Unemployment, Vacancies, and Social Networks

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January 14, 2014

(Job Market Paper)

Abstract

I incorporate social networks into a search and matching model. The model predicts that the presence of network externalities (i) increases the volatility of unemployment and (ii) can lead to multiple equilibria. I demonstrate that, when social ties are fixed, aggregate matching functions exhibit decreasing returns to scale, and unemployment, vacancies, tightness, and matching rates have a larger response to productivity shocks. Numerical examples suggest that unemployment is up to twice as volatile and productivity shocks exhibit more propagation than when network effects are absent. When social ties can sever and attach over time I find that, depending on the network formation process, multiple equilibria can arise creating large and persistent shifts in the Beveridge curve. The model also predicts higher average wages for those hired through network search than through random search and increased persistence of productivity shocks.

Keywords: Social Networks, Unemployment, Search and Matching.

JEL Classification: D85, E24, J64.

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1 Introduction

The use of social networks is pervasive in the matching of workers and firms. Petrongolo and Pissarides (2001) declare, “The matching function summarizes a trading technology between agents who place advertisements, read newspapers and magazines, go to employment agencies, and mobilize local networks\(^1\) that eventually bring them together into productive matches.” Nonetheless, there are few attempts to explicitly incorporate local networks into standard labour search models\(^2\). This paper models social networks in the labour market and thereby provides microfoundations for the matching function. This gives insights into several stylized facts.

First, search models of the labour market do a poor job of explaining the short-run volatility of unemployment (and other variables) in the post-war era. In fact, observed unemployment rates are an order of magnitude more volatile than a benchmark search model predicts. Furthermore, observed vacancies exhibit more persistence than such models predict.

Second, the long run relationship between vacancies and unemployment, known as the Beveridge curve, is subject to sudden and persistent shifts. For instance, there has been a recent shift in the U.S. Beveridge curve\(^3\). The search and matching literature provides several reasons as to why such shifts occur, such as skill mismatch and lack of aggregate demand.

I develop a model of unemployment and social networks to analyze the above facts. Unemployed workers utilize local networks to search for job openings. The resulting matching function has different properties than is often assumed in the literature. My model predicts that when the network structure is fixed the equilibrium unemployment rates, vacancy rates, tightness, and matching rates have a larger response (in absolute value) to productivity shocks than in baseline search models. Vacancies also exhibit persistence outside the steady-state.

When links form and sever over time there can be multiple equilibria corresponding to high unemployment and low unemployment. I characterize the equilibria when the rate of link growth does not depend on employment status. I also provide a robustness result to the latter assumption. The model allows the possibility of discontinuities in the set of equilibria. Small changes in certain labour market parameters can lead to large, persistent changes in equilibrium behaviour.

When examining unemployment volatility, the results rely on variables

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\(^1\)Emphasis my own.

\(^2\)There are a few exceptions discussed in the literature review section.

\(^3\)As well as other OECD countries since 2009.
being in steady-state and are qualitative in nature. To check the robustness of the analytical results several numerical examples are provided. I show that the standard deviations of several labour market variables are up to twice as large with the inclusion of network effects. Vacancies exhibit higher autocorrelation and the contemporaneous correlation of tightness and productivity is lower.

The model is consistent with several other stylized facts. First, in equilibrium there is wage dispersion amongst similarly productive workers. The model predicts heterogeneous wages due to heterogeneity in workers’ number of friends. Second, the model predicts that the average wage of those referred by friends is (weakly) higher than the average wage of those who obtain jobs through their own search effort. The result resembles a wage premium.

The paper is organized as follows. Section 2 presents the relevant stylized facts and literature. Section 3 presents the model with fixed networks and the results on the volatility of unemployment and vacancies. Section 4 presents the model with an endogenous network and the results on the shifting Beveridge curve. Section 5 concludes. Appendix 1 contains proofs and Appendix 2 provides technical details related to network analysis.

## 2 Evidence and Literature

### 2.1 Social Networks and Employment Transitions

There are several studies that document the use of social networks in the labour market. In a study of 2553 Quebec government workers, Langois (2007) finds that 42.7% of workers found their current job through a contact. Erickson and Yancey (1980) finds that 57.7% found their current position through a strong or weak tie. Granovetter (1983) provides a detailed survey,
and argues that weak ties (acquaintances) are the primary sources of job information.

There is a large literature that supports the notion that social networks improve labour market outcomes\(^4\). Laschever (2009) uses data on World War I draftees and the 1930 U.S. census to identify the impact of social networks on employment likelihood and finds that an additional employed peer increases employment likelihood by 0.8 percent. Beaman (2012) looks at the reallocation of refugees and finds that “tenured” members of the social network improve outcomes and members that are new arrivals harm employment outcomes.

The idea that links are created and destroyed over time is supported by theory and evidence. A robust finding across several types of networks is that the number of friends follows a Pareto distribution. Such distributions follow what is referred to as a power law and exhibit “fat tails.” The primary explanation for the finding is that networks grow and the probability of forming a link is increasing in one’s current stock of links (known as preferential attachment). Newman (2010) and Jackson (2008) provide detailed accounts of the topic.

2.2 Unemployment and Vacancy Dynamics

There are several labour market variables that will be analyzed. The unemployment rate is the fraction of unemployed workers in the labour force, denoted \( u \). The vacancy rate is the number of vacancies as a fraction of the labour force, denoted \( v \). The labour market tightness is the ratio of vacancies to unemployed workers, or \( \frac{v}{u} \). The number of matches per period is denoted \( m \). Finally, \( p \) is the level of worker productivity.

Table 1 shows some basic properties of U.S. labour market data from 1951-2005. Models of the labour market (search models in particular) have difficulty replicating labour market data, including the standard deviation\(^5\) of unemployment (0.19), and the autocorrelation of vacancies (0.94).

Another statistic that is difficult to replicate is the job finding rate \( m \). Table 2 describes the job finding rate during 1951-2003 in the U.S. Similar to unemployment, matching rates are more volatile in the data than standard

\(^4\)Not every study supports the hypothesis that network size improves labour market outcomes. Khan and Lehrer (2012) use data from a field market experiment in Cape Breton, Canada and find that although the Community Employment Innovation Project tends to increase an individual’s weak ties, long run employment outcomes do not improve.

\(^5\)Standard deviations are of logged variables taken from detrended data using the HP filter with smoothing parameter 10\(^5\).
<table>
<thead>
<tr>
<th>Std. Dev.</th>
<th>Autocorrelation</th>
<th>u</th>
<th>v</th>
<th>v/u</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>0.118</td>
<td>0.908</td>
<td>-0.949</td>
<td>0.897</td>
<td>0.948</td>
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Table 2: U.S. Job Finding Rate, 1951-2003 (Source: Shimer, 2005)

search models predict. Several studies find matching rates to be procyclical; the probability of finding a job, given the vacancy-unemployment ratio, varies positively with the business cycle. Sedlacek (2010) finds that match efficiency is procyclical and explains 26-35% of job finding rate variation.

### 2.3 Beveridge Curve

The empirical relationship between vacancies and unemployment, known as the Beveridge curve, is subject to sudden changes. Hobijn and Sahin (2012) document the changing relationship across 14 OECD countries since 2007.

Figure 1 shows the Beveridge curve from December 2003 until April 2013. It is clear that the relationship between vacancies and unemployment changes in 2009.

The U.S. Beveridge curve has a history of shifting. Fujita and Ramey (2006), while addressing the sluggish response of vacancies, provide a table of the shifting Beveridge curve from 1951-2004.

### 2.4 Literature Review

There is a theoretical literature on social networks in the labour market. There are several successful explanations of labour market facts with social network models. First, several authors explore negative duration dependence. Calvo-Armengol and Jackson (2004) look at static social networks and find that employment statuses are correlated across time and path-connected individuals. Bramoullé and Saint Paul (2010) take the argument further by allowing networks to evolve over time. If social ties are created at a higher rate between workers of the same employment status then the model produces duration dependence.

There is plenty of evidence (Granovetter, 1995) that those referred to a job by others obtain a wage premium. Montgomery (1991) argues that if firms can observe a current worker’s productivity, and productive (unproductive) workers are more likely to refer productive (unproductive) workers, then firms will hire workers referred by productive workers. Because of homophily, among worker types, firms are willing to pay referred workers more.

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6Homophily is the idea that agents with similar characteristics tend to be friends. The
Figure 1: Shift in the U.S. Beveridge Curve

The Beveridge Curve (job openings vs. unemployment rate), seasonally adjusted, December 2000–April 2013

than the market wage. Furthermore, there is a non-degenerate distribution of wages because the optimal wage setting strategy is a mixed strategy.

Galenianos (2013(a)) looks at the role of referrals in a search framework with heterogeneous workers in which a wage premium exists. My model also exhibits the appearance of wage premia, but for different reasons. In my model heterogeneity in bargaining power and the statistical properties of the social network lead to wage heterogeneity.

Finally, some work has been done on matching functions with a network component. Fontaine (2007) uses an urn-ball framework with a series of complete networks. Calvo-Armengol and Zenou (2005) uses an urn-ball matching function with regular random networks to discuss congestion in search. Galenianos (2013(b)) utilizes a matching function in an environment with new firms and expanding firms to discuss procyclical matching efficiency. In every case the matching function exhibits decreasing returns to scale in unemployment and vacancies.

My model has both similarities and differences to Fontaine (2007) and Calvo-Armengol and Zenou (2005). The papers are similar to mine in that the resulting matching functions exhibit similar properties, including decreasing return to scale. Furthermore, all papers employ an urn-ball matching foundation. My paper is different in that I do not rely on restricting the network topology. To overcome the complexity I use a mean-field approximation.

There are similarities and differences between my model and Galenianos (2013(b)) as well. The resulting matching functions have similar properties. However, Galenianos (2013(b)) abstracts from the details of the network. Furthermore, the foundation is not based on an urn-ball approach.

