Missing Bids and Other Bidding Anomalies

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August 25, 2017

Preliminary - please do not quote, do not distribute.

Abstract

We document a novel bidding pattern observed in procurement auctions from Japan: winning bids tend to be isolated. We prove that in a general class of models, missing bids robustly indicate non-competitive behavior. In addition, we provide evidence that missing bids coincide tightly with known cartel activity. Finally, we show that missing bids are consistent with efficient collusion in environments where it is difficult for bidders to coordinate on precise bids.

KEYWORDS: missing bids, collusion, isolated winner strategies, cartel enforcement, procurement.

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†We are especially indebted to Steve Tadelis for encouragement and detailed feedback. The paper benefited from discussions with Pierpaolo Battigali, Eric Budish, Yeon-Koo Che, Francesco Decarolis, Emir Kamenica, Roger Myerson and Ariel Pakes, as well as comments from seminar participants at Bocconi, the 2017 Berkeley-Sorbonne workshop on Organizational Economics, the University of Chicago, and the 2017 NYU CRATE conference on theory and econometrics.

1 Introduction

This paper documents a novel bidding pattern in procurement auctions in Japan: there is essentially no mass of bids just above the winning bid. Put differently, winning bids tend to be isolated. This missing mass of bids makes it a profitable stage-game deviation for bidders to increase their bids. We argue that this unusual departure of stage-game best-response is a robust indicator of collusion, and investigate what this tells us about the strategic behavior of bidders.

We show that these bidding patterns are a robust indicator of non-competitive behavior. In a competitive equilibrium, firms cannot have a short-term incentive to increase their bids. We show that this incentive constraint implies that the elasticity of firms' *counterfactual demand* (i.e., the probability of winning an auction with any given bid) must bounded above by -1. This is true under general informational environments, including settings with private values and common values. Bidding behavior in our data contradicts this result: since the distribution of Δ has essentially no mass at zero, the elasticity of counterfactual demand is approximately zero in our data.

In addition we derive a bound on the fraction of competitive auctions in our data. When the average counterfactual demand in a given dataset has elasticity larger than $-\kappa$ for some $\kappa \in [0, 1]$, the fraction of competitive auctions is bounded above by κ. An immediate implication is that most auctions in our data are non-competitive.

Finally, we propose an interpretation of missing bids as part of robust non-competitive schemes. We observe that missing bids are not rationalized by standard models of tacit collusion [XXX: include references]. In all these models, the cartel's main concern is to incentivize losers not to undercut the winning bid. Isolated winning bids make the allocation robust to small perturbations.

[XXX Sylvain: Include discussion of Che et al. (2010)]

2 Motivating Facts

[XXX Sylvain: include description of datasets in which an anomaly is visually noticeable, include information about context – size of auctions, number of bidders, frequency of participation. Discuss possibility of renegotiation, inexperienced bidders, if large renegotiation possible we might have the wrong profit function]

Our motivating data comes from two prior papers. Kawai and Nakabayashi (2014) study procurement auctions run by Japan's national government. The auction format in their sample is a first-price auction with a secret reserve price, and with re-bidding in case there is no successful winner. Chassang and Ortner (2016) study procurement auctions run by cities in the Japanese prefecture of Ibaraki. Cities allocate their projects using a standard first-price auction with public reserve price.

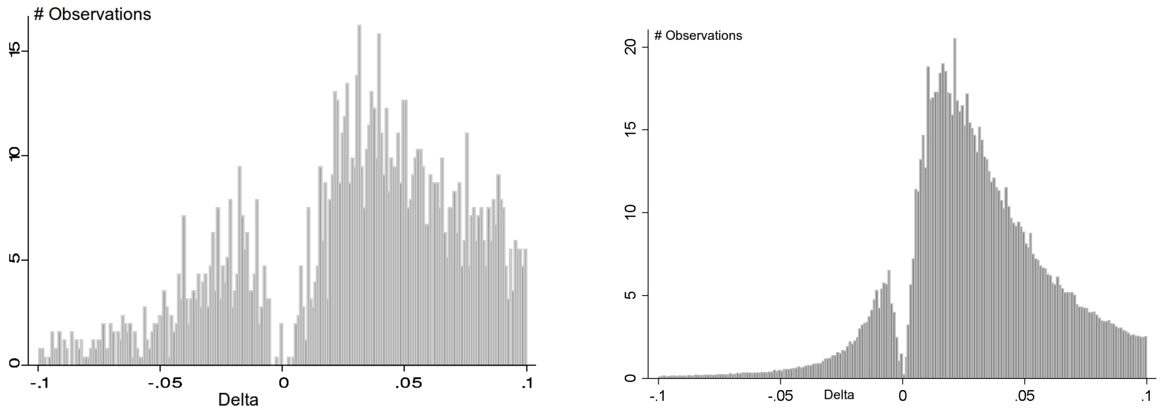
For any given firm, we investigate the distribution of

$$\Delta = \frac{\text{own bid - most competitive bid}}{\text{reserve price}}.$$

The left panel of Figure 1 plots the distribution of bid differences Δ for a large firm in the sample of auctions in Kawai and Nakabayashi (2014). The right panel aggregates bid differences over the sample firms in the data. The mass of missing bids around a difference of 0 is significant. It is widespread and remains stark when aggregating over all auctions in our sample.[1]

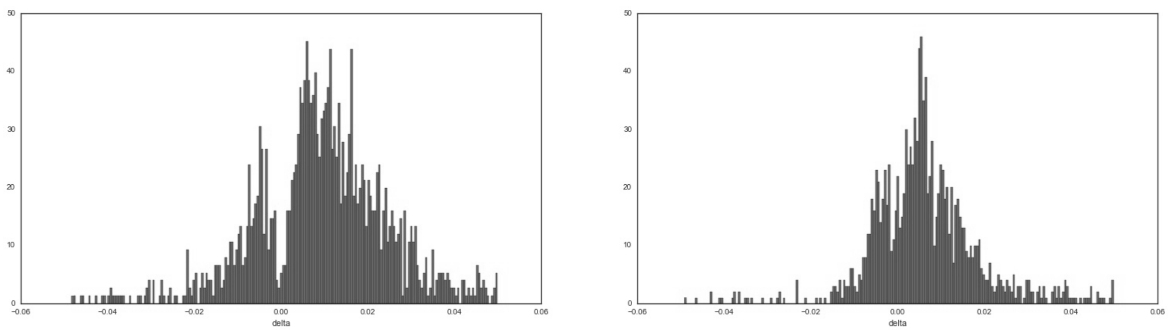
Figure 2 presents plots the distribution of Δ for the data in Chassang and Ortner (2016). The left panel uses all the bids in the sample. Again, we see a significant mass of missing bids around zero. The right panel shows that these missing bids disappear when we exclude winning bids from the analysis.

[1]Note that the distribution is of normalized bid-differences is skewed to the right since the most competitive alternative bid is a minimum over other bidders' bids.



(a) single large firm (b) all firms

Figure 1: Distribution of bid-difference Δ – national data.



(a) all firms (b) non-winners

Figure 2: Distribution of bid-difference Δ – city data.

3 Framework

We consider a dynamic setting in which, at each period $t \in \mathbb{N}$, a buyer needs to procure a single project. The auction format is a first-price auction with reserve price r. In what follows, we normalize $r = 1$.

In each period $t \in \mathbb{N}$, a set $\widehat{N}_t \subset N$ of bidders is able to participate in the auction, where N is the overall set of bidders. We think of this set of participating firms as those eligible to produce in the current period.[2] The set of eligible bidders is independently and identically distributed over time (i.i.d.).

[2]See Chassang and Ortner (2016) for a treatment of endogenous participation by cartel members.

Realized costs of production for eligible bidders $i \in \widehat{N}_t$ are denoted by $\mathbf{c}_t = (c_{i,t})_{i \in \widehat{N}_t}$. Each bidder $i \in \widehat{N}_t$ submits a bid $b_{i,t}$. Without significant loss in generality, we assume that bids are bounded within $[\underline{b}, r]$, with $\underline{b} > 0$. Profiles of bids are denoted by $\mathbf{b}_t = (b_{i,t})_{i \in \widehat{N}_t}$. We let $\mathbf{b}_{-i,t} \equiv (b_{j,t})_{j \neq i}$ denote bids from firms other than firm i, and define $\wedge \mathbf{b}_{-i,t} \equiv \max_{j \neq i} b_{j,t}$. The procurement contract is allocated to the bidder submitting the lowest bid at a price equal to her bid.

In the case of ties, we follow Athey and Bagwell (2001) and let the bidders jointly determine the allocation. This simplifies the analysis but requires some formalism (which can be skipped at moderate cost to understanding). We allow bidders to simultaneously pick numbers $\gamma_t = (\gamma_{i,t})_{i \in \widehat{N}_t}$ with $\gamma_{i,t} \in [0,1]$ for all i, t. When lowest bids are tied, the allocation to a lowest bidder i is

$$x_{i,t} = \frac{\gamma_{i,t}}{\sum_{\{j \in \widehat{N}_t \text{ s.t. } b_{j,t} = \min_k b_{k,t}\}} \gamma_{j,t}}.$$

Participants discount future payoffs using common discount factor $\delta < 1$. Bids are publicly revealed at the end of each period.

