Privacy Preserving Signals*

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Abstract

A signal is *privacy-preserving* with respect to a collection of *privacy sets*, if the posterior probability assigned to every privacy set remains unchanged conditional on any signal realization. We characterize the privacy-preserving signals for arbitrary state space and arbitrary privacy sets. A signal is privacy-preserving if and only if it is a garbling of a *reordered quantile signal*. These signals are equivalent to couplings, which in turn lead to a characterization of optimal privacy-preserving signals as solutions to an optimal transport problem. We discuss the economic implications of our characterization for statistical discrimination, the revelation of sensitive information in auctions, monopoly pricing, and price discrimination.

Keywords: Privacy-preserving signal, privacy sets, independence, reordered quantile signal, protected characteristics.

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1 Introduction

In many economic settings, there are constraints on what information can be used or revealed: Characteristics such as race, gender, and sexual orientation are protected in many contexts, and the information that can be revealed about them is limited due to legal, regulatory, or social norms. Motivated by this, we study the set of signals (Blackwell experiments) which are constrained to not reveal certain information.

For example, consider the case of a bank determining whether to grant a loan to an individual. In making this decision, the bank benefits from using an individual's characteristics to predict whether they will default. However, the Equal Credit Opportunity Act prohibits discrimination against loan applicants on the basis of their "protected characteristics", such as race, gender, or age. As a result, the bank is legally required to ensure that its loan decisions are not influenced by these protected characteristics. In other words, the information used by the bank in making loan decisions cannot be based on these characteristics.

This paper presents a framework for understanding information disclosure in situations where certain aspects of the state of the world must be kept private, or equivalently, where decisions must be taken independent of certain characteristics. Following Blackwell (1953), we model information as a signal about an abstract state of the world $\omega \in \Omega$. To capture a notion of protected information, we introduce a collection of events called *privacy sets*, which represent the information that cannot be disclosed. For instance, in the context of a bank loan, the privacy sets would include all the protected characteristics. We define a signal as *privacy-preserving* if, for any signal realization, the posterior probability of any privacy set remains unchanged and equals its prior probability. In other words, a privacy-preserving signal does not reveal any information about events that belong to the privacy sets.

We characterize all privacy-preserving signals. The privacy sets can always be represented as a random variable θ defined on the same state space Ω, so that a signal s is privacy-preserving if and only if it is independent of θ. In the bank loan example, the random variable θ would indicate an applicant's protected characteristics. A privacy-preserving signal s can be informative about the applicant in many aspects, including their default probabilities, but must be independent of the applicant's protected characteristics θ.

Our first main result characterizes all privacy-preserving signals: For any one-dimensional statistic $\phi : \Omega \to \mathbb{R}$ of the state ω, we define the *ϕ-quantile signal* as the signal that reveals the

empirical quantile of ϕ conditional on θ, plus potentially some noises when it has an atom. This signal is privacy-preserving as it is uniformly distributed on $[0,1]$ for every value of θ. A *reordered ϕ-quantile signal* is a signal obtained by further (possibly randomly) reordering the unit interval for each realization of θ while preserving its distribution.

Theorem 1 establishes that for any statistic ϕ such that (ϕ, θ) fully reveals the state ω, a signal is privacy-preserving if and only if it is Blackwell-dominated by a reordered ϕ-quantile signal. As a result, the set of *all* privacy-preserving signals can essentially be generated by a *single* ϕ-quantile signal, via garbling and reordering. Moreover, these reordered ϕ-quantile signals are Blackwell undominated among all privacy-preserving signals. Therefore, the reordered ϕ-quantile signals are exactly the frontier of all privacy-preserving signals in Blackwell's sense.

Although privacy-preserving signals in general do not have a single Blackwell-maximum, when only the posterior *means* of a one-dimensional statistic ϕ are relevant, Theorem 3 shows that the distribution of posterior means induced by the ϕ-quantile signal is a mean-preserving spread of the distribution of posterior means induced by any other privacy-preserving signal. Consequently, in settings where the only economically relevant variables are the posterior means (e.g., when a decision-maker has a payoff that is affine in a statistic ϕ), *every* privacy-preserving signal leads to a lower expected payoff than the ϕ-quantile signal. Thus, regardless of how complex the privacy sets are, these privacy constraints can be completely summarized by a majorization constraint when only posterior means are relevant.

Our results extend to the setting where privacy-preserving is defined *conditional* on another given random variable y. A signal is *conditionally privacy-preserving* if it is independent of θ *conditional on* y. This extension allows us to analyze settings where signals (or decisions) are allowed to reveal some information about the protected characteristics, as long as it based on information contained in y. For example, banks cannot directly base discriminatory loan decisions on race, but can make predictions using applicants' credit history, which is deemed "materially relevant", even though credit histories may be correlated with race.

Having characterized the set of privacy-preserving signals, we then explore how to optimize over this set. Consider a decision-maker who chooses an action $a \in A$ to maximize their payoff $u(\omega, a)$, after observing a privacy-preserving signal. Our results imply that it is without loss to restrict attention to reordered quantile signals. Proposition 2 shows that the resulting optimization problem is equivalent to a an optimal transport problem. When

payoffs depends only on a single dimensional statistic ϕ and on θ, we fully characterize the optimal privacy-preserving signals for a wide class of decision and persuasion problems, including when the payoff is single-crossing in (ϕ, a), or linear in ϕ, when the statistic ϕ takes binary values, and when actions are binary. This characterization covers a broad range of decision problems that appeared in the literature.

To illustrate the usefulness of our mathematical results, we