There is a large body of literature on the shortcomings of search models. Shimer (2005) develops a stochastic version of Pissarides (2000), calibrates the model to U.S. data, and finds that the model departs from U.S. data in important ways. In particular, predicted unemployment volatility is too low, and predicted vacancies have low (quarterly) autocorrelation compared to U.S. data. Andolfatto (1996) embeds search frictions in a real business

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7 A complete network is a network in which all nodes are connected. Fontaine (2007) has finite groups in which agents are connected to everyone in the same group and no one from another group.

8 An exception is Calvo-Armengol and Zenou (2005) in which increasing network density decreases the total number of matches above a certain threshold.

9 I discuss the approach in detail in Appendix 2 and Section 3.

10 The approach of Galenianos (2013(b)) has firms being born, or expanding and dividing. A firm that expands accepts applications from the current employee’s contacts.
cycle model and finds that vacancies in U.S. data are larger than the model can account for. Costain and Reiter (2007) investigate a search model with stochastic unemployment benefits and find a fundamental tradeoff between unemployment volatility and the impact of unemployment benefits on unemployment.

Cardullo (2010) provides a survey of the attempts to model unemployment volatility and vacancy creation in a manner that overcomes the Shimer (2005) critique. Recalibration (Hagedorn and Manovskii, 2005) and rigid wages (Hall, 2005; Pissarides, 2010) fail to satisfactorily match U.S. data. Barnichon (2012) proposes a model with endogenous productivity. Several other microfoundations have been proposed, each with varying degrees of success (see Cardullo, 2010). In the same survey, several explanations of vacancy persistence are presented.

The work presented here belongs in the microfoundation classification. I propose a mechanism (social networks) that leads to matching rate volatility and persistence, which drives volatility and persistence in unemployment and vacancies.

There is a literature on shifts in the Beveridge curve. Some (for example, Hobijn and Sahin, 2012) suggest an increase in skill mismatch (a structural explanation) as a cause for the shift. Another argument is lack of aggregate demand (a cyclical explanation).

My approach is structural\textsuperscript{11}. I show that under certain social network formation processes an exogenous change in labour market parameters can lead to multiple equilibria. These equilibria correspond to high and low unemployment. I provide a numerical example in which changing the job separation rate can cause a large persistent shift in the Beveridge curve.

3 Model with Fixed Networks

Here I examine networks with fixed links to analyze short-run changes in unemployment, vacancies, and matching rates. First, elementary examples and definitions related to network analysis are provided. Second, I outline the modelling methodology. In particular, (i) mean-field approximations, (ii) urn-ball matching, and (iii) the telephone-line queuing process are discussed. Finally, a search model equilibrium is defined and results are provided.

\textsuperscript{11}Changes in skill mismatch are similar to changes in social structure as both affect the matching technology.
3.1 Networks in the Labour Market

Consider the following scenario. An unemployed agent has $d^E$ employed friends and is filling out applications. The agent engages in random search by applying directly to firms. The agent also engages in network search by having employed friends fill out applications on his behalf \footnote{Random and network search are often referred to as formal and informal search, respectively.}

If $\mu_R$ random search applications are filled per period and each employed friend fills out $\mu_N$ applications per period, total applications from the worker per period are $\mu_R + \mu_N d^E$. Thus, the number of applications, and consequently the rate at which an unemployed worker receives job offers depends on a property of the network.

The following example illustrates the interaction. Figure 2 depicts a society. Agents B and C are unemployed. However, agent B has access to an employed friend. Thus, there is a total of $\mu_R + \mu_N$ applications from B and a total of $\mu_R$ from C per period.

In the short run links are fixed. However, over time agents lose and gain contacts. Economic homophily is the idea that agents of similar employment status tend to be friends. Homophily would imply that, supposing employment status remains unchanged, B and C are more likely to remain friends than A and B.

To abstract from the problem that the distribution of firm size and vacancies over firms matter, I suppose that all vacancies are individual firms and that employed agents randomly communicate with these firms. The details will be described when discussing the nature of vacancies.

I also abstract from the issue that the number of unemployed friends of
employed friends may matter. If B were employed instead of A one could imagine that the network application rate would fall due to competition between A and C for B’s vacancy. In the model it is assumed that employed agents send out applications randomly and independently (as opposed to the same vacancy). Thus, B would send A’s application to a vacancy, and C’s to a (with probability 1) different vacancy.

These two assumptions greatly improve the tractability of the model while maintaining important features. Comparing our model to models that violate the second assumption highlights the traction gained. For instance, Calvo-Armengol and Jackson (2004) can only analytically solve the model for a small set of network topologies.

I will now provide a series of formal definitions needed for the following analysis.

**Definition 3.1 (Network)**

A network is a pair \((L, T)\), where \(L\) is the set of nodes and \(T \subset L \times L\) is the set of ties.

A typical element of \(L\) is \(i\), and an element of \(T\) is \((i, j)\).\(^{13}\) The network can be interpreted as a set of workers and corresponding friendships. I use \(L\) to denote the labour force which is fixed throughout the analysis. The set \(T\) is fixed in this section, but is allowed to vary over time in later sections.

Agents are either employed or unemployed. Thus, the labour force \(L\) is partitioned into an employed set \(E\) and unemployed set \(U\). This partition induces a partition of the set of ties \(T\).

**Definition 3.2 (Tie Types)**

The set \(T\) is partitioned as follows:

(i) \(T_{UU} = \{(i, j) \in T | i \in U, j \in U\}\)

(ii) \(T_{EE} = \{(i, j) \in T | i \in E, j \in E\}\)

(iii) \(T_{UE} = \{(i, j) \in T | i \in U, j \in E\}\)

Notice that \(T_{UU} \subset U \times U, T_{EE} \subset E \times E,\) and \(T_{UE} \subset U \times E\). The partitioning of ties into types is important for describing nodes. The following definition allows one to characterize nodes by network location.

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\(^{13}\)There is no distinction between \((i, j)\) and \((j, i)\). This is referred to as an *undirected* network.
Definition 3.3 (Neighbourhood and Degree)

(i) The neighbourhood of an unemployed agent \(i\) is the set \(n_i = \{j|(i,j) \in T_{UE} \cup T_{UU}\}\). An employed agent’s neighbourhood is denoted \(N_i = \{j|(i,j) \in T_{EU} \cup T_{EE}\}\).

(ii) The degree of an unemployed agent \(i\) is \(d_i = |n_i|\). An employed agent \(i\) has degree \(D_i = |N_i|\).

I adopt the convention that neighbourhoods and degree in lowercase (uppercase) will refer to unemployed (employed) agents. Like ties, neighbourhoods and degrees can be distinguished by employment status.

Definition 3.4 (Neighbourhood Types and Degree Types)

(i) An unemployed agent \(i\) has an \(S\)-neighbourhood denoted \(n_i^S = \{j|(i,j) \in T_{US}\}\), where \(S \in \{E,U\}\). Similarly an employed agent \(i\) has an \(S\)-neighbourhood denoted \(N_i^S = \{j|(i,j) \in T_{ES}\}\).

(ii) An unemployed agent \(i\) has an \(S\)-degree denoted \(d_i^S = |n_i^S|\), where \(S \in \{E,U\}\). Similarly an employed agent \(i\) has an \(S\)-degree denoted \(D_i^S = |N_i^S|\).

A tie’s type is determined by the status of the agents at each end. Notice that \(d_i^E + d_i^U = d_i\) for every \(i \in U\) and \(D_i^E + D_i^U = D_i\) for every \(i \in E\). Similarly, \(n_i^E \cup n_i^U = n_i\) and \(n_i^E \cap n_i^U = \emptyset\) for all \(i \in U\). It is the set \(T_{UE}\), and thus \(d_i^E\), that is important for job search.

Most of the results in this paper rely on the size of individual neighbourhoods being small relative to the network as a whole (i.e. the ratio of an individual’s degree to the population being 0). To ensure this, \(N_i\) and \(n_k\) are finite for all \(i \in E\) and \(k \in U\) and \(|L| = \infty\).

Up until this point I have described the individuals (nodes), the bilateral relationships (ties), and employment status. The following definition provides a description of a network’s aggregate properties.

Definition 3.5 (Degree Distributions and Degree Sets)

(i) The degree distribution of a network gives, for each degree \(d\), the fraction of agents with that degree, and is denoted \(f(d)\).

(ii) The \(S\)-degree distribution of a network gives, for each \(S\)-degree \(d^S\), the fraction of agents that \(S\)-degree, and is denoted \(f^S(d^S)\), where \(S \in \{U,E\}\).

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14The function \(|\cdot|: \mathbb{X} \rightarrow \mathbb{Z}_+\) denotes the cardinality of the set \(X\).

15Due to the network being undirected, \(T_{UE} = T_{EU}\).

16I also impose this property on the set of vacancies \(V\).
The degree set of a network is the pair \( \{(f(d), d)\}_{d \in \mathbb{R}^+} \equiv \chi \). The S-degree set is the pair \( \{(f^S(d^S), d^S)\}_{d^S \in \mathbb{R}^+} \equiv \chi^S \).

Every network can be described by its degree distribution, or, equivalently, degree set. However, the degree distribution does not uniquely identify the network\(^{17}\).

There is one more important property of the network. An agent is described by an employment status and neighbourhood characteristics. Therefore, a neighbour can be described by an employment status and neighbourhood characteristics. The reason that it is important is that, as will be described in subsequent sections, neighbours with many employed friends are more likely to find jobs and be employed. This affects one’s neighbourhood composition and job prospects.

The following definition describes the number of neighbours and \( S \) – neighbours that have \( k \) employed friends.

**Definition 3.6 (\( k \)-neighbours)**

(i) (Unemployed) worker \( i \)'s \( k \)-neighbourhood is the subset of worker \( i \)'s neighbourhood whose members have \( k \) employed friends, and is denoted \( n_i(k) = \{ j | j \in n_i, d_j^E = k \} \).

(ii) (Unemployed) worker \( i \)'s \( k_S \)-neighbourhood is the subset of worker \( i \)'s \( S \)-neighbourhood whose members have \( k \) employed friends, and is denoted \( n_{i}^S(k) = \{ j | j \in n_{i}^S, d_j^E = k \} \), where \( S \in \{ U, E \} \).