Costs. We allow for costs that are serially correlated. Denoting by $\langle ., . \rangle$ the usual dot-product we assume that costs take the form

$$c_{i,t} = \langle \alpha_i, \theta_t \rangle + \varepsilon_{i,t} > 0 \tag{1}$$

where

- parameters $\alpha_i \in \mathbb{R}^k$ are fixed over time;
- $\theta_t \in \mathbb{R}^k$ may be unknown to the bidders at the time of bidding, but is revealed to bidders at the end of period t; we assume that θ_t follows a Markov chain;
- $\varepsilon_{i,t}$ is i.i.d. with mean zero.

In period t, bidder i obtains profits

$$\pi_{i,t} = x_{i,t} \times (b_{i,t} - c_{i,t}).$$

Note that costs can include both direct costs of production and the opportunity cost of backlog.

Information. In each period t, bidder i gets a signal $z_{i,t}$ that is conditionally i.i.d. given $(\theta_t, (c_{j,t})_{j\in\widehat{N}_t})$. This allows our model to nest many informational environments, including asymmetric information private value auctions, common value auctions, as well as complete information. Bids $\mathbf{b}_t$ are observable at the end of the auction.

Transfers. Bidders are able to make positive transfers from one to the other at the end of each period. A transfer from i to j is denoted by $T_{i\to j,t} \geq 0$. Transfers are costly, and we denote by $K\left(\sum_{j\neq i} T_{i\to j,t}\right)$ the cost to player i of the transfers she makes. We assume that K is positive, increasing and convex. Altogether, flow realized payoffs to player i in period t take the form

$$u_{i,t} = \pi_{i,t} + \sum_{j\neq i} T_{j\to i,t} - K\left(\sum_{j\neq i} T_{i\to j,t}\right).$$

Solution Concepts. The public history h_t at period t takes the form

$$h_t = (\theta_{s-1}, \mathbf{b}_{s-1}, \mathbf{T}_{s-1})_{s\leq t},$$

where $\mathbf{T}_s$ are the transfers made in period s. Our solution concept is perfect public Bayesian equilibrium (Athey and Bagwell (2008)), with strategies

$$\sigma_i : h_t \mapsto (b_{i,t}(z_{i,t}), (T_{i\to j,t}(z_{i,t}, \mathbf{b}_t))_{j\neq i}),$$

where bids $b_{i,t}(z_{i,t}) \in \Delta([\underline{b}, r])$ and transfers $(T_{i\to j,t}(z_{i,t}, \mathbf{b}_t))_{j\neq i} \in \Delta(\mathbb{R}^{n-1})$ depend on the public history available at the time of decision making and on the current private signal $z_{i,t}$. We let H denote the set of all public histories.

We emphasize the class of competitive equilibria, or in this case Markov perfect equilibria (Maskin and Tirole, 2001). In a competitive equilibrium, players condition their play only on payoff relevant parameters.

Definition 1 (competitive strategy). *We say that a strategy profile $\sigma = (\sigma_i)_{i\in N}$ is competitive (or Markov perfect) if and only if $\forall i \in N$ and $\forall h_t \in H$, $\sigma_i(h_t, z_{i,t})$ depends only on $(\theta_{t-1}, z_{i,t})$.*

We say that a strategy profile $\sigma = (\sigma_i)_{i\in N}$ is a competitive equilibrium if it is a perfect public Bayesian equilibrium in competitive strategies.

We note that in a competitive equilibrium, firms must be playing a stage-game Nash equilibrium at every period; that is, firms must play a static best-reply to the actions of their opponents. Generally, an equilibrium may include periods in which firms collude and periods in which they compete. This leads us to define competitive histories.

Definition 2 (competitive histories). *Fix a strategy profile $\sigma = (\sigma_i)_{i\in N}$ and a public history h_t. We say that σ is competitive at history h_t if play at h_t is stage-game optimal for all firms.*

We say that auction a is competitive under equilibrium σ if the public history associated with a is non-collusive.

Fix a perfect public Bayesian equilibrium σ. For all public histories $h_t \in H$, all private signals $z_{i,t}$, and all bids $b' \geq \underline{b}$, player i's *counterfactual demand* at $h_{i,t} = (h_t, z_{i,t})$ is

$$D_i(b'|h_{i,t}) \equiv \mathsf{prob}_\sigma(\wedge \mathbf{b}_{-i,t} > b'|h_{i,t}).$$

For any finite set of histories $E = \{(h_t, z_{i,t})\} = \{h_{i,t}\}$, weights $w \in [0,1]^E$ and scalar $\rho \geq 0$, define

$$\overline{D}(\rho|E,w) \equiv \sum_{(h_{i,t})\in E} w(h_i) D_i((1+\rho)b_{i,t}|h_{i,t})$$

to be the average counterfactual demand for histories in E.

By default, if no weighting function is specified, $w(h) = \frac{1}{|E|}$.

Data and counterfactual demand. Fix a set of histories $E = \{(h_t, z_{i,t})\}$ and weights $w(h_t, z_{i,t}) = \frac{1}{|E|}$, and define

$$\widehat{D}(\rho|E) \equiv \sum_{(h_t, z_{i,t})\in E} \frac{1}{|E|} \mathbf{1}_{\wedge \mathbf{b}_{-i,t} > (1+\rho) b_{i,t}}.$$

Lemma 1. *For any perfect public Bayesian equilibrium σ, $\widehat{D}(\rho|E) - \overline{D}(\rho|E)$ converges in probability to zero as $|E| \to \infty$.*

Proof. Let E be a set of histories, and fix $\rho > 0$. For each history $h_{i,t} = (h_t, z_{i,t}) \in E$, define

$$\begin{aligned}\varepsilon_{i,t} &\equiv \mathbb{E}_\sigma[\mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)}|h_{i,t}] - \mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)} \\ &= \mathsf{prob}_\sigma(\wedge \mathbf{b}_{-i,t} > b_{i,t}(1+\rho)|h_{i,t}) - \mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)}.\end{aligned}$$

Note that $\widehat{D}(\rho|E) - \overline{D}(\rho|E) = \frac{1}{|E|}\sum_{h_{i,t}\in E} \varepsilon_{i,t}$.

Since costs are i.i.d conditional on θ_t and since signals are i.i.d. conditional on $(\theta_t, (c_{j,t})_{j\in \widehat{N}_t})$, it follows that for all histories $h_{j,t-s} \in E$ with $s \geq 0$, by the law of iterated expectations $\mathbb{E}_\sigma[\varepsilon_{i,t}|h_{j,t-s}] = \mathbb{E}_\sigma[\mathbb{E}_\sigma[\mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)}|h_t, z_{i,t}] - \mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)}|h_{t-s}, z_{j,t-s}] = 0$.[3]

Let $K = |E|$. Number the histories in E as $1, ..., K$ such that, for any pair of histories $k = (h_s, z_{i,s}) \in E$ and $k' = (h_{s'}, z_{j,s'}) \in E$ with $k' > k$, $s' \geq s$. For each history $k = (h_t, z_{i,t})$, let $\varepsilon_k = \varepsilon_{i,t}$, so that

$$\widehat{D}(\rho|E) - \overline{D}(\rho|E) = \frac{1}{K}\sum_{k=1}^{K} \varepsilon_k.$$

[3] This holds since, in a perfect public Bayesian equilibrium, bidders' strategies at any time t depend solely on the public history and on their private information at time t.

Note that, for all $\hat{k} \leq K$, $S_{\hat{k}} \equiv \sum_{k=1}^{\hat{k}} \varepsilon_k$ is a Martingale, with increments $\varepsilon_{\hat{k}}$ whose absolute value is bounded above by 1. By the Azuma-Hoeffding Inequality, $\mathsf{prob}(|S_K| \geq K\alpha) \leq 2\exp\{-\alpha^2 K/2\}$ for every $\alpha > 0$. Therefore, $\frac{1}{K} S_K = \widehat{D}(\rho|E) - \overline{D}(\rho|E)$ converges in probability to zero as $K \to \infty$. ■

4 Inference

4.1 Inference from upward deviations

The pattern of bids illustrated in Figures 1 and 2 is striking. Our first main result shows that it is inconsistent with competitive behavior.

Proposition 1. *Let σ be a competitive equilibrium. Then,*

$$\forall h_i, \quad \frac{\partial \log D_i(b'|h_i)}{\partial \log b'}\Big|_{b'=b_i^+(h_i)} \leq -1, \tag{2}$$

$$\forall E, \quad \frac{\partial \log \overline{D}(\rho|E)}{\partial \rho}\Big|_{\rho=0^+} \leq -1. \tag{3}$$

In other terms, under any non-collusive equilibrium, the elasticity of counterfactual demand must be less -1 at every history. We stress that bidding behavior in our data contradicts the results in Proposition 1. Note that for every $i \in N$ and every h_i,

$$\begin{aligned} D_i(b'|h_i) &= \mathsf{prob}_\sigma(b' - \wedge b_{-i} < 0|h_i) \\ &= \mathsf{prob}_\sigma(b' - b_i + \Delta_i < 0|h_i), \end{aligned}$$

where we used $\Delta_i = \frac{b_i - \wedge b_{-i}}{r} = b_i - \wedge b_{-i}$ (since we normalized $r = 1$). Since the density of Δ_i at 0 is essentially 0, the elasticity of demand is approximately zero as well.

Proof. Consider a competitive equilibrium σ. Let u_i denote the flow payoff of player i, and let $V(h_{i,t}) \equiv \mathbb{E}_\sigma \left(\sum_{s \geq t} \delta^{s-t} u_{i,s} \middle| h_{i,t} \right)$ denote her equilibrium payoff at history $h_{i,t} = (h_t, z_{i,t})$.