(iii) (Unemployed) worker \( i \)'s \( k \)-degree is the number of \( k \)-neighbours that worker \( i \) has, and is denoted \( d_i(k) = |n_i(k)| \).

(iv) (Unemployed) worker \( i \)'s \( k_S \)-degree is the number of \( k_S \)-neighbours that worker \( i \) has, and is denoted \( d_{i}^S(k) = |n_{i}^S(k)| \) where \( S \in \{ U, E \} \).

The definition of \( k \)-neighbours demonstrates how complex network analysis can become. The partitioning of \( L \) based on employment status and neighbourhood (which itself depends on employment status and degree) can be carried further by identifying a neighbour with his \( k \)-neighbourhood, etc. Our previous assumptions about the role of networks in applying for jobs allow us to ignore these issues.

\(^{17}\)The networks \( L_1 \) and \( L_2 \) with the same degree distribution will be isomorphic.
3.2 Mean-Field Approximation

The goal of the model is to analyze the labour market dynamics. Time is continuous and denoted by $t$. The dynamics of the model presented thus far can get very complicated. For instance, the evolution of $S$-degrees are stochastic as workers lose and gain jobs randomly\textsuperscript{18}. I call the model of the previous section the true model. To overcome these issues I apply an approximation model called a mean-field approximation\textsuperscript{19}. To see the idea, let $\dot{d}_E^i(k) \equiv \frac{\partial d_E^i(k)}{\partial t}$. The true model would have the change in $d_E^i(k)$ be a stochastic process on the set of integers. The mean-field approximation (i) allows $d_E^i(k)$ to be any non-negative real number, (ii) imposes a deterministic law of motion, and (iii) has the law of motion depend on aggregate variables (ie. the mean). For instance, the law of motion for $d_E^i(k)$ in the mean-field approximation is

$$
\dot{d}_E^i(k) = m_u(k)(d_i(k) - d_E^i(k)) - \lambda d_E^i(k) \tag{1}
$$

where $m_u(k)$ is the matching rate of an unemployed agent with $k$ employed friends and $\lambda$ is the separation rate. Notice that $d_U^i(k) = d_i(k) - d_E^i(k)$. The first term describes the number of $U$-neighbours that gain employment whereas the second term describes the number of $E$-neighbours that lose jobs. Similarly, the other laws of motion are:

$$
\dot{d}_U^i(k) = -m_u(k)(d_i(k) - d_E^i(k)) + \lambda d_E^i(k) \tag{2}
$$

$$
\dot{D}_E^i(k) = m_u(k)(D_i(k) - D_E^i(k)) - \lambda D_E^i(k) \tag{3}
$$

$$
\dot{D}_U^i(k) = -m_u(k)(D_i(k) - D_U^i(k)) + \lambda D_E^i(k) \tag{4}
$$

\textsuperscript{18}Even if the means of aggregate variables move deterministically, the changes in network topology depend on the distribution and are therefore random.

\textsuperscript{19}Calvo-Armentegol and Zenou (2005) overcomes these issues by randomly drawing the set of links every period. Galenianos (2013) overcomes these issues by having neighbourhoods be infinitely large.
Notice that there is no net loss or gain in total ties. Mean-field approximations are shown to be good approximation under most circumstances. I discuss these issues in Appendix 2. The analysis in the paper is done with a mean-field approximation\textsuperscript{20}.

### 3.3 Urn-Ball Matching

A useful approach to describing the total number of matches in a given period is with a matching function. Normally, matches are exogenously given as a function of the total number of searchers on both sides of the market, in this case \( m(u, v) \). Furthermore, it is standard to assume that the function has several desirable properties, such as constant returns to scale.

There is a large literature that derives the matching function from first principles. One common approach is referred to as the urn-ball method. I utilize the urn-ball method to provide foundations for the matching function.

Consider a set of urns, \( V \), and a set of agents \( U \). Each agent possesses \( \mu_R > 0 \) balls and places each ball in a particular urn with probability \( \frac{1}{V} \). Thus, agents pick the urn in which to drop a ball randomly with replacement.

Now suppose that each urn belongs to a different firm. Once every ball is placed in some urn the firm draws a ball from its urn at random. Some firms receive no applications and thus draw zero balls. Therefore, a firm draws a ball at random conditional on its urn containing at least one ball. If a firm draws a ball belonging to worker \( i \) then the worker gets the job. In the case of worker \( i \) getting several balls drawn he chooses the job at random.

The above environment describes a common urn-ball process. Our process is slightly different. Each employed agent also places balls in urns. An employed agent samples an urn with replacement \( \mu_N \) times for each of his employed friends. The following definition allows us to discuss matching.

**Definition 2.7 (Matches)**

(i) The matching function gives the number of matches for unemployed workers per worker as a function of the unemployment rate \( (u) \), vacancy rate \( (v) \), and period length \( (\Delta t) \), and is denoted by \( m(u, v, \Delta t) \).

\textsuperscript{20}Simulations not presented here show that our analysis approximates the true model well. Bramoulle and Saint Paul (2010) use a similar model and come to the same conclusion.
(ii) The k-matching function gives the number of matches for unemployed workers with k employed friends per worker with k employed friends as a function of unemployment, vacancies, and period length, and is denoted by \( m(u, v, \Delta t, k) \).

(iii) The matching rate is the number of matches per worker per unit time, and is denoted \( \frac{m(u, v, \Delta t)}{\Delta t} \).

(iv) The k-matching rate is the number of matches per worker with k employed friends per unit time, and is denoted \( \frac{m(u, v, \Delta t, k)}{\Delta t} \).

I will often suppress notation by having \( m(u, v, \Delta t, k) \equiv m(\Delta t, k) \) and \( \lim_{\Delta t \to 0} \frac{m(u, v, \Delta t)}{\Delta t} \equiv m(u, v) \). Furthermore define

\[
\begin{align*}
    m_u(u, v) &\equiv \frac{m(u, v)}{u} \\
    m_u(u, v, k) &\equiv \frac{m(u, v, k)}{u(k)} \\
    m_v(u, v) &\equiv \frac{m(u, v)}{v}
\end{align*}
\]

The following proposition provides a matching function with network effects in a continuous time environment. It augments the result of Albrecht, Gautier, and Vroman (2004) to be used in our environment. Let \( d^E \equiv \sum_{i \in U} \frac{I^E(d_i)}{u} \).

**Proposition 1**

Let the number of matches over a time interval \( \Delta t \) be given by \( m(\Delta t) \), and suppose that \( |V|, |U| \to +\infty \) where \( \frac{|V|}{|U|} = \theta < +\infty \).

If:

(i) Each unemployed agent sends out \( \mu_R \) job applications,

(ii) Each employed agent sends out \( \mu_N \) job applications on behalf of each unemployed friend, and

(iii) When an unemployed agent sends out an application there is a probability \( 1 - \xi \) that the application is destroyed,
then when $\Delta t \rightarrow 0$ the aggregate matching rate is

$$m(u, v) = (\mu_R \xi + \mu_N \bar{d}^E)u$$

The assumption (iii) adds uncertainty to the application process. The reason for the assumption will become clear in the next section. The uncertainty is only applied to the random search process (as opposed to the network search process). The justification is that unemployed agents send applications to a firm not knowing whether a vacancy is available. In contrast, employed agents know whether the firm is ready to hire. Our results do not depend on $\xi$ applying to the unemployed searchers only.

The matching function has several desirable properties. First, it is linear in $u$. As is evident in Petrongolo and Pissarides (2001), linearity is a standard property of urn-ball matching functions in continuous time. Another desirable property is the linearity in $\bar{d}^E$.

Our result is new to the literature. Although several papers examine matching functions with networks, all are non-linear functions of the network topology, restrict the set of possible network topologies, or are only partially microfounded.

### 3.4 Telephone Line Queuing Process

There are well-established shortcomings with the urn-ball matching function. In continuous time the function does not depend on the number of vacancies. Here I follow Stephens (2007) and use a telephone-line queuing process to endogenize $\xi$.

To illustrate the idea suppose that vacancies come in two types: processing, waiting. Processing vacancies are those vacancies that have been created but are unready to be filled. The justification for this is that time and effort is involved in between the decision to create a vacancy and the interview process. Waiting vacancies are those vacancies that are ready to be filled. Let $V_w \subset V$ be the set of waiting vacancies and $v_w = |V_w|/|L|$.

Random search occurs according to a telephone-line queuing process. Workers call a firm (uniform) randomly. If the firm has a processing vacancy it does not pick up the telephone. If the firm has a waiting vacancy it picks up the phone and a match is created. Thus, the probability that a phone call.

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21The topology of a network with $n$ nodes refers to the specific pattern of ties.

22It is possible to include vacancies but then the function fails to satisfy $m(0, v) = m(u, 0) = 0$. 

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15
reaches a waiting vacancy is \( \frac{v_w}{v} \). The matching function can be rewritten as

\[
(\mu_R \frac{v_w}{v} + \mu_N \bar{d}^E)u.
\]

Network search has an added benefit. Namely, employed workers know that a vacancy is waiting. Thus, all network applications reach a waiting vacancy\(^{23}\).

To pin down \( \frac{v_w}{v} \) one must discuss the determinants of \( v_w \). If processing vacancies become waiting vacancies at rate \( \mu_v \) then the inflow of waiting vacancies is \( \mu_v(v - v_w) \). Furthermore, vacancies are being filled at the matching rate \( m_u(u,v) \). Therefore, waiting vacancies evolve according to

\[
\dot{v}_w = \mu_v(v - v_w) - \mu_R u \frac{v_w}{v} - \mu_N \bar{d}^E u \quad (5)
\]

where \( \bar{d}^E \) is the mean \( E \)-degree. Note that the term as written can be negative. I assume that, in steady-state, the network effect is small enough so as not to fill all vacancies. For the rest of the paper we assume that \( \mu_v = 1 \).