Let $b_{i,t} = b$ be the bid that bidder i places at history $h_{i,t}$. Since $b_{i,t} = b$ is an equilibrium bid, it must be that for all bids $b' > b$,

$$\mathbb{E}_\sigma \left[(b - c_{i,t}) \mathbf{1}_{\wedge \mathbf{b}_{-i,t} > b} + \delta V(h_{i,t+1}) \middle| h_{i,t}, b_{i,t} = b \right] \\ \geq \mathbb{E}_\sigma \left[(b' - c_{i,t}) \mathbf{1}_{\wedge \mathbf{b}_{-i,t} > b'} + \delta V(h_{i,t+1}) \middle| h_{i,t}, b_{i,t} = b' \right]$$

Since σ is a competitive equilibrium, $\mathbb{E}_\sigma[V(h_{i,t+1})|h_{i,t}, b_{i,t} = b] = \mathbb{E}_\sigma[V(h_{i,t+1})|h_{i,t}, b_{i,t} = b']$. Hence, we must have

$$bD_i(b|h_{i,t}) - b'D_i(b'|h_{i,t}) = \mathbb{E}_\sigma \left[b\mathbf{1}_{\wedge \mathbf{b}_{-i,t} > b} - b'\mathbf{1}_{\wedge \mathbf{b}_{-i,t} > b'} \middle| h_{i,t} \right] \\ \geq \mathbb{E}_\sigma \left[c_{i,t} (\mathbf{1}_{\wedge \mathbf{b}_{-i,t} > b} - \mathbf{1}_{\wedge \mathbf{b}_{-i,t} > b'}) \middle| h_{i,t} \right] \geq 0. \tag{4}$$

The inequality above implies that, for all $b' > b$,

$$\frac{\log D_i(b'|h_i) - \log D_i(b|h_i)}{\log b' - \log b} \leq -1.$$

Inequality (2) follows from taking the limit as $b' \downarrow b$. Inequality (3) follows from summing (4) over histories in E, and performing the same computations. ■

As the proof highlights, this result exploits the fact that in procurement auctions, zero is a natural lower bound for costs. In contrast, for auctions where bidders have a positive value for the good, there is no obvious upper bound to valuations to play that role.

An implication of Proposition 1 is that, in our data, bidders have a short-term incentive to increase their bids. To keep participants from bidding higher, for every $\epsilon > 0$ small, there exists $\nu > 0$ and a positive mass of histories $h_{i,t} = (h_t, z_{i,t})$ such that,

$$\delta \mathbb{E}_\sigma \left[V(h_{i,t+1}) \middle| h_{i,t}, b_i(h_{i,t}) \right] - \delta \mathbb{E}_\sigma \left[V(h_{i,t+1}) \middle| h_{i,t}, b_i(h_{i,t})(1+\epsilon) \right] > \nu. \tag{5}$$

In other terms, equilibrium σ must give bidders a dynamic incentive not to overcut the winning bid.

Gauging the share of competitive histories. Fix a perfect public Bayesian equilibrium σ and consider a finite set of histories E. Let $E^{\mathsf{comp}} \subset E$ be the set of competitive histories in E, and let $E^{\mathsf{coll}} = E \backslash E^{\mathsf{comp}}$ be the set of collusive histories in E. Define $\mathbf{s}_{\mathsf{comp}}(E) \equiv \frac{|E^{\mathsf{comp}}|}{|E|}$ to be the fraction of competitive histories in E.

For all public histories $h_t \in H$, all private signals $z_{i,t}$, and all bids $b' \geq \underline{b}$, player i's *counterfactual revenue* at $h_{i,t} = (h_t, z_{i,t})$ is

$$R_i(b'|h_{i,t}) \equiv b' D_i(b'|h_{i,t}).$$

For any finite set of histories E and scalar $\rho \geq 0$, define

$$\overline{R}(\rho|E) \equiv \sum_{h_{i,t} \in E} \frac{1}{|E|} (1+\rho) b_{i,t} D_i((1+\rho) b_{i,t} | h_{i,t})$$

to be the average counterfactual revenue for histories in E.

Since the elasticity of counterfactual demand is bounded above by -1 for all auctions in E^{comp} (Proposition 1), it follows that $\overline{R}(\rho|E^{\mathsf{comp}}) - \overline{R}(0|E^{\mathsf{comp}}) \leq 0$ for all $\rho > 0$.

Proposition 2. *Let E be a set of histories. Then,*

$$\frac{\overline{R}(\rho|E) - \overline{R}(0|E)}{\rho} \geq \kappa \Longrightarrow \mathbf{s}_{\mathit{comp}}(E) \leq 1 - \kappa.$$

Proof. Fix a set of histories E such that

$$\frac{\overline{R}(\rho|E) - \overline{R}(0|E)}{\rho} \geq \kappa$$

for some $\rho > 0$. Note that

$$\begin{aligned}
\kappa &\leq \frac{1}{\rho}[\overline{R}(\rho|E) - \overline{R}(0|E)] \\
&= \mathbf{s}_{\mathsf{comp}}(E)\frac{1}{\rho}\left[\overline{R}(\rho|E^{\mathsf{comp}}) - \overline{R}(0|E^{\mathsf{comp}})\right] \\
&\quad + (1 - \mathbf{s}_{\mathsf{comp}}(E))\frac{1}{\rho}\left[\overline{R}(\rho|E^{\mathsf{coll}}) - \overline{R}(0|E^{\mathsf{coll}})\right] \\
&\leq (1 - \mathbf{s}_{\mathsf{comp}}(E)).
\end{aligned}$$

The last inequality follows from two observations. First, by Proposition 1, $\overline{R}(\rho|E^{\mathsf{comp}}) - \overline{R}(0|E^{\mathsf{comp}}) \leq 0$. Second,

$$\frac{1}{\rho}[\overline{R}(\rho|E^{\mathsf{coll}}) - \overline{R}(0|E^{\mathsf{coll}})] \leq \frac{1}{\rho}((1+\rho)\overline{R}(0|E^{\mathsf{coll}}) - R(0|E^{\mathsf{coll}})) = \overline{R}(0|E^{\mathsf{coll}}) \leq r = 1.$$

■

In words, for any set of histories E for which total revenue is increases by more than κ when bids are uniformly increased by $(1+\rho)$, the share of competitive auctions in E is bounded above by $1 - \kappa$.

For each set of histories E and each $\rho > 0$, define

$$\widehat{R}(\rho|E) \equiv \sum_{h_{i,t} \in E} \frac{1}{|E|}(1+\rho) b_{i,t} \mathbf{1}_{\wedge \mathbf{b}_{-i,t} > (1+\rho) b_{i,t}}.$$

Lemma 2. *For any perfect public Bayesian equilibrium σ, $\widehat{R}(\rho|E) - \overline{R}(\rho|E)$ converges in probability to zero as $|E| \to \infty$.*

4.2 Inference from upward and downward deviations

Missing bids documented in Section 2 imply that bidders can increase their bids by a small amount without affecting their likelihood of winning the auction. Interestingly, although the residual demand is inelastic immediately around winning bids, it is very elastic for larger deviations. Moderate drops in price (a few percent) lead to large increases in the likelihood of winning the contract.

This suggests that jointly considering upward and downward deviations will provide a tighter bound on the share of competitive histories than the bound in Proposition 2. [XXX Sylvain: this lets us detect inconsistencies with competitive equilibrium other than missing bids – i.e. mass at 0 type anomaly]

Take as given a set of histories E, and $\overline{D}(\rho|E)$ the corresponding residual demand curve. Lemma 1 shows that these objects can be estimated using realized bidding data. However, subjective residual demand $D_i(\rho|h_{i,t})$ at a given history $h_{i,t}$ cannot be estimated.

Take as given $\rho^-, \rho^+ \in (0,1)$. For each history $h_{i,t} \in E$, let $D_{h_{i,t}}$, $P^-_{h_{i,t}}$, $P^+_{h_{i,t}}$ denote

$$\begin{aligned} D_{h_{i,t}} &\equiv D_i(b_{h_{i,t}}|h_{i,t}) \\ P^-_{h_{i,t}} &\equiv D_i\left((1-\rho^-)b_{h_{i,t}}|h_{i,t}\right) - D(b_{h_{i,t}}|h_{i,t}) \\ P^+_{h_{i,t}} &\equiv D_i(b_{h_{i,t}}|h_{i,t}) - D_i\left((1+\rho^+)b_{h_{i,t}}|h_{i,t}\right). \end{aligned}$$

Finally, define

$$D \equiv \frac{1}{|E|}\sum_{h\in E} D_h, \quad P^- \equiv \frac{1}{|E|}\sum_{h\in E} P^-_h, \quad P^+ \equiv \frac{1}{|E|}\sum_{h\in E} P^+_h.$$

[XXX Sylvain: for now I'm assuming private values. We can replace with affiliation later on.]

In any competitive equilibrium, subjective residual demand at all histories must satisfy four types of constraints: feasibility constraints, individual optimality constraints, aggregate consistency constraints, and ad hoc economic plausibility constraints. Formally, for every history h there must exist c_h, D_h, P_h^-, P_h^+ satisfying the following conditions

Feasibility.

$$c_h \in [0, b_h], \quad D_h \in [0, 1], \quad P_h^- \in [0, 1 - D_h], \quad P_h^+ \in [0, D_h] \tag{6}$$

Individual optimality.

$$\left[(1 - \rho^-)b_h - c_h\right](D_h + P_h^-) \leq (b_h - c_h)D_h \tag{7}$$

$$\left[(1 + \rho^+)b_h - c_h\right](D_h - P_h^+) \leq (b_h - c_h)D_h \tag{8}$$

Economic plausibility. In addition to incentive compatibility constraints, one may be able to impose plausible ad hoc constraints on the bidder's economic environment at history h. For now we focus on two intuitive constraints on the bidder's costs c_h and interim beliefs D_h, P_h^-, P_h^+:[4]

$$\frac{b_h}{c_h} \leq 1 + m \tag{9}$$

and

$$\begin{aligned} \left|\log \frac{D_h}{1 - D_h} - \log \frac{D}{1 - D}\right| &\leq k \\ \left|\log \frac{P_h^-}{1 - P_h^-} - \log \frac{P^-}{1 - P^-}\right| &\leq k \\ \left|\log \frac{P_h^+}{1 - P_h^+} - \log \frac{P^+}{1 - P^+}\right| &\leq k \end{aligned} \tag{10}$$

where $m \in [0, +\infty]$ is a maximum markup, and $k \in [0, +\infty)$ provides an upper bound

[4]We discuss general constraints in the appendix.

to the information contained in any signal.[5]

One focal case in which $k = 0$ is that of i.i.d. types.

Aggregate consistency. Finally, the bidders' subjective demand must be consistent with aggregate data. In addition to $\widehat{D} \equiv \frac{1}{|E|}\sum_{h_{i,t}\in E} \mathbf{1}_{b_{h_{i,t}} < \wedge b_{-i,h_{i,t}}}$, we define

$$\widehat{P}^{-} \equiv \frac{1}{|E|}\sum_{h_{i,t}\in E} \mathbf{1}_{\wedge b_{-i,h_{i,t}} \in [(1-\rho^{-})b_{i,h_{i,t}}, b_{i,h_{i,t}})} \quad \text{and} \quad \widehat{P}^{+} \equiv \frac{1}{|E|}\sum_{h_{i,t}\in E} \mathbf{1}_{\wedge b_{-i,h_{i,t}} \in (b_{i,h_{i,t}}, (1+\rho^{+})b_{i,h_{i,t}}]}$$

Given a tolerance level $T > 0$ demand at histories $h \in E$ is consistent with the data if and only if

$$\frac{1}{|E|}\sum_{h\in E} D_h \in \left[\widehat{D} - T, \widehat{D} + T\right] \tag{11}$$

$$\frac{1}{|E|}\sum_{h\in E} P_h^{-} \in \left[\widehat{P}^{-} - T, \widehat{P}^{-} + T\right] \tag{12}$$

$$\frac{1}{|E|}\sum_{h\in E} P_h^{+} \in \left[\widehat{P}^{+} - T, \widehat{P}^{+} + T\right] \tag{13}$$

From now on, we assume that, for each $h \in E$, there exists (D_h, P_h^{+}, P_h^{-}) and a cost $c_h \in [b_h/(1+m), b_h]$ satisfying (6), (7), (8) and (10).[6] We let $K \equiv |E|$ be the number of histories in E.

Proposition 3. *Consider an economic environment in which conditions* (9) *and* (10) *hold. If a strategy profile σ is competitive at all histories $h \in E$, then there exists constants $\beta > 0$ and $\lambda > 0$ such that, with probability at least $1 - \beta\exp(-\lambda K)$, conditions* (6), (7), (8), (11), (12) *and* (13) *hold simultaneously.*

We define the share of non-competitive histories as the share of histories that must be excluded for the data to become consistent with competitive play.

[5]Specifically, we have that $\log \frac{P_h^{-}}{1-P_h^{-}} - \log \frac{P^{-}}{1-P^{-}} = \log \frac{\mathsf{prob}(h|Z)}{\mathsf{prob}(h|\neg Z)}$ for Z the event that $\wedge b_{-i} \in [(1-\rho^{-})b_h, b_h]$. Hence, k is a bound on the log-likelihood ratio of signals.

[6]If this was not true, then history h would not be not competitive.

Definition 3 (share of competitive histories)**.** *We define the maximum share of competitive histories as*

$$\widehat{s}_{comp} \equiv \frac{1}{|E|} \max_{\mathbf{p}^C=(p_h^C)_{h\in E}\in[0,1]^{|E|}} \sum_{h\in E} p_h^C \tag{14}$$

such that there exists $(D_h, P_h^-, P_h^+, c_h)_{h\in E}$ *satisfying history-level constraints* (6), (7), (8), (9), (10) *and modified aggregate constraints*

$$\frac{1}{\sum_{h\in E} p_h^C} \sum_{h\in E} p_h^C D_h \in [\widehat{D}(\mathbf{p}^C) - T, \widehat{D}(\mathbf{p}^C) + T] \tag{11$'$}$$

$$\frac{1}{\sum_{h\in E} p_h^C} \sum_{h\in E} p_h^C P_h^- \in [\widehat{P}^-(\mathbf{p}^C) - T, \widehat{P}^-(\mathbf{p}^C) + T] \tag{12$'$}$$

$$\frac{1}{\sum_{h\in E} p_h^C} \sum_{h\in E} p_h^C P_h^+ \in [\widehat{P}^+(\mathbf{p}^C) - T, \widehat{P}^+(\mathbf{p}^C) + T] \tag{13$'$}$$

where

$$\widehat{D}(\mathbf{p}^C) \equiv \frac{1}{\sum_{h\in E} p_h^C} \sum_{h\in E} p_h^C \mathbf{1}_{b_h < \wedge b_{-i,h}}$$

$$\widehat{P}^-(\mathbf{p}^C) \equiv \frac{1}{\sum_{h\in E} p_h^C} \sum_{h\in E} p_h^C \mathbf{1}_{\wedge b_{-i,h} \in [(1-\rho^-) b_{i,h},\, b_{i,h})}$$

$$\widehat{P}^+(\mathbf{p}^C) \equiv \frac{1}{\sum_{h\in E} p_h^C} \sum_{h\in E} p_h^C \mathbf{1}_{\wedge b_{-i,h} \in (b_{i,h},\, (1+\rho^+) b_{i,h}]}$$

Lemma 3. *Consider a strategy profile* σ *and an economic environment in which conditions* (9) *and* (10) *hold. Let* E *be a set of realized histories such that a share* $s_{comp} \in (0, 1]$ *is competitive. Then, there exists* $\beta > 0$ *and* $\lambda > 0$ *such that, with probability at least* $1 - \beta \exp(-\lambda K)$*,* $\widehat{s}_{comp} \geq s_{comp}$*.*

A difficulty with Problem (14) is that the optimization variable $\mathbf{p}^C$ belongs to $[0, 1]^{|E|}$ and the set of constraints is non-convex. The following convex relaxation is more amenable to computation. First note that the vector of bidding outcomes

$$y_h \equiv (\mathbf{1}_{b_h < \wedge b_{-i,h}}, \mathbf{1}_{\wedge b_{-i,h} \in [(1-\rho^-)b_{i,h}, b_{i,h})}, \mathbf{1}_{\wedge b_{-i,h} \in (b_{i,h}, (1+\rho^+)b_{i,h}]})$$

can only take four values: $Y \equiv \{(1,0,0),(1,0,1),(0,1,0),(0,0,0)\}$. It turns out that $(y_h)_{h\in E}$ is a sufficient statistic of data for Problem (14). This allows us to consider solution functions $q^C : Y \to [0,1]$. Note that, for any $\mathbf{p}^C \in [0,1]^{|E|}$,

$$\begin{aligned}(\widehat{D}(\mathbf{p}^C), \widehat{P}^-(\mathbf{p}^C), \widehat{P}^+(\mathbf{p}^C)) &= \frac{1}{|E|}\sum_{y\in Y}\left(y \times \sum_{h|y_h=y} p_h^C\right) \\ &= \frac{1}{|E|}\sum_{y\in Y} y \times |\{h|y_h = y\}| q^C(y)\end{aligned}$$

for $q^C(y) = \frac{\sum_{h|y_h=y} p_h^C}{|\{h|y_h=y\}|}$.

Given D, P^-, P^+, let Z be the set of beliefs $z_h = (D_h, P_h^-, P_h^+)$ such that there exists a cost $c_h \in \left[\frac{1}{1+m}b_h, b_h\right]$ satisfying (6), (7) and (8).[7] Let

$$A(\mathbf{p}^C) = [\widehat{D}(\mathbf{p}^C) - T, \widehat{D}(\mathbf{p}^C) + T] \times [\widehat{P}^-(\mathbf{p}^C) - T, \widehat{P}^-(\mathbf{p}^C) + T] \times [\widehat{P}^+(\mathbf{p}^C) - T, \widehat{P}^+(\mathbf{p}^C) + T]$$

denote the set of aggregate constraints. Finally, for $X \in \{D, P^-, P^+\}$, define

$$\underline{B}(X,k) \equiv \frac{\frac{X}{1-X}\exp(-k)}{1+\frac{X}{1-X}\exp(-k)} \quad \text{and} \quad \overline{B}(X,k) \equiv \frac{\frac{X}{1-X}\exp(k)}{1+\frac{X}{1-X}\exp(k)},$$

and let

$$I = [\underline{B}(D,k), \overline{B}(D,k)] \times [\underline{B}(P^-,k), \overline{B}(P^-,k)] \times [\underline{B}(P^+,k), \overline{B}(P^+,k)]$$

denote the information constraints.

[7]In Appendix B we characterize the extreme points of set Z.

For any function $q : Y \to [0,1]$ define $\tilde{s}_{comp}(q) \equiv \frac{1}{|E|} \sum_{y \in Y} q(y) |\{h | y_h = y\}|$, and let $\mathbf{p}(q) \in [0,1]^{|E|}$ be such that, for all $h \in E$, $p(q)_h = q(y_h)$. Let $\mathsf{Conv}(Z)$ denote the convex hull of Z.