### 3.5 Unemployment and Matching

Here I highlight an important property of our network matching function. Let \( \lambda \) be the separation rate. The unemployment rate for workers with \( k \) employed friends is as follows

\[
\dot{u}(k) = \lambda(1 - u(k)) - u(k)m_u(k) \quad (6)
\]

Average unemployment evolves according to \( \dot{u} = \sum_{d=0}^{+\infty} f^E(k)\dot{u}(k) \).

To highlight the role of the networks mechanism in matching I provide a preliminary result. The following proposition states the properties of the matching function when waiting vacancies and unemployment are in steady-state.

**Proposition 2 (Steady-State Matching Function)**

Suppose \( \dot{v}_w = 0, d_i^E = 0 \) for all \( i \in U \), and \( \dot{u}(k) = 0 \) for all \( k \in \mathbb{R}_+ \).

(i) The matching function \( m(u,v) \) is decreasing returns to scale in \( (u,v) \).

(ii) For sufficiently small \( \mu_N \bar{d} \), \( m(u,v) \) is strictly increasing in \( \bar{d} \) and \( v \).

(iii) There exists a \( \hat{u} \in [0,1] \) such that the matching function is increasing in \( u \) on \([0,\hat{u}]\) and decreasing on \((\hat{u},1]\).

\(^{23}\)As mentioned earlier, the results do not rely on the assumption. The difference to Proposition 1 is a matching function of \((\mu_R + \mu_N \bar{d}^E)\xi_u \).
The result characterizes the matching function when the network is in steady-state. The key result is that the matching function exhibits decreasing returns to scale. When the number of vacancies and unemployed double, the non-steady state matching function exhibits constant returns to scale. However, the number of intermediaries (employed agents) decreases. There are more workers searching, but each with less intensity.

The number of matches increases with $v$ and $\bar{d}$. It is conceivable that the network effect is big enough to clear the market, which justifies the bound on $\mu_N \bar{d}$. Finally, the unemployment rate always increases matches at low unemployment rates. However, it is possible for unemployment to decrease the number of matches at high levels of unemployment.

The properties derived are common to the literature. One difference is that the steady-state conditions imposed endogenize the $S$-neighbourhoods. Furthermore, our result demonstrates that the properties hold with a mean-field approximation, and abstracting from friends of friends in the matching process.

### 3.6 Equilibrium with a Fixed Network

A matching function has been derived, and now a full-fledged search and matching model may be analyzed. Here I augment Pissarides (2000) by incorporating the network effect. An equilibrium definition is given and results on volatility, persistence, and wages are stated.

One can imagine a scenario in which wage set through bargaining and workers are heterogeneous. A non-cooperative bargaining solution can be complex and obscure the role of networks. To maintain the focus on changes in social networks (as opposed to wage setting) I determine wages with Nash bargaining as in Pissarides (2000). Furthermore, the analysis assumes that both firms and workers can observe a worker’s current network position. Although I believe many of the results are robust to changes in this specification the verification is beyond the scope of this paper.

Unless otherwise mentioned, I assume that waiting vacancies, $S$-neighbourhoods, and unemployment are in steady-state. This assumption implies that each unemployed worker of degree $d_i$ will have the same neighbourhood composition, which makes the analysis much easier. In case of doubt, $d$ refers to the total number of friends and $k$ refers to the total number of employed.

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24 Galenianos (2013) provides a matching function with the above properties when workers have a continuum of friends. Calvo-Armengol and Zenou (2005), with the exception of point $(ii)$, derives the results with random regular networks.

25 This includes neighbourhood composition and degree.
friends.

Let \( r \) be the (common) rate of return, \( b \) be unemployment benefits, \( p \) be productivity, \( \theta = \frac{v}{d} \) be labour market tightness, and \( u(d) \) be the unemployment rate of workers with degree \( d \). Notice that unemployment depends on the total number of friends an individual has instead of the number of employed friends. This is because the steady-state conditions pin down \( d_i^E \) for all \( i \) in terms of \( d_i \).

Abusing notation, \( U(d) \) is the flow utility of being unemployed with degree \( d \). \( W(d) \) the flow utility of working with degree \( d \). The flow utility of a vacancy is \( V \). Similarly, filled jobs have flow utility depending on the type of worker hired, denoted \( J(d) \). Because wages can be conditioned on a worker’s degree value functions are too. The value of vacancies, \( V \), does not depend on any individual worker’s degree because of the uncertainty in the matching process.

The value functions\(^{26} \) associated with employment status and vacancy status are

\[
\begin{align*}
    rU(d) &= b + m_u(d)(W(d) - U(d)) \\
    rW(d) &= w(d) + \lambda(U(d) - W(d)) \\
    rJ(d) &= p - w(d) + \lambda(-J(d)) \\
    rV &= -cp + \sum_{i \in U} m_u(d_i)(J(d_i) - V)
\end{align*}
\]

for all \( d \) where \( c > 0 \), \( p > 0 \), and \( b > 0 \).

The value functions differ from the baseline Pissarides (2000) model in an important respect. Worker utility depends on degree \( d \). This is because the future probability of becoming employed (potentially) depends on \( d \). This means wages will depend on degree, which means the value of a job depends on \( d \). Also notice the decomposition of the matching function into network and random matching components.

The steady-state conditions and Nash bargaining allow us to derive expressions for unemployment, wages, \( S \)-neighbourhoods, and \( \frac{v}{d} \). To complete the model one must determine the vacancy rate \( v \). Each vacancy is a firm and profit maximization involves decided between creating a vacancy or not. The final condition is the free-entry condition, \( V = 0 \).

The following definition is the equilibrium concept for fixed networks.

\(^{26} \)These are valid assuming \( v_w, d_i^E \), and \( u \) are in steady-state. Otherwise the value functions have additional terms related to the changes in state-variables.
Definition 2.8 (Fixed Network Equilibrium)

Given a network \((L, T)\), a Fixed Network Equilibrium (FNE) satisfies:

(i) Steady-State Unemployment: \(\dot{u}(d) = 0 \ \forall d\)

(ii) Steady-State Neighbourhood: \(\dot{d}_i = \dot{d}_i^u = \dot{D}_i^e = \dot{D}_j^u = 0, \ \forall i \in U, j \in E\)

(iii) Steady-State Waiting Vacancies: \(\dot{v}_w = 0\)

(iv) Nash-Bargaining: \(w(d) = \arg\max(W(d) - U(d))\beta(J(d) - V)^{1-\beta}\)

(v) Free-Entry: \(V = 0\)

Conditions (i), (iv), and (v) are similar to Pissarides (2000). The main difference is that wages, and thus value functions, depend on an agent’s degree.\(^{27}\)

Conditions (ii) and (iii) are steady-state conditions on \(S\)-neighbourhoods and waiting vacancies, respectively.

3.7 Results on Volatility and Persistence

A FNE can explain some labour market statistics due to the decreasing returns to scale matching function. First, changes in productivity, \(p\), lead to changes in unemployment rate, \(u\). The changes in \(u\) affect the \(E\)-neighbourhoods of unemployed workers. This leads to a lower matching rate and lower steady-state unemployment. There is a feedback effect as changes in \(u\) are reinforced by less network matching.

Second, vacancy rate \(v\) is a jump variable. Thus, changes in \(p\) lead to jumps in \(v\). However, now \(u\) has a larger effect on \(v\). To see this examine \(\theta\). In a baseline search model \(\theta\) exhibits no persistence; exogenous changes in \(p\) lead to a one-time jump in \(\theta\). In the model presented here, exogenous changes in \(p\) lead to an initial jump in \(\theta\) followed by a gradual change towards a steady-state.

When looking at the effect of \(p\) on \(u\) one must look at the direct effect and the indirect effect (through \(\theta\)). The indirect effect is assessed by examining the elasticity of \(\theta\) with respect to \(p\). The next result states that the equilibrium response in \(\theta\) (and \(u\)) to changes in productivity is larger with network search than with no network search.

\(^{27}\)Technically, they depend on \(d^E\). Conditions (i)-(iii) imply that \(d_i^E\) is entirely determined by \(d_i\).
Proposition 3 (Volatility)

Let $\epsilon(\mu_N)x_{(p-b)}$ be elasticity of $x$ with respect to $p - b$ for $\mu_N$ and that $\mu_R + \mu_NdE$ is constant. If $\mu_N > 0$ then in FNE

$$\epsilon(\mu_N)_{\theta,(p-b)} > \epsilon(0)_{\theta,(p-b)}$$

and

$$\epsilon(\mu_N)_{u,(p-b)} > \epsilon(0)_{u,(p-b)}$$

The result demonstrates that both equilibrium unemployment and tightness can exhibit larger volatility. The feedback effect of losing intermediaries (employed workers) between firms and the unemployed magnifies that response of each variable. Granted the results are limited to the steady-state, if transitory dynamics are of little consequence then the result is a good approximation.

The next proposition looks at the non-equilibrium dynamics of vacancy creation. Essentially, if steady-state waiting vacancies, wage determination, and free-entry ((iii)-(v) of FNE definition) remain then $\theta$ is a function of past $p$ and $v$ exhibits more persistence. Let $\theta(\mu_N)$ be the tightness that satisfies (iii) at $\mu_N \geq 0$.

Proposition 4 (Persistence)

Suppose (iii)-(v) of FNE continue to hold, and at $t_0$ $\dot{d}E \neq 0$ and $\dot{u} \neq 0$. Then $\dot{\theta}(\mu_N) \neq 0$ for $t > t_0$ if and only if $\mu_N > 0$.

The result can be strengthened in that one-time positive (negative) jumps in $p$ lead to positive (negative) $\dot{\theta}(\mu_N)$. The labour market tightness in the original model is not a function of flow variables. Now changes in $dE$ and $u$ have an impact on $\theta$ by steadily increasing (decreasing) the matching efficiency when $p$ increases (decreases).

3.8 Numerical Example

The analytical results provide a mechanism to amplify and extend the effect of shocks. The results are qualitative, and the quantitative importance of the mechanism can only be determined through calibration.