Proposition 4. *[equivalent representations]*

$$\widehat{s}_{\mathsf{comp}} \leq \max_{q:Y \to [0,1]} \{\tilde{s}_{comp}(q) \mid \mathsf{Conv}[Z] \cap A(\mathbf{p}(q)) \cap I \neq \emptyset\}$$

5 Empirical Findings

- Elasticity in baseline data, implied share of non-competitive auctions/bidders
- Elasticity for 'likely non-competitive' and 'likely competitive' subsets

  Plot $b_{(2)} - b_{(1)}$ in both cases
- Documenting 'missing bids' in other data sets [with formal metrics rather than visual gap]

Non-competition and collusion. Figure 3 shows the bidding behavior of a group of firms in our national data that were convicted for bid-rigging. The left panel plots the distribution of Δ prior to their conviction, while the right panel plots the same distribution after the conviction. Consistent with Proposition 1, the distribution of Δ has missing mass around zero only during the non-competitive period.

6 Interpreting Missing Bids

This section has two objectives. First, we want to highlight that the bidding behavior we observe in our data is not easily explained by standard models of collusion. Second, we put forward two explanations that rationalize these bidding patterns.

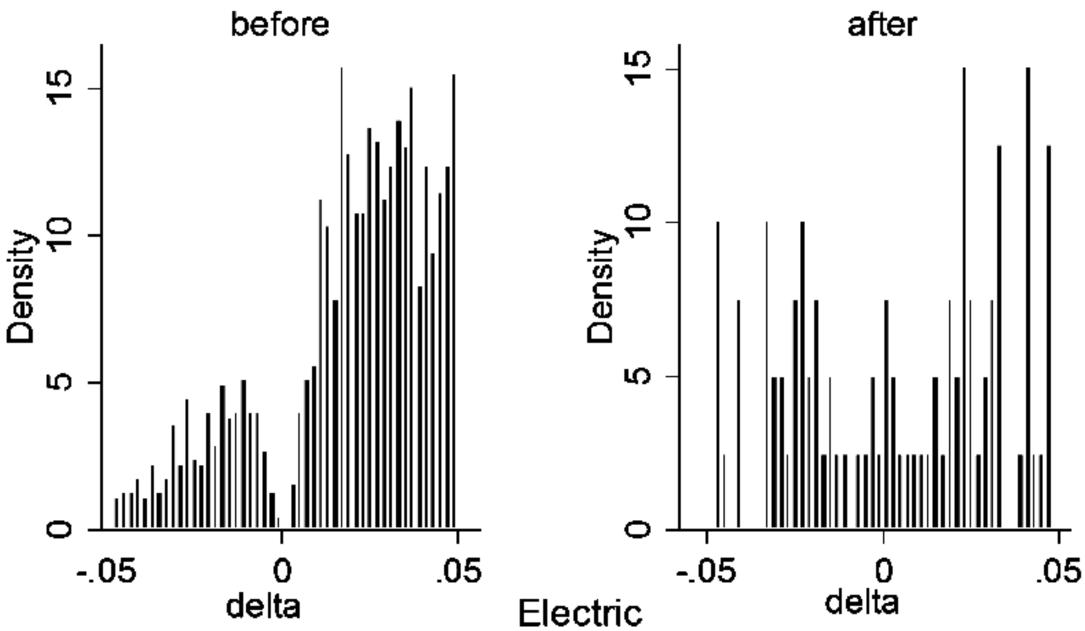


Figure 3: Distribution of normalized bid-difference Δ.

Workhorse model. We specialize the model in Section 3 as follows. We assume: (i) costs are i.i.d. across firms and across periods, (ii) cost realizations are publicly observed by all firms, and (iii) utility is perfectly transferable.

Definition 4. *Equilibrium σ exhibits common-knowledge bids (CKB) on the equilibrium path if and only if, for all on-path histories, the winning bid is common knowledge among bidders.*

We denote by Σ^{CKB} the set of Subgame Perfect Equilibrium in CKB. For any $\sigma \in \Sigma^{CKB}$ and any history h_t, let

$$V(\sigma, h_t) = \mathbb{E}_\sigma \left[\sum_{i \in N} \sum_{s \geq 0} \delta^s x_{i,t+s}(b_{i,t+s} - c_i) \Big| h_t \right]$$

denote the total surplus generated by σ at history h_t. Define

$$\overline{V} \equiv \max_{\sigma \in \Sigma^{CKB}} V(\sigma, h_0)$$

to be the highest equilibrium surplus sustainable in equilibrium.

For any cost realization $\mathbf{c} = (c_i)_{i \in N}$, we denote by $\mathbf{x}^*(\mathbf{c}) = (x_i^*(\mathbf{c}))_{i \in N}$ the efficient allocation (i.e., the allocation that assigns the contract to the lowest cost bidder and breaks ties randomly). We denote by $b_{(1)}$ and $b_{(2)}$ the lowest and second lowest bids. The following result, which is proved in Chassang and Ortner (2016), characterizes bidding behavior in any equilibrium that attains $\overline{V}$.

Proposition 5. *Let σ be an equilibrium that attains $\overline{V}$. Then:*

(i) equilibrium σ is stationary on path, and generates surplus $\overline{V}$ at every history.

(ii) for any cost realization $\mathbf{c} = (c_i)_{i \in N}$, the lowest cost bidder wins at bid $b^*(\mathbf{c})$ defined by

$$b^*(\mathbf{c}) \equiv \sup \left\{ b \le r : \sum_{i \in N} (1 - x_i^*(\mathbf{c}))[b - c_i]^+ \le \delta \overline{V} \right\}.$$

(iii) there is no money left on the table under equilibrium σ: $b_{(2)} - b_{(1)} \approx 0$ at all periods.

By Proposition 5, the bidding patterns in our data cannot be rationalized by collusion in CKB. In an optimal equilibrium in CKB, firms never use strategies under which the winning bidder has a short-run incentive to overcut the winning bid. Indeed, this would mean that firms have to spend continuation surplus to provide incentives to the winner not to bid higher. This creates efficiency losses relative to equilibria in which the winner is given incentives not to overcut by having the second lowest bid right on top of the winning bid.

As a result, bids will be clustered in a non-competitive equilibria in CKB, and the "money left on the table" (i.e., the difference between the winning bid and the second lowest bid) will be negligible. This is in sharp contrast with the bidding patterns we observe in our data, under which winning bids are isolated and the money left on the table is significant.

Missing bids as coordination challenges. The fact that winning bids are isolated implies that the allocation that this bidding behavior induces is robust to trembles or imprecisions in the communication among cartel members. Isolated winning bids may emerge as a response to such imperfections when the cartel wants to guarantee that the designated winner wins each auction. If this is the cartel's reason for using bidding profiles that give rise to isolated winning bids, and if the size of the trembles are small, we would expect bids to still be highly clustered and the difference between the second lowest bid $b_{(2)}$ and the lowest bid $b_{(1)}$ to be distributed in a narrow interval just above zero.

However, as Figure 4 shows, the distribution of the bid difference $b_{(2)} - b_{(1)}$ in our data shows substantial dispersion. In other words, bids tend not to be clustered in our data. Interestingly, the literature has identified highly clustered bids as a marker for collusion (see, for instance, (Harrington, 2008)). By this metric, Figure 4 suggests that auctions in our data are competitive, when in fact they are not.

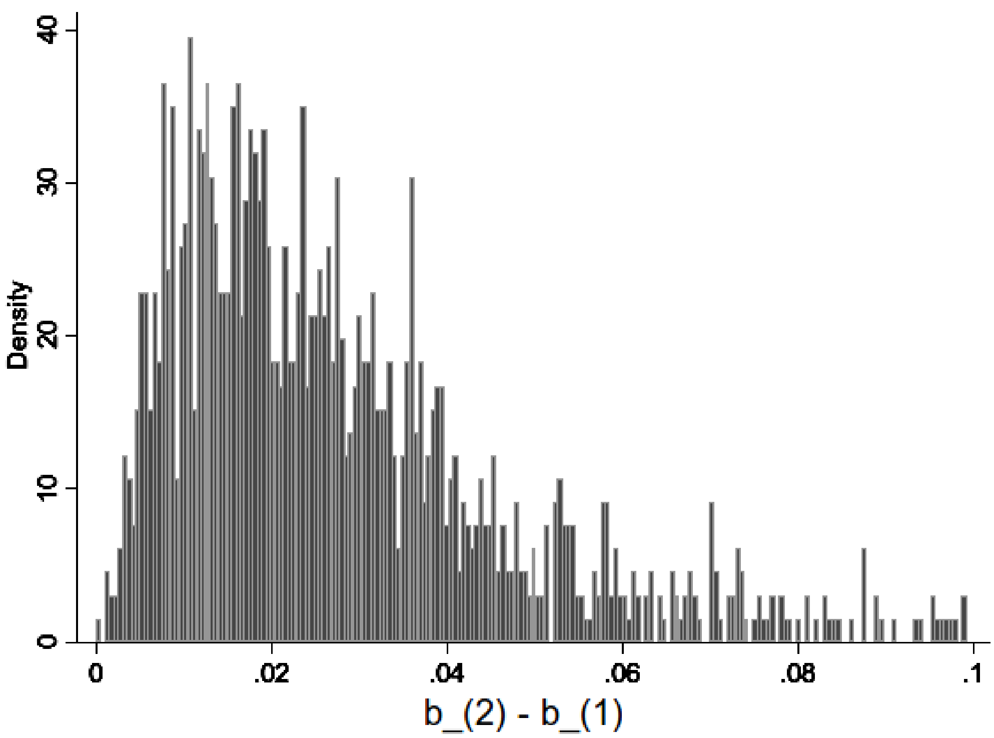


Figure 4: Distribution of normalized bid difference $b_{(2)} - b_{(1)}$ – national data.