I undertake a calibration exercise similar to Shimer (2005) to verify our analytical results. Our comparative statics rely on the model being in equilibrium. In a stochastic dynamic environment, being in the steady-state is
not the norm. It is conceivable that transitory dynamics yield qualitatively opposite results, or quantitatively insignificant results.

Our methodology is as follows. First, I choose a simple network topology. I look at 1-regular networks\(^{28}\); agents are organized in pairs. I make this choice because the environment is as simple as possible without giving up the network effect.

Second, I simulate the mean-field approximation model\(^{29}\). Third, I run two sets of simulations. One set with \(d_i = 0\) for all \(i\) and one set with \(\mu_N > 0\) and \(d_i = 1\) for all \(i\).

Fourth, Figure 4 displays our choice of calibration for each simulation. Most of the parameters are unchanged from Shimer (2005), though there are slight differences. The parameters are chosen to target a mean unemployment rate of 0.057 and mean matching rate of 1.5. To keep the response of vacancies to productivity shocks comparable, I fix \(c\) across simulations.

Some results are presented below. I will use the use the suffix US to refer to the U.S. data, \(N\) to refer to the simulation with networks, and \(S\) to refer to the simulation without networks.

\[
\begin{align*}
\sigma_{US}^{\theta} &= 0.190 > \sigma_{U}^{N} = 0.020 > \sigma_{U}^{S} = 0.009 \\
\sigma_{US}^{m} &= 0.118 > \sigma_{m}^{N} = 0.108 > \sigma_{m}^{S} = 0.087 \\
\sigma_{US}^{\theta} &= 0.382 > \sigma_{\theta}^{N} = 0.171 > \sigma_{\theta}^{S} = 0.137
\end{align*}
\]

The results on volatility are an improvement on previous results. Although these are simple simulations, they suggest that it is possible for network effects to generate up to twice as much volatility in important variables. Furthermore, simulations in the style of Hagedorn and Manovskii (2008) may require less extreme values of \(b\) and \(\beta\).

The following results look at correlations. Let \(\rho_v\) be the autocorrelation of \(v\) and \(r_{x,y}\) be the contemporaneous correlation of \(x\) and \(y\).

\[
\begin{align*}
\rho_v^{US} &= 0.940 > \rho_v^{N} = 0.886 > \rho_v^{S} = 0.817 \\
r_{\theta,p}^{US} &= 0.396 < r_{\theta,p}^{N} = 0.976 < r_{\theta,p}^{S} = 0.999
\end{align*}
\]

Other correlations are also improved upon. The simulation results suggest that the steady-state conclusions of Proposition 4 and Proposition 5 can be trusted.

\(^{28}\)A \(k\)-regular network is one in which every agent has degree \(k\).

\(^{29}\)Simulations and analytics not presented here verify that it is a good approximation for large or dense networks.
with Networks        without Networks
\( p \)    Ornstein-Uhlenbeck Process
\( \lambda \)  0.09     0.09
\( m \)    1.5      1.5
\( c \)    0.02     0.02
\( \beta \)   0.72    0.72
\( b \)    0.4      0.4
\( \mu_R \) \( \frac{5}{2} \) –

Figure 3: Calibrated Variables

3.9 Wages

Referred workers tend to be paid higher wages (Granovetter, 1995). Several authors provide an asymmetric information explanation (Montgomery, 1991; Galenianos, 2012). My model has no asymmetric information, yet wage heterogeneity exists in non-regular networks.

The next proposition states that there can be an appearance of wage premia.

Proposition 5 (Equilibrium Wages)

Let \( \bar{w}(R) \) and \( \bar{w}(N) \) be the average wages of jobs found through random search and network search respectively. In a FNE, \( \bar{w}(N) \geq \bar{w}(R) \). The inequality is strict when \( d_i \neq d_j \) for some \( i \neq j \).

Only regular networks (same degree for all nodes) give the same outside option for all agents. The result is distinct from other literature in that outside options alone are driving differences in wages.

4 Model with Evolving Networks

Observed social networks evolve over time. Ties break and new connections are formed while employment statuses change. Here I present a coevolutionary model of networks and unemployment. Our main result is that multiple equilibria can arise when the rate of link formation depends on the number of agents available to link to. For instance, one accumulates links to employed agents faster when the unemployment rate is low. These multiple equilibria can exhibit tipping points.\(^{30}\)

\(^{30}\) A previous version of the paper analyzed unemployment volatility in an evolving network. The results of the previous section are largely robust to changes in the set \( T \).
Before continuing, I will discuss a few concepts in the formation of networks. Links are accumulated in two important ways. First, agents meet randomly or in ways that do not depend on network structure. Second, agents often find friends through current friends, which suggests that the rate of an agent’s link accumulation depends positively on the number of links he currently has. *Preferential attachment* (PA) is the idea that agents with more links gain more friends per period. In my model, PA is essential for the existence of multiple equilibria.

There also exist biases in the network formation process that depend on an agent’s employment status. The tendency of agents with the same employment status to be linked is called *economic homophily.* In the presence of preferential attachment it is important to distinguish between the different manners in which homophily might present itself. *First-Order Economic Homophily* (FH) is the idea that agents of a particular employment status are more likely to accumulate links with agents of a their status. For instance, if \( i \) is employed then \( i \) is more likely to meet \( j \) if \( j \) is also employed.

*Second-Order Economic Homophily* (SH) is the idea that an employed friend of a particular status is more likely to refer a friend of the same status. For instance, independent of agent \( i \)’s status, if \((i, j) \in T \) and \( j \) is employed then \( i \) is more likely to meet \( k \) through \( j \) if \( k \) is also employed. This means that employed friends are more useful for meeting new employed friends.

The results in this section rely on PA and SH. To make things clear we assume that there is *pure* SH in the sense that only employed friends are useful for gaining employed friends, though the assumption is not required.

### 4.1 Mean-Field Approximation

Much like the previous section, I apply a mean-field approximation. Now the laws of motion for workers’ \( S \)-neighbourhoods are different. An unemployed worker \( i \) has \( d_E^E(k) \) employed friends each of which have \( k \) employed friends, which evolves according to:

\[
\frac{d_E^E(k)}{dt} = -\delta d_E^E(k) + h \rho g(d_E^E(k))(1 - u(k)) + m_u(k) d_U^E(k) - \lambda d_E^E(k) \quad (11)
\]

Notice the similarities with the fixed network model. The final two terms, \( m_u(k) d_U^E(k) - \lambda d_E^E(k) \), describe the current friends who lose and gain friends. However, when matching rates and/or separation rates are sufficiently high the model can exhibit instabilities and qualitatively counterfactual results.

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31 I borrow this terminology from Bramoullé and Saint Paul (2010).
employment. The other two terms, $-\delta d_i^E + h\rho g(d_i^E)(1 - u)$, describe the attaching and severing of links.

The term $\delta$ describes the rate at which links become detached. Because the mean-field approximation is a deterministic approximation of a stochastic process, $\delta$ can also be thought of as the probability that a link is destroyed. Thus, $-\delta d_i^E(k)$ is (approximately) the total number of links destroyed over a unit interval.

To understand link formation first examine $g(d_i^E(k))(1 - u(k))$. The second term, $(1 - u(k))$ says that the more agents (conditional on $k$) that are employed the faster the rate at which other befriends them. Simply put, the more employed agents with $k$ employed friends that are in the economy, the faster one connects to them.

The term $g(d_i^E(k))$ says that the rate of employed friend accumulation depends on the number of friends (with $k$ employed friends) that and agent is connected to.

The parameter $h$ captures FH. Namely, the creation of links is penalized if the workers are of different employment statuses. $\rho$ captures the rate of link formation common to all individuals. The non-linearity of the link formation process can lead to multiple equilibria. Figure 4 shows an example of a law of motion in one dimension. Notice that there are three absorbing states.

$\dot{d}_i^U(k) = -\delta d_i^U(k) + \rho g(d_i^U(k))u(k) - m_u(k)d_i^U(k) + \lambda d_i^E(k)$ (12)

$\dot{D}_i^E(k) = -\delta D_i^E(k) + \rho g(D_i^E(k))(1 - u(k)) + m_u(k)D_i^U(k) - \lambda D_i^E(k)$ (13)

$\dot{D}_i^U(k) = -\delta D_i^U(k) + h\rho g(D_i^E(k))u(k) - m_u(k)D_i^U(k) + \lambda D_i^E(k)$ (14)

The parameter $h$ captures FH. Namely, the creation of links is penalized if the workers are of different employment statuses. $\rho$ captures the rate of link formation common to all individuals. The non-linearity of the link formation process can lead to multiple equilibria. Figure 4 shows an example of a law of motion in one dimension. Notice that there are three absorbing states.

$^32$Normally $g$ would be a function of total links. First, our results are robust to this when $h < 1$. Second, one can think of this as another type of homophily. Namely, friends are more likely to connect you with their friend when the friends are of a similar type.
Figure 4: The law of motion $\dot{d}E$ plotted against $dE$. There are three absorbing states.

4.2 Equilibrium

Because the set of links, $T$, can change over time one needs a more general equilibrium definition than a FNE. Here we define an equilibrium to accommodate the evolving nature of the network.

Network Equilibrium

Given a set of nodes $L$, a network equilibrium (NE) satisfies:

(i) Steady-State Unemployment: $\dot{u}(d) = 0 \forall d$

(ii) Steady-State Degree Distributions: $F^S_t(dE) = F^S(dE) \forall t, S \in \{U, E\}$

(iii) Steady-State Waiting Vacancies: $\dot{v}_w = 0$

(iv) Nash-Bargaining: $w(dE) = \arg\max(W(dE) - U(dE))^{\beta}(J(dE) - V)^{1-\beta}$.

(v) Free-Entry: $V = 0$

Let the set of NE be denoted $\Gamma(L)$. Notice that a NE is a FNE with a endogenous distribution of total links.