Missing bids as sophisticated bidding. We now propose a novel explanation for missing bids that is consistent with dispersed bids. Our explanation is based on the idea that a cartel may benefit from maintaining designated losers uninformed about the bid of the designated bidder, to reduce their incentives to undercut the winning bid. This relates to Myerson (1986), and more recently Sugaya and Wolitzky (2016), who emphasize the idea that providing more information to agents that what is strictly necessary may make implementation harder.[8]

To illustrate our idea in the simplest framework, we consider a model with two bidders, $i = 1, 2$. Bidders' costs are drawn i.i.d. from c.d.f. F, with support $[\underline{c}, \overline{c}]$. There are no informational asymmetries: costs are publicly observed by both bidders. We further assume that firms cannot transfer money, and that bidders have access to a randomization device which is private before they place their bids, but becomes public after the auction. In practice, a mediator may play the role of this randomization device.[9] We focus on strongly symmetric equilibria of this game, i.e., subgame perfect equilibria in which, at every history, both bidders use symmetric bidding strategies.

Let $\overline{V}^{CKB}$ and $\underline{V}^{CKB}$ be, respectively, the highest and lowest payoffs that a bidder can obtain in a strongly symmetric CKB equilibrium. The assumption that costs are publicly observed implies that $\underline{V}^{CKB} = 0$.[10] In a strongly symmetric CKB equilibrium that attains $\overline{V}^{CKB}$, for each cost vector $\mathbf{c} = (c_1, c_2)$ the firm with the lowest cost wins the auction, and the winning bid is set equal to

$$b^{CKB}(\mathbf{c}) = \min\{r, c_{(2)} + \delta\overline{V}^{CKB}\}.$$

Indeed, winning bid $b^{CKB}(\mathbf{c})$ is the highest bid such that the firm with the highest cost does not have an incentive to undercut.

[8]Similar ideas also appear in Ortner and Chassang (2015).
[9]Mediators are frequently used by bidding rings. See, for instance, Asker (2010).
[10]Indeed, bidders can push each other's payoff to their min-max value of 0 by both bidding $b(\mathbf{c}) = c_{(1)}$, and letting the lowest cost firm win.

Let $\overline{V} \geq \overline{V}^{CKB}$ denote the highest payoff that a firm can obtain in a strongly symmetric equilibrium (which may or may not be in CKB). The following result holds.

Proposition 6. *Suppose* $\overline{V}^{CKB} < \frac{1}{1-\delta}\mathbb{E}[r - c_{(1)}]$*. Then,* $\overline{V} > \overline{V}^{CKB}$.

In words, whenever perfect collusion is not sustainable in a CKB equilibrium, firms can sustain strictly higher payoffs by using non-CKB strategies.

Proof. Suppose $\overline{V}^{CKB} < \frac{1}{1-\delta}\mathbb{E}[r - c_{(1)}]$, and note that this implies that, with probability bounded away from zero, $b^{CKB}(\mathbf{c}) < r$.

Consider the following strategy profile. For each cost vector $\mathbf{c}$ such that $b^{CKB}(\mathbf{c}) = r$, both firms place bid equal to r and the lowest cost firm wins (recall our assumption that firms can break ties). For each cost $\mathbf{c}$ such that $b^{CKB}(\mathbf{c}) < r$, the firm with the lowest cost submits bid $b^{CKB}(\mathbf{c})+\nu(\mathbf{c})$, where $\nu(\mathbf{c}) \in \Delta([0, \overline{\nu}(\mathbf{c})])$ with $\overline{\nu}(\mathbf{c}) = \min\{r - b^{CKB}(\mathbf{c}), \delta\overline{V}^{CKB}\}$. In particular, the private randomization device draws a number $\nu(\mathbf{c})$ from a distribution $G \in \Delta([0, \overline{\nu}(\mathbf{c})])$, which is observed only by the lowest cost firm. After firms place their bids, the outcome of this randomization becomes public. The firm with the highest cost bids $b^{CKB}(\mathbf{c}) + \overline{\nu}(\mathbf{c})$. If a firm ever deviates at some period t, from $t + 1$ onwards they play an equilibrium that delivers payoffs $\underline{V}^{CKB} = 0$ to both firms.

Note that this strategy profile generates a payoff for each bidder equal to $V = \overline{V}^{CKB} + \frac{1}{1-\delta}\mathbb{E}_{\mathbf{c}}[\mathbb{E}_{\nu}[\nu(\mathbf{c})]] > \overline{V}^{CKB}$. Note further that the winning firm does not have an incentive to deviate: if she is recommended to bid $b^{CKB}(\mathbf{c}) + \nu$ for some $\nu \in [0, \overline{\nu}(\mathbf{c})]$ and instead bids $b^{CKB}(\mathbf{c}) + \nu'$ with $\nu' \in [0, \overline{\nu}(\mathbf{c})]$,[11] she obtains at most

$$b^{CKB}(\mathbf{c}) + \overline{\nu}(\mathbf{c}) - c_{(1)} + \delta 0 \leq b^{CKB}(\mathbf{c}) - c_{(1)} + \delta V \leq b^{CKB}(\mathbf{c}) + \nu - c_{(1)} + \delta V,$$

where the first inequality uses $\overline{\nu}(\mathbf{c}) \leq \delta\overline{V}^{CKB} \leq \delta V$. The losing firm does not have an

[11]Clearly, deviating to a bid $b^{CKB}(\mathbf{c}) + \nu'$ with $\nu' \notin [0, \overline{\nu}(\mathbf{c})]$ is not optimal.

incentive to deviate either provided that

$$\forall b < b^{CKB}(\mathbf{c}) + \overline{\nu}(\mathbf{c}), \quad \mathsf{prob}(b < b^{CKB}(\mathbf{c}) + \nu(\mathbf{c}))(b - c_{(2)}) \leq \delta V. \tag{15}$$

Since $V \geq \overline{V}^{CKB}$, (15) will be satisfied if, for all $b < b^{CKB}(\mathbf{c}) + \overline{\nu}(\mathbf{c})$, $\mathsf{prob}(b < b^{CKB}(\mathbf{c}) + \nu(\mathbf{c}))(b - c_{(2)}) \leq \delta\overline{V}^{CKB}$, or equivalently, $\mathsf{prob}(b < b^{CKB}(\mathbf{c}) + \nu(\mathbf{c})) \leq \frac{\delta\overline{V}^{CKB}}{b - c_{(2)}}$. Using the change of variable $u = b - b^{CKB}(\mathbf{c})$ and recalling that $b^{CKB}(\mathbf{c}) = \delta\overline{V}^{CKB} + c_{(2)}$, we obtain that, for all $u \in [0, \overline{\nu}(\mathbf{c})]$,

$$\mathsf{prob}(u < \nu(\mathbf{c})) \leq \frac{\delta\overline{V}^{CKB}}{u + \delta\overline{V}^{CKB}}. \tag{16}$$

By first-order stochastic dominance, the distribution that generates the highest surplus sets (16) with equality for all $u \in [0, \overline{\nu}(\mathbf{c}))$, i.e., $\mathsf{prob}(\nu(\mathbf{c}) < u) = \frac{u}{u + \delta\overline{V}^{CKB}}$, and places mass point $1 - \frac{\overline{\nu}(\mathbf{c})}{\overline{\nu}(\mathbf{c}) + \delta\overline{V}^{CKB}}$ at $\overline{\nu}(\mathbf{c})$. ■

7 Conclusion

This paper documents a novel pattern in bids from Japanese procurement auctions: winning bids tend to be isolated. We show that this bidding behavior is a strong marker for collusion. Exploiting the structure of competitive equilibria, we provide a way to estimate the share of competitive auctions in our data.

We argue that, in the presence of trembles or imperfections in communication, isolated winning bids may emerge as a way to guarantee that the contracts will be allocated to the designated winner. Under this explanation, we would expect the amount of money left on the table to be narrowly concentrated around zero when trembles and imperfections are small. This is largely contradicted by our data: bid differences tend to be substantial.

This motivates an alternative explanation for missing bids. We show that a cartel may be able to sustain higher profits by letting the designated winner bid according to a distribution, and letting the non-winners know only the highest point in the support of this distribution.

Appendix

A Proofs

Proof of Proposition 3. Note first that conditions (6), (7), (8) must automatically hold at every competitive history $h \in E$.

Next, we show that,

$$\begin{aligned} \mathsf{prob}(|\hat{P}^- - P^-| \geq T) &\leq 2\exp(-T^2K/2) \qquad (17) \\ \mathsf{prob}(|\hat{P}^+ - P^+| \geq T) &\leq 2\exp(-T^2K/2) \\ \mathsf{prob}(|\hat{D} - D| \geq T) &\leq 2\exp(-T^2K/2), \end{aligned}$$

Note that these inequalities imply that

$$\mathsf{prob}(|\hat{P}^- - P^-| \geq T \text{ and } |\hat{P}^+ - P^+| \geq T \text{ and } |\hat{D} - D| \geq T) \leq 6\exp(-T^2K/2).$$

Setting $\beta = 6$ and $\lambda = T^2/2$, it follows that conditions (6), (7), (8), (11), (12) and (13) hold simultaneously with probability at least $1 - \beta\exp(-\lambda K)$.