4.3 Network Equilibrium with $h = 1$

The special case of no first-order economic homophily ($h = 1$) leads to a conveniently navigable model. The following proposition states some of the basic properties of the equilibrium.
Proposition 6 (Properties of Network Equilibrium)

Let \( h = 1 \) and \( \text{supp}(x) \) be the support of distribution \( f(x) \). Let \( \phi(x) \) be the set of absorbing states of \( x \). Then for any NE:

(i) Any equilibrium distribution of \( x \) has support in \( \phi(x) \). For instance, \( \text{supp}^*(d^E_k) \subset \phi(d^E_k) \)

(ii) Any distribution over absorbing states is an equilibrium distribution.

(iii) \( \text{supp}^*(d^E_i(k)) = \text{supp}^*(D^E_i(k)) \) and \( \text{supp}^*(d^U_i(k)) = \text{supp}^*(D^U_i(k)) \).

(iv) There exists a function \( g(d^E_i(k)) \) and parameters \( (\delta, \rho) \) such that \( |\phi(d^E_i(k))| = 3 \).

The proposition makes several important points. First, (i) limits the set of equilibrium distributions. The set of distributions is restrict to the set of absorbing states. Second, (ii) says that if \( |\phi(d^E)| > 1 \) then more than one NE can exist. If there are several points on which an individual’s network can settle then there is a possibility of multiple equilibria. Third, (iii) says that the type degree distributions are independent of one’s employment status. Once an individual is in steady-state he stays there, even if his employment status changes. Finally, (iv) says that there may be multiple absorbing states for \( d^E_i(k) \).

Therefore, the model is very tractable and multiple equilibria can arise. Point (ii) means that a large set \( \phi(x) \) can lead to a large set of NE. To keep the analysis clear a refinement of the set of NE, \( \Gamma(L) \), is presented. Namely, I concentrate on networks in which individuals have the same number of links.

Definition 4.1: Regular Equilibria

The set of network equilibria such that \( d^*_i = d^*_e \) for all \( i \) are referred to as regular equilibria. The set of regular equilibria is denoted by \( \Gamma_r(L) \subset \Gamma(L) \).

Notice that when \( |\phi(d^E)| = 1 \) then \( \Gamma_r(L) = \Gamma(L) \). The refinement is not useful for settings with a unique absorbing state. However, the refinement is very useful for dealing with multiple absorbing states.

The following proposition discusses equilibrium Beveridge curves.
Proposition 7

Consider \((d^E, u, v)\) and \((d^E', u', v')\) associated with distinct Regular Equilibria at \(h = 1\). Let the equilibrium relationship be described by \(u = \omega(u, d^E)\).

If \(d^E' > d^E\) and \(\frac{\mu_R}{\mu_N}\) is sufficiently small then \(\omega(v, d^E') < \omega(v, d^E)\).

The result says that each equilibrium network is associated with “high” and “low” unemployment-vacancy relationship. There a several equilibrium Beveridge curves. The following numerical example illustrates how a change in \(\lambda\) can reduce the set of equilibria from three to one. In particular, a high unemployment equilibrium can disappear temporarily, but the change can be permanent.

4.4 Example

Consider the function \(g(x) = 1 + (x - 1)^\frac{1}{3}, \delta = 1, \) and \(\rho = 1\). Notice that \(g(x)\) is increasing, strictly concave for \(x > 1\), and strictly convex for \(x \in [0, 1)\). When the parameters are calibrated to have \(u\) close to 0 then there are obviously two (stable) steady states and thus regular equilibria. However, when the parameters are chosen to set \(u\) close to 1, then there is only one equilibrium.

Figure 5 shows \(\dot{d}^E\) under the two scenarios. Figure 6 shows two Beveridge curves associated with the larger and smaller \(d^E\) respectively.

A similar effect can be demonstrated for changes in productivity.
4.5 Network Equilibrium with $h < 1$

The benefit of working with no homophily is the ease of calculation. The drawback is that (as Bramoulle and Saint Paul (2010) point out) there is zero duration dependence from link destruction. As both duration dependence and homophily are observed in the data, it is good to know whether the results of the previous section extend to $h < 1$.

To see how homophily affects the equilibrium suppose only one equilibrium exists. It is easy to see from the laws of motion (the system of equations (11) – (14)) that the absorbing value $d^E_\ast$ is smaller than the absorbing value $D^E_\ast$. In fact, an equilibrium distribution is on $I \equiv [d^{E_\ast}, D^{E_\ast}]$.

Now suppose that there are multiple absorbing states $d^E_L < d^E_M < d^E_H$ and $D^E_L < D^E_M < D^E_H$. There is are two associated intervals $I_H \equiv [d^E_H, D^E_H]$ and $d^E_L, D^E_L$. Any equilibrium distribution is on a subset of these two intervals.

The next result says that multiple equilibria can exist when $I_1$ and $I_2$ do not overlap. If the intervals overlap then agents end up in the intersection. Furthermore, the intervals will not overlap if there is not ‘too much” homophily.

**Proposition 8**

*Suppose the laws of motion are such that when $h = 1$ there are three absorbing states for $d^E_i$ and $D^E_i$ each. Let $I_L$ and $I_H$ be the relevant intervals. The*
following are equivalent:

(i) There are multiple equilibria.

(ii) $I_L \cap I_H = \emptyset$.

(iii) $h > \bar{h}$ for some $\bar{h} \in (0, 1)$.

A similar result holds for heterophily, though such an assumption produces the counterfactual prediction of positive duration dependence. The result says that a necessary and sufficient condition for multiple equilibria is that the rate of link accumulation not be “too dependent” on employment status. Notice an implication of our result: $h = 0$ implies a unique equilibrium.

5 Conclusion

The role of network effects in labour market has yet to be fully explored. I contribute to the literature by developing two related models. First, I look at models with fixed networks and evaluate network effects on the volatility and persistence of important labour market variables. I characterize the equilibrium and find that the existence of network effects increases the volatility of unemployment, vacancies, tightness, and matching rates. Networks also increase the propagation of exogenous changes in productivity.

To examine the unemployment-vacancy relationship I allow links to evolve over time. I find that the existence of preferential attachment can produce multiple equilibria leading to several potential Beveridge curves. Changes in labour market variables, such as separation rates and productivity, can eliminate the high unemployment equilibrium. Such changes can lead to sudden, large, and persistent changes in the vacancy-unemployment relationship.

Our results motivate the further investigation of network effects in search and matching models of the labour market. First, I rely on Nash bargaining by imposing an information structure that is unrealistic. How can bargaining with asymmetric information affect the model’s predictions? Second, our simulations are conducted in a highly stylized environment. The implications of network topology and stability for unemployment volatility are not yet known. These issues are left for further research.
6 References


7 Appendix A: Proofs

7.1 Proof of Proposition 1

The proof involves investigating an urn-ball application process. Recall that an unemployed agent sends out $\mu_R$ applications per period of length $\Delta t$ and each of his employed friend sends out $\mu_N$ applications on his behalf. Therefore, the total number of applications sent for worker $i$ per period is $\mu_R + \mu_N d_i^E$. Let $\mu_R + \mu_N d_i^E \equiv \kappa_i$.

Let the probability of $i$ getting a job offer over $\Delta t$ be denoted by $p_i(\Delta t)$. All heterogeneity in $p_i(\Delta t)$ comes from heterogeneity in $d_i$ and thus $\kappa_i$. Therefore, the total number of matches, $M(\Delta t)$, is described by

$$M(\Delta t) = \sum_{i \in U} p_i(\Delta t)$$

Albrecht, Gautier, and Vroman (2004) calculate the probability $p_i(\Delta t)$ for $\kappa_i = \kappa \forall i \in U$, $\kappa$ is a positive integer, and $\Delta t = 1$. They demonstrate that in the limit of $|U|, |V| \to +\infty$ where $|V|/|U| = \theta$ that

$$p_i(\Delta t) = 1 - \frac{\theta}{\kappa_i \Delta t} (1 - e^{-\kappa_i \Delta t/\theta})^{\kappa_i \Delta t}$$

where I augmented the expression with $\Delta t$.

Due to heterogeneity one cannot simply multiply the above by $u$. Due to non-linearity of $p_i(\Delta t)$ in $\kappa_i$ one cannot simply substitute in the mean. However, as one approaches continuous time ($\Delta t \to 0$) the function (per unit time) becomes linear in $\kappa_i$.

$$\frac{M(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \sum_{i \in U} \frac{p_i(\Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \sum_{i \in U} (1 - \frac{\theta}{\kappa_i \Delta t} (1 - e^{-\kappa_i \Delta t/\theta})^{\kappa_i \Delta t})/\Delta t$$

$$= \sum_{i \in U} \lim_{\Delta t \to 0} (1 - \frac{\theta}{\kappa_i \Delta t} (1 - e^{-\kappa_i \Delta t/\theta})^{\kappa_i \Delta t})/\Delta t$$

$$= \sum_{i \in U} \kappa_i = \bar{\kappa} |U|$$

The calculation of the limit is done by applying L’Hopital’s rule. By dividing each side by $|L|$, noting that $|L| \to +\infty$, and remembering the
definition of \( m(u, v) \) we get

\[
m(u, v) = \kappa u
\]

The result follows by substituting for \( \kappa \).

\[\blacksquare\]

7.2 Proof of Proposition 2

The proof has three steps: impose \( \dot{v}_w = 0 \) and solve, impose \( \dot{u} = \dot{d}^E = 0 \) and solve, show \((i) - (iii)\) are true.

**Step 1:** First I derive the aggregate matching function as a function of the network. Setting \( \dot{v}_w = 0 \) yields a steady-state fraction of waiting vacancies \( v_w = \frac{v - \mu_N \bar{d}^E u}{\mu_R u + v} \). I restrict attention to values where the fraction is non-negative.

The total number of matches (over a small interval) is

\[
m(u, v, \bar{d}^E) = \mu_R u v_w + \mu_N \bar{d}^E u.
\]

Notice the decomposition into a random search and network search component. Rearranging gives the aggregate matching function as

\[
m(u, v, \bar{d}^E) = \mu_R u \frac{v_w^*}{v} + \mu_N \bar{d}^E u
\]

\[
= \frac{\mu_R u (v - \mu_N \bar{d}^E u)}{\mu_R u + v} + \mu_N \bar{d}^E u
\]

\[
= \frac{v u}{\mu_R u + v} (\mu_R + \mu_N \bar{d}^E)
\]

\[
= \frac{v (\mu_R + \mu_N \bar{d}^E)}{\mu_R + \theta}
\]

(16)

**Step 2:** Imposing \( \dot{u}(k) = \dot{d}^E(k) = 0 \) gives steady-state \( u(k) \) and \( d^E(k) \).

\[
u^*(k) = \frac{\lambda}{\lambda + m_u(u, v, k)}
\]

\[
d^E^*(k) = \frac{m_u(u, v, k)}{\lambda + m_u(u, v, k)}
\]

\[
\Rightarrow
\]

\[
d^E^*(k) = (1 - u^*(k))d(k)
\]

(17)
**Step 3:** To verify the properties, one must first substitute $d^E$ into the matching function.

$$m(u, v, k) = \frac{v}{\mu_R + \theta}(\mu_R + \mu_N d^E(k))$$

$$= \frac{v}{\mu_R + \theta}(\mu_R + \mu_N(1 - u(k)) d(k))$$

(18)

To show (i) let $\gamma > 1$.

$$m(\gamma u, \gamma v, k) = \frac{\gamma v}{\mu_R + \theta}(\mu_R + \mu_N(1 - \gamma u(k)) d(k))$$

$$= \frac{\gamma v}{\mu_R + \theta}(\mu_R + \mu_N\left(\frac{1}{\gamma} - u(k)\right)d(k))$$

$$< \frac{\gamma v}{\mu_R + \theta}(\mu_R + \mu_N(1 - u(k)) d(k))$$

$$= \gamma m(u, v, k)$$

(19)

To show (ii) one need only differentiate $m(u, v)$ with respect to $v$ and $d$ respectively. The proof is sufficiently trivial to be left to the reader.

To show (iii) one need only take the derivative of $m(u, v)$ with respect to $u$. Equivalently, we look at each component of the matching function.

$$\frac{\partial m(u, v, k)}{\partial u(k)} = \frac{m\theta}{(\mu_R + \theta)u} - \frac{v\mu_N d(k)}{(\mu_R + \theta)}$$

$$= \frac{\theta}{\mu_R + \theta}\left(\theta\mu_R + (\theta(1 - u) - u)\mu_N d(k)\right)$$

(20)

This term will be positive or negative depending on the size of the parameters, $\theta$, and $(1 - u)$. The cutoff $\hat{u}$ is

$$0 = \theta\mu_R + \theta\mu_N d(k) - (1 + \theta)\hat{u}\mu_N d(k)$$

$$\Leftrightarrow$$

$$0 = u^2 + vu - \frac{v(\mu_R + \mu_N d(k))}{\mu_N d(k)}$$

(21)

The quadratic equation always has a positive real root and a negative real root. The (positive) solution is

$$\hat{u} = \frac{v}{2} \left(-1 + \left[1 + \frac{\left(\mu_R + \mu_N d(k)\right)}{\mu_N d(k)v}\right]^\frac{1}{2}\right)$$

Therefore, if $u > \hat{u}$ then matches decrease in $u$. 

■
7.3 Proof of Proposition 3

The proof has two steps. First, the equilibrium is shown to be reducible to two equations. Second, given the equation I can calculate $\epsilon_{\theta,p-b}$ and $\epsilon_{\theta,p-b}$.

**Step 1:** The FNE conditions establish several equations. Substituting $dE^*$ and $\frac{v}{\theta}$ into the matching function reduces the equilibrium conditions to the following equations. Note that in equilibrium each agent with degree $d$ will have the same $dE^*$. I will suppress the * notation. The remaining conditions are

\begin{align*}
0 &= \lambda(1 - u(d)) - \frac{\theta}{\mu_R + \theta}(\mu_R + \mu_N(1 - u(d))d)u(d) \quad (22) \\
0 &= (1 - \beta)(W(d) - U(d)) - \beta(J(d) - V) \quad (23) \\
0 &= p - \bar{w}(d) + \frac{(r + \lambda)pc}{m_v} \quad (24)
\end{align*}

for all $d$. The above equation are analogous to the equations of Pissarides (2000). Equation (19) imposes steady-state unemployment, equation (20) is the first-order condition for Nash bargaining, and equation (21) is the job-creation equation. The first two condition on degree $d$ whereas the final evaluates averages. \( \bar{w}(d) \) comes from the fact that when a vacancy is created it is unknown which type of worker $d$ will be hired.

Although there is wage heterogeneity, it is easy to verify that the average wage equation is similar to Pissarides (2000).

\[ \bar{w}(d) = m_u(1 - \beta)b + m_u\beta p(1 + c\theta) \]

Substituting this term into the job creation equation reduces the system to two equations to work with.

\begin{align*}
0 &= \lambda(1 - u) - E\left[\frac{\theta}{\mu_R + \theta}(\mu_R + \mu_N(1 - u(d))d)u(d)\right] \quad (25) \\
\frac{(1 - \beta)(p - b)}{c} &= (r + \lambda) \frac{(\mu_R + \theta)}{E[(\mu_R + \mu_N(1 - u(d))d)]} + \beta \theta \quad (26)
\end{align*}
The expectation is taken over $d$ with a probabilities\textsuperscript{33} $m_u(d)$ for a worker $m_v(d)$ for a firm.

**Step 2:** Taking the total derivative of the system one can solve for the elasticities. Let $\eta_x$ be the elasticity of $m_u$ with respect to $x$.

\begin{align}
\epsilon_{u,p-b} &= \frac{-\eta_\theta m_u^2(r + \lambda)}{\lambda((r + \lambda)(1 + \eta_\theta)m_u + \beta) - \eta_u m_u((r + \lambda)(1 + 2\eta_\theta)m_u + \beta)} \\
\epsilon_{\theta,p-b} &= \frac{(r + \lambda)m_u}{(r + \lambda)(1 + \eta_\theta)m_u + \beta} \left(1 + \frac{(r + \lambda)m_u^2\eta_u\eta_\theta}{\lambda((r + \lambda)(1 + \eta_\theta)m_u + \beta) - \eta_u m_u((r + \lambda)(1 + 2\eta_\theta)m_u + \beta)}\right) 
\end{align}

Increasing $\mu_N$ from zero to a positive number increases $\eta_u$ from zero to a positive number and $m_u$. In this case the absolute values of both equations increase as proposed.

As a side note another way to view the equations is to see what happens when $\mu_N$ increases but $m_u$ stays fixed. To accomplish this $\mu_N$ and $\mu_R$ must change (in opposite directions). Decreasing $\mu_R$ decreases $\eta_\theta$. It is not guaranteed that the magnitudes will change with these parameters for small changes, but they do for large changes.

### 7.4 Proof of Proposition 4

The proof involves keeping the job creation equation and wage equation from the previous proof and taking the time derivative of $\theta$. The idea is that the network effect, which shows up in the matching function, will create persistence in changes in $p$. Totally differentiating the job creation equation gives

\begin{equation}
\frac{(r + \lambda)}{E[\mu_R + \mu_NdE]} + \beta \frac{\partial \theta}{\partial t} = \frac{(r + \lambda)(\mu_R + \theta)}{E[\mu_R + \mu_NdE]^2} \mu_N E[\frac{\partial dE}{\partial t}] 
\end{equation}

It is clear that $\frac{\partial \theta}{\partial t} > 0$ requires a network effect $\mu_N > 0$. Being outside of the steady-state guarantees that $E[\frac{\partial dE}{\partial t}] > 0$.

\textsuperscript{33}These need not sum to 1 in continuous time.
7.5 Proof of Proposition 5

To show that the average wage is higher for referred agents I (Step 1) show that wages are higher for those of higher $E$-degree, and (Step 2) show the degree distribution conditional on being hired through network search first order stochastically dominates (FOSD) the degree distribution conditional on being hired through random search.

**Step 1:** A property of Nash bargaining is that those with a higher outside option get a higher share of the surplus. In FNE, the outside option of a worker of $E$-degree $d_i^E$ is described by the value functions

$$rU(d_i^E) = b + m_u(d_i^E)(W(d_i^E) - U(d_i^E))$$

and

$$rW(d_i^E) = w(d_i^E) + \lambda(U(d_i^E) - W(d_i^E))$$

Furthermore, in FNE all workers of the same $d$ have degree $d_i^E$ (i.e. a one to one mapping). The matching rate $m_u(d_i^E)$ is higher for those of higher degree and thus the wage is higher for those of higher degree.

**Step 2:** Now it must be demonstrated that the conditional distribution of a network-hired person, denoted $p(d|N)$, first-order stochastically dominates that of a randomly hired person, $p(d|R)$. The trick is to find the probability of being hired through a channel $H \in \{R, N\}$ given degree, denoted $p(H|d)$. Knowledge of the degree distribution gives $p(d)$ and $p(H) = \sum_d p(H|d)p(d)$. We apply Bayes’ rule to obtain $p(d|H)$ and can verify $p(d|N)$ first-order stochastically dominates $p(d|R)$.

Recall that the matching function $m(d)$ can be decomposed into $\mu_R u(d)\xi$ and $\mu_N d$. Because the population is large these give the number of workers of degree $d$ matched randomly and through the network respectively. The telephone-line queuing process endogenizes $\xi$ making it depend only on aggregate variables (independent of $d$), and thus taken as a constant.