To see why the inequalities in (17) hold, note that

$$\begin{aligned} P^- - \hat{P}^- &= \frac{1}{|E|}\sum_{h_{i,t}\in E} \mathbb{E}_\sigma[\mathbf{1}_{\wedge b_{-i,h_{i,t}}\in[(1-\rho^-)b_{i,h_{i,t}},b_{i,h_{i,t}})}|h_{i,t}] - \mathbf{1}_{\wedge b_{-i,h_{i,t}}\in[(1-\rho^-)b_{i,h_{i,t}},b_{i,h_{i,t}})} \\ &= \frac{1}{K}\sum_{h_{i,t}\in E} \varepsilon_{i,t}, \end{aligned}$$

where $\varepsilon_{i,t} \equiv \mathbb{E}_\sigma[\mathbf{1}_{\wedge b_{-i,h_{i,t}}\in[(1-\rho^-)b_{i,h_{i,t}},b_{i,h_{i,t}})}|h_{i,t}] - \mathbf{1}_{\wedge b_{-i,h_{i,t}}\in[(1-\rho^-)b_{i,h_{i,t}},b_{i,h_{i,t}})}$ and where we used

$|E| = K$. By the law of iterated expectations, for any history $h_{j,s}$ with $s \leq t$,

$$\mathbb{E}_\sigma[\varepsilon_{i,t}|h_{j,s}] = \mathbb{E}_\sigma[\mathbb{E}_\sigma[\mathbf{1}_{\wedge b_{-i,h_{i,t}} \in [(1-\rho^-)b_{i,h_{i,t}}, b_{i,h_{i,t}})}|h_{i,t}] - \mathbf{1}_{\wedge b_{-i,h_{i,t}} \in [(1-\rho^-)b_{i,h_{i,t}}, b_{i,h_{i,t}})}|h_{j,s}] = 0.$$

Number the histories in E as $1, ..., K$ such that, for any pair of histories $k = h_{i,s} \in E$ and $k' = h_{j,s'} \in E$ with $k' > k$, $s' \geq s$. For each history $k = h_{i,t}$, let $\varepsilon_k = \varepsilon_{i,t}$, so that

$$P^- - \widehat{P}^- = \frac{1}{K} \sum_{k=1}^{K} \varepsilon_k.$$

Note that, for all $\hat{k} \leq K$, $S_{\hat{k}} \equiv \sum_{k=1}^{\hat{k}} \varepsilon_k$ is a Martingale, with increments $\varepsilon_{\hat{k}}$ whose absolute value is bounded above by 1. By the Azuma-Hoeffding Inequality, for every $T > 0$,

$$\mathsf{prob}(|\widehat{P}^- - P^-| \geq T) = \mathsf{prob}(|S_K| \geq KT) \leq 2\exp\{-T^2K/2\}.$$

The other two inequalities in (17) follow from the same arguments. ■

Proof of Lemma 3. Let $E^{\mathsf{comp}} \subset E$ be the set of competitive histories in E, so that $s_{\mathsf{comp}} = \frac{|E^{\mathsf{comp}}|}{|E|}$. Consider the vector $\mathbf{p}^{\mathsf{comp}} = (p_h^{\mathsf{comp}})_{h \in E}$ with $p_h^{\mathsf{comp}} = 1$ for all $h \in E^{\mathsf{comp}}$ and $p_h^{\mathsf{comp}} = 0$ otherwise. We show that there exists $\beta > 0$ and $\lambda > 0$ such that, with probability at least $1 - \beta\exp(-\lambda K)$, vector $\mathbf{p}^{\mathsf{comp}}$ satisfies constraints (6), (7), (8), (11′), (12′) and (13′). This implies that, with probability at least $1 - \beta\exp(-\lambda K)$, $\hat{s}_{\mathsf{comp}} \geq \frac{1}{|E|}\sum_h p_h^{\mathsf{comp}} = s_{\mathsf{comp}}$, establishing our result.

Note first that, for all histories $h \in E^{\mathsf{comp}}$, there exists (D_h, P_h^-, P_h^+, c_h) such that conditions (6), (7), (8), (9) and (10) hold. For all $h \in E^{\mathsf{coll}}$, pick D_h, P_h^+, P_h^- and c_h satisfying (9), (10), (6), (7) and (8).[12]

[12]By assumption, such D_h, P_h^+, P_h^- and c_h exist.

For every $\mathbf{p}^C \in [0,1]^{|E|}$, define

$$D(\mathbf{p}^C) \equiv \frac{1}{\sum_{h\in E} p_h} \sum_{h\in E} p_h^C D_h,$$
$$P^-(\mathbf{p}^C) \equiv \frac{1}{\sum_{h\in E} p_h} \sum_{h\in E} p_h^C P_h^-$$
$$P^+(\mathbf{p}^C) \equiv \frac{1}{\sum_{h\in E} p_h} \sum_{h\in E} p_h^C P_h^+.$$

Note that

$$D(\mathbf{p}^{\mathsf{comp}}) = \frac{1}{|E^{\mathsf{comp}}|} \sum_{h\in E^{\mathsf{comp}}} D_h,$$
$$P^-(\mathbf{p}^{\mathsf{comp}}) = \frac{1}{|E^{\mathsf{comp}}|} \sum_{h\in E^{\mathsf{comp}}} P_h^-,$$
$$P^+(\mathbf{p}^{\mathsf{comp}}) = \frac{1}{|E^{\mathsf{comp}}|} \sum_{h\in E^{\mathsf{comp}}} P_h^+.$$

Similarly,

$$\widehat{D}(\mathbf{p}^{\mathsf{comp}}) = \frac{1}{|E^{\mathsf{comp}}|} \sum_{h\in E^{\mathsf{comp}}} \mathbf{1}_{b_h < \wedge b_{-i,h}}$$
$$\widehat{P}^-(\mathbf{p}^{\mathsf{comp}}) = \frac{1}{|E^{\mathsf{comp}}|} \sum_{h\in E} \mathbf{1}_{\wedge b_{-i,h} \in [(1-\rho^-)b_{i,h},\, b_{i,h})}$$
$$\widehat{P}^+(\mathbf{p}^{\mathsf{comp}}) = \frac{1}{|E^{\mathsf{comp}}|} \sum_{h\in E^{\mathsf{comp}}} \mathbf{1}_{\wedge b_{-i,h} \in (b_{i,h},\, (1+\rho^+)b_{i,h}]}$$

By the same arguments as in the proof of Proposition 3, and using $|E^{\mathsf{comp}}| = s_{\mathsf{comp}}K$,

$$\mathsf{prob}(|\hat{P}^-(\mathbf{p}^{\mathsf{comp}}) - P^-(\mathbf{p}^{\mathsf{comp}})| \geq T) \leq 2\exp(-T^2 K s_{\mathsf{comp}}/2) \tag{18}$$
$$\mathsf{prob}(|\hat{P}^+(\mathbf{p}^{\mathsf{comp}}) - P^+(\mathbf{p}^{\mathsf{comp}})| \geq T) \leq 2\exp(-T^2 K s_{\mathsf{comp}}/2)$$
$$\mathsf{prob}(|\hat{D}(\mathbf{p}^{\mathsf{comp}}) - D(\mathbf{p}^{\mathsf{comp}})| \geq T) \leq 2\exp(-T^2 K s_{\mathsf{comp}}/2)$$

These inequalities imply that, for $\mathbf{p}^C = \mathbf{p}^{\mathsf{comp}}$, conditions (11), (12) and (13) hold simultaneously with probability at least $1 - \beta \exp(-\lambda K)$, with $\beta = 6$ and $\lambda = T^2 s_{\mathsf{comp}}/2$. ■

Proof of Proposition 4. Let $\mathbf{p}^C \in [0,1]^C$ be a solution to Problem (14), and let $(D_h, P_h^-, P_h^+)_{h \in E}$ be the corresponding beliefs satisfying all the constraints of the problem. For every $y \in Y$, define $E(y) \equiv \{h \in E : y_h = y\}$.

Consider any permutation $\alpha : E \to E$ such that, for all $y \in Y$ and all $h \in E(Y)$, $\alpha(h) \in E(Y)$. Let $\tilde{\mathbf{p}}^C = (\tilde{p}_h^C)_{h \in E}$ be such that, for all $h \in H$, $\tilde{p}_h^C = p_{\alpha(h)}^C$. Note that $\tilde{\mathbf{p}}^C = (\tilde{p}_h^C)_{h \in E}$ is also a solution (14), together with beliefs $(\tilde{D}_h, \tilde{P}_h^-, \tilde{P}_h^+)_{h \in E}$ such that, for all $h \in H$, $(\tilde{D}_h, \tilde{P}_h^-, \tilde{P}_h^+) = (D_{\alpha(h)}, P_{\alpha(h)}^-, P_{\alpha(h)}^+)$.

Since this is true for any such permutation α, it follows that there exists $\overline{\mathbf{p}}^C \in [0,1]^{|E|}$ and corresponding beliefs $(\overline{D}_h, \overline{P}_h^-, \overline{P}_h^+)_{h \in E}$, such that

(i) for all $y \in Y$ and all $h, h' \in E(Y)$, $\overline{p}_h^C = \overline{p}_{h'}^C = p_y$,

(ii) $\frac{1}{|E|} \sum_h \overline{p}_h^C = \frac{1}{|E|} \sum_h p_h^C = \widehat{s}_{\mathsf{comp}}$,

(iii) $\frac{1}{\sum_{h \in E} \overline{p}_h^C} \sum_{h \in E} \overline{p}_h^C \times (\overline{D}_h, \overline{P}_h^-, \overline{P}_h^+) \in A(\overline{\mathbf{p}}^C)$,

(iv) for all $h \in E$, $(\overline{D}_h, \overline{P}_h^-, \overline{P}_h^+) \in \mathsf{Conv}\,[Z] \cap I$.