The expressions are

$$p(R|d) = \frac{\mu_R \xi u(d)}{\mu_R \xi u(d) + \mu_N E[(1-u(d))d]}$$

and

$$p(N|d) = \frac{\mu_N (1-u(d))d}{\mu_R \xi u(d) + \mu_N E[(1-u(d))d]}$$

Applying Bayes’ rule gives

$$p(d|N) = \frac{f(d) \mu_R \xi u(d) + \mu_N E[(1-u(d))d]}{E[\mu_R \xi u(d) + \mu_N E[(1-u(d))d]]}$$

38
and
\[ p(d|R) = \frac{f(d) u(d)}{\mu_R \xi u(d) + \mu_N (1-u(d))d} E[\frac{u(d)}{\mu_R \xi u(d) + \mu_N (1-u(d))d}] \]

To show that the top FOSD the bottom we calculate the distribution of each, denoted \( P(d|H) \), by summing over \( d \) from 0 to \( x > 0 \).

\[ P(d|R) = \sum_{d=0}^{x} p(d|R) \] (29)

\[ P(d|H) = \sum_{d=0}^{x} p(d|H) \] (30)

It is easy to verify that (26) < (27) for all \( x > 0 \) and therefore (26) FOSD (27). Because \( w(d) \) is weakly increasing in degree and by definition of FOSD it must be the case the conditional average is weakly larger. Furthermore, differences in (26) and (27) disappear when \( d_i = d \) for all \( i \).

\[ \square \]

7.6 Proof of Proposition 6

Setting \( h = 1 \) means that \( \phi(d^E_i) = \phi(D^E_i) \) for all \( i \in L \). Any \( i \) with stock of employed friends \( x \in \phi(d^E_i) \) will maintain \( x \) employed friends independent of employment status (which is not true for \( h \neq 1 \)).

Furthermore, any agent not at an \( x \in \phi(x) \) moves toward the nearest \( x \). Thus, it is apparent that the support of equilibrium distribution \( f^E(d^E) \) is in \( \phi(d^E) \) and any such distribution will do. The same can be said for \( d^U \) and \( D^U \). Thus, (i) – (iii) are demonstrated.

Part (iv) is demonstrated by the example in the text.

\[ \square \]

7.7 Proof of Proposition 7

Because attention is restricted to regular equilibria, high connectivity equilibrium involves all agents at \( d^E \) and the low has all agents at \( d^E \).
Fixing $v$ can compare unemployment rates.

$$u(dE') = \frac{\lambda}{\lambda + \frac{\theta'}{\mu_R + \theta}(\mu_R + \mu_N dE')} \equiv \omega(v, dE')$$ (31)

$$u(dE) = \frac{\lambda}{\lambda + \frac{\theta'}{\mu_R + \theta}(\mu_R + \mu_N dE')} \equiv \omega(v, dE)$$ (32)

Because we are fixing $v$ it is not enough that $dE$ is changing. The term $\theta$ also changes. It is the case that (31) > (32) if $dE' > dE$ and

$$dE > u + \frac{v \mu_N}{\mu_R - \mu_N}$$

Thus, for $\frac{\mu_N}{\mu_R}$ large enough the inequality holds.

7.8 Proof of Proposition 8

The proof has two steps. First we show $(iii) \Rightarrow (ii)$ by arguing that decreasing $h$ below 1 by some amount will not cause the intervals $I_l$ and $I_h$ to overlap if they are disjoint at $h = 1$ and there are three absorbing states. Next we show $(ii) \Rightarrow (i)$ by contradiction. Finally, we show $(i) \Rightarrow (iii)$ by contrapositive.

A fact that is easy to verify is that any ergodic subset of $\mathbb{R}_+$ in this context has a unique stationary distribution over elements in that set.

$(iii) \Rightarrow (ii)$:

To begin the proof let us consider $\phi(x, h)$ as a correspondence in $h$. The following are true:

By assumption, $|\phi(dE, 1)| = |\phi(D^E, 1)| = 3$. As demonstrated previously, at $h = 1$ there are multiple equilibria. The correspondence is not convex-valued when $|\phi(x, h)| > 1$.

The question of whether $I_L \equiv [d_L^{E*}, D_L^{E*}]$ overlaps with $I_H \equiv [d_H^{E*}, D_H^{E*}]$ is equivalent to whether $D_L^{E*} > d_H^{E*}$. By assumption, $D_L^{E*} < d_H^{E*}$ when $h = 1$.

We can recast the question as whether $\phi(x, h)$ is continuous as $h = 1$. To see this, $h > \tilde{h}$ can be equivalently cast as $H$ being in a neighbourhood around 1.

Discontinuities in $\phi(x, h)$ can only happen when $|\phi(x, h)|$ is even, which rules out $h = 1$ having a discontinuity.
$(ii) \Rightarrow (i)$:

Suppose $I_L \cap I_H \neq \emptyset$ and there are multiple equilibria. Any agent that begins outside $I_L \cap I_H$ ends up in $I_L \cap I_H$ with probability 1 asymptotically. The set $I_L \cap I_H$ is ergodic and therefore has a unique stationary distribution, which is a contradiction.

$(i) \Rightarrow (iii)$:

Suppose $h = 0$. Then all agents end up in $[0,D^E_L]$ with probability 1. Furthermore the set is ergodic and therefore has a unique stationary distribution.

\[\blacksquare\]

8 Appendix 2: Concepts

8.1 Urn-Ball Matching and Telephone-Line Queuing Processes

The urn-ball matching process is a class of statistical processes that provide microfoundations for the matching function. The process captures two important aspects of search. First, search effort is bounded. A worker can only apply for a fraction of available jobs. In the simple case workers are limited to one application per time period. Second, a decentralized application process suffers from lack of coordination. If workers could coordinate they may wish to apply to separate jobs. To possibility of workers applying to the same job leads to congestion in the market.

The idea is that workers each send out one application (ball) per period in a uniform random fashion. Each firm (urn) who receives at least one application draws one application. The worker associated with that application is hired. The probability of a worker getting at least one job offer is

\[p \equiv 1 - (1 - \frac{1}{|V|})^{|U|}\]

Notice that $m(u,v) = vp$. Let $\theta = \frac{|V|}{|U|}$ be finite and $|V|,|U| \rightarrow +\infty$. It is easy to verify that the limiting number of matches is

\[m(u,v) = v(1 - e^{-\theta})\]
A wide range of models in economics, including networked labour markets, use some form of urn-ball process. However, the model suffers from several drawbacks. First, the process described is in discrete time. The continuous time analog yields

\[ m(u, v) = u \]

The function in continuous time does not depend on the number of vacancies, which is a desirable property. If the process is augmented to allow firms to actively recruit workers one obtains

\[ m(u, v) = u + v \]

This function fails to satisfy \( m(0, v) = m(u, 0) = 0 \). A solution proposed by Stephens (2007) is to require new vacancies to have a processing time. Once a vacancy is processed it begins waiting to be filled. If processing vacancies become waiting vacancies \( (v_w) \) at Poisson rate \( \gamma \) the evolution of vacancies can be described by the differential equation

\[ \dot{v}_w = \gamma(v - v_w) - m(u, v) \]

The telephone-line queuing process can be explained in the following way. A worker applies by making a telephone call. If made randomly, there is a probability \( \frac{v_w}{v} \) the call reaches a waiting vacancy and an application (ball) is given to the firm (urn). There is also a probability \( \frac{v - v_w}{v} \) that a call is made to a processing vacancy, which case the firm does not answer the phone and an application cannot be made.

Because only a fraction of balls get put into urns, the aggregate matching function from the process is

\[ \frac{v_w}{v} u \]

Combining the matching function with the law of motion for \( v_w \) yields

\[ \dot{v}_w = \gamma(v - v_w) - \frac{v_w}{v} u \]

One can calculate the steady state \( \frac{v_w}{v} = \frac{\gamma v}{\gamma v + u} \) and obtain

\[ m(u, v) = \frac{\gamma vu}{\gamma v + u} \]
The steady-state matching function exhibits several desirable properties: homogenous of degree 1 in \((u, v)\), \(m(0, v) = m(u, 0) = 0\), increasing and unbounded in \((u, v)\).

8.2 Mean-Field Approximation

The use of mean-field approximations are widespread in statistical physics. Furthermore, several economists and social scientists have employed the technique (Bramoulle and Saint Paul, 2010; Jackson and Rogers, 2007; Vega-Redondo, 2007; Price, 1964). Nonetheless, no introductory resource aimed at economists exists\(^{34}\). Here I cover the general ideas. The exposition borrows from McComb (2004) and Newman (2010).

To get an idea for the use of mean-field theory one can look at Weiss's theory of ferromagnetism. Suppose that there are a large number of particles interacting with one another. Any particle \(i\) can be in one of two states, \(s_i \in \{-1, 1\}\). Let \(M\) denote the average state, and \(T\) denote temperature. Let \(M\) depend on the environmental forces \(B_E\):

\[
M = f(BT)
\]

A particle’s state is normally a function of external forces \(B_E\) and internal forces \(B_I\), where \(B = B_E + B_I\). \(B_E\) can be thought of as magnetism originating from outside sources, such as the experimenter. \(B_I\) can be thought of as the magnetism of other particles in the system. External forces are generally easy to measure and control. However, the exact manner of particle interaction determining \(B_I\) can be quite complicated. The assumption used is that \(B_E\) depends on the mean state \(M\) across all particles. Thus, the magnetic field with no external influence \((B_E = 0)\) is \(B = B_I = g(M)\). Therefore, the solution to the model is the fixed point

\[
M^* = f(g(M^*)T)
\]

One can solve for the critical temperature \(T_C\) above which the system exhibits a phase transition from \(M^* = 0\) to \(M^* > 0\). When \(f\) and \(g\) are concave and increasing this is equivalent to \(T\) such that

\[
1 = f'(g(M)T_C)g'(M)T_C
\]

\(^{34}\)Jackson (2008) provides a minor introduction
The assumption $B_I = g(M)$ is likely not to be true on a microfounded level. However, the approximation is often a good one. The averages of both models will be approximately the same when (i) $L$ is large and/or (ii) interaction is sufficiently rich.

To see the application to networks, consider our fixed network model. Matching rates depend on one's number of employed friends, which is changing over time. For instance, the probability of losing an employed friend is $\lambda$. A particular individual may lose no employed friends or all of his employed friends. However, I deterministically impose that every unemployed worker $i$ loses exactly $\lambda d_i^E$ employed friends.

---

The basis for the approximation is the Bogoliubov Inequality. The idea is that deviations of the approximated model from the actual model are bounded above by the amount of free-energy in the system. In thermodynamic equilibrium the entropy is (constrained) maximized equivalently minimizing the free-energy.