Let $q : Y \to [0,1]$ be such that $q(y) = p_y$, so that $\overline{\mathbf{p}}^C = \mathbf{p}(q)$. Since $\sum_{h \in E} \frac{\overline{p}_h^C}{\sum_{h \in E} \overline{p}_h^C} \times (\overline{D}_h, \overline{P}_h^-, \overline{P}_h^+) \in \mathsf{Conv}\,[Z] \cap A(\mathbf{p}(q)) \cap I$, it follows that

$$\widehat{s}_{\mathsf{comp}} \leq \max_{q : Y \to [0,1]} \left\{ \tilde{s}_{comp}(q) \,|\, \mathsf{Conv}\,[Z(\mathbf{p}(q))] \cap A(\mathbf{p}(q)) \neq \emptyset \right\}.$$

■

B Computational Appendix

Requirements from competitive equilibrium. For every history h in data set E, there must exist $z_h = (D_h, P_h^-, P_h^+)$ and $c_h \in [0, b_h]$ such that $D_h \in [0,1]$, $P_h^- \in [0, 1 - D_h]$ and $P_h^+ \in [0, D_h]$ such that

IC

$[(1-\rho^-)b - c_h](D_h + P_h^-) \leq (b_h - c_h)D_h$

$[(1+\rho^+)b - c_h](D_h - P_h^+) \leq (b_h - c_h)D_h$

Costs

$\frac{c_h}{b_h} \geq \frac{1}{1+m}$

Let $Z(\hat{D}, \hat{P}^+, \hat{P}^-)$ denote the set of such triplets z. Note that I'm using realized event frequencies rather than true ones – a priori the true frequencies would be more natural, but it's less tractable.

Information constraints For $X \in \{\hat{D}, \hat{P}^+, \hat{P}^-\}$, $\left|\log \frac{X_h}{1-X_h} - \log \frac{X}{1-X}\right| \leq k$

Consistency with data In addition, with high probability, aggregate demand at competitive auctions must be consistent with observed data.

Let

$$\hat{z}_h = (1_{\wedge b_{-i,h} > b_h}, 1_{\wedge b_{-i,h} \in ((1-\rho^+)b_h, b_h)}, 1_{\wedge b_{-i,h} \in (b_h, (1+\rho^+)b_h)})$$

and $Y = \{(0,0,0), (1,0,0), (0,1,0), (1,0,1)\}$

For any mapping $p^C : Y \to [0,1]$ we define

$$(\hat{D}(p^C), \hat{P}^-(p^C), \hat{P}^+(p^C)) = \frac{1}{|E|} \sum_{y \in Y} p^C(y) \times |E_y| \times y$$

where $E_y = \{h \in E \; s.t. \; y_h = y\}$

Let $A_T(p^C) = [\hat{D}(p^C) \pm T] \times [\hat{P}^-(p^C) \pm T] \times [\hat{P}^+(p^C) \pm T] \cap [0,1]^3$

Let $\tilde{s}_{comp}(p^C) = \frac{1}{|E|}\sum_{y \in Y} p^C(y) \times |E_y|$

The maximum share of competitive auctions consistent with data is

$$\max_{p^C}\{\tilde{s}_{comp}(p^C) \mid [\tilde{s}_{comp} \times \textsf{convex } Z(p^C)] \cap A_T(p^C) \neq \emptyset\}$$

Simplifying the computation of convex $Z(p^C)$ First observe that the IC constraint and the cost constraint can be rewritten as

$$\left[1 - \rho^- - \rho^- \frac{D_h}{P_h^-}\right] \vee \frac{1}{1+m} \leq 1 + \rho^+ - \rho^+ \frac{D_h}{P_h^+} \qquad (IC)$$

The information constraints can be rewritten as

$$X_h \in [\underline{B}(X,k), \overline{B}(X,k)]$$

for $X \in \{D, P^-, P^+\}$ and

$$\underline{B}(X,k) = \frac{\frac{X}{1-X}\exp(-k)}{1 + \frac{X}{1-X}\exp(-k)} \quad \text{and} \quad \overline{B}(X,k) = \frac{\frac{X}{1-X}\exp(k)}{1 + \frac{X}{1-X}\exp(k)}$$

Let $B(p^C) \equiv \times_{X \in \{D, P^-, P^+\}}[\underline{B}(X,k), \overline{B}(X,k)]$.

Note that there exists P_h^- such that (IC) holds iff

$$1 + \rho^+ - \rho^+ \frac{D_h}{P_h^+} > \frac{1}{1+m} \iff \frac{D_h}{P_h^+} < \left(1 + \rho^+ - \frac{1}{1+m}\right) \Big/ \rho^+$$

Hence, the set of acceptable P_h^+ given D_h is

$$P_h^+ \in \left[\frac{\rho^+}{1 + \rho^+ - 1/(1+m)} D_h, D_h\right]$$

Given D_h and P_h^+ such that a solution P_h^- to (IC) exists, the range of values P_h^- solving (IC) is the set of values $P_h^- \in [0, 1 - D_h]$ such that

$$1 - \rho^- - \rho^- \frac{D_h}{P_h^-} \leq 1 + \rho^+ - \rho^+ \frac{D_h}{P_h^+} \tag{19}$$

$$\iff -\rho^- - \rho^+ + \rho^+ \frac{D_h}{P_h^+} \leq \rho^- \frac{D_h}{P_h^-} \tag{20}$$

$$\iff P_h^- \leq \overline{P}_h^-. \tag{21}$$

where

$$\overline{P}_h^- = \min \left\{ 1 - D_h, \frac{\rho^- D_h}{\left[-\rho^- - \rho^+ + \rho^+ \frac{D_h}{P_h^+} \right]^+} \right\}.$$

The Problem We define

$Z(p^C)$ the set of points satisfying (IC), and cost constraint.

We are interested in solving

$\max_{p^C} \sum_{y \in Y} p^C$ such that

$$\mathsf{convex}(Z(p^C)) \cap B(p^C) \cap A(p^C) \neq \emptyset$$

Claim 1 The set $Z(p^C)$ admits as extreme points - $D_h = 0$, $P_h^+ = 0$, $P_h^- = 0$ - $D_h = 0$, $P_h^+ = 0$, $P_h^- = 1$ - $D_h = 1$, $P_h^+ = 1$, $P_h^- = 0$ - $D_h = 1$, $P_h^+ = \frac{\rho^+}{1+\rho^+-1/(1+m)}$, $P_h^- = 0$

It's convex hull is therefore the corresponding pyramid

Claim 2 Sets A and $\mathsf{convex}(Z)$ are both convex. If they do not have an intersection, they must be separated by a hyperplane.

If A and $\mathsf{convex}(Z)$ are separated, then they are separated by one of the faces of the pyramid that is not the plan $(D_h, P_h^+, P_h^- = 0)$.

B.1 deprectated

Lemma B.1. *The extreme points of set* $\mathsf{Conv}[Z]$ *are*

$$(D_h, P_h^-, P_h^+) = (0, 0, 0)$$
$$(D_h, P_h^-, P_h^+) = (0, 1, 0)$$
$$(D_h, P_h^-, P_h^+) = (1, 0, 1)$$
$$(D_h, P_h^-, P_h^+) = (1, 0, \frac{\rho^+}{1 + \rho^+ - \frac{1}{1+m}})$$

Proof. The individual optimality constraints (7) and (8) can be re-written as

$$\max\left\{1 - \rho^- - \rho^- \frac{D_h}{P_h^-}, \frac{1}{1+m}\right\} \leq 1 + \rho^+ - \rho^+ \frac{D_h}{P_h^+}. \tag{22}$$

Given D_h and P_h^+, there exists a feasible P_h^- such that (22) holds if and only if

$$1 + \rho^+ - \rho^+ \frac{D_h}{P_h^+} > \frac{1}{1+m}.$$

Hence, for a given $D_h \in [0, 1]$, the set of acceptable P_h^+ is $[\frac{\rho^+}{1+\rho^+-\frac{1}{1+m}} D_h, D_h]$.

Given D_h and $P_h^+ \in [\frac{\rho^+}{1+\rho^+-\frac{1}{1+m}} D_h, D_h]$, the range of P_h^- satisfying (22) is the set of $P_h^- \in [0, 1 - D_h]$ such that

$$P_h^- \leq \overline{P}_h^-(D_h, P_h^+) \equiv \min\left\{1 - D_h, \frac{\rho^- D_h}{-\rho^- - \rho^+ + \rho^+ \frac{D_h}{P_h^+}}\right\}.$$

The arguments above imply that set Z is given by the intersection of the following sets: (i) $[0, 1]^3$, (ii) $\{(D_h, P_h^-, P_h^+) : P_h^+ \in [\frac{\rho^+}{1+\rho^+-\frac{1}{1+m}} D_h, D_h]\}$, and (iii) $\{(D_h, P_h^-, P_h^+) : P_h^- \leq P_h^-(D_h, P_h^+)\}$. The extreme points of this set are $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 1)$ and $(1, 0, \frac{\rho^+}{1+\rho^+-\frac{1}{1+m}})$. ■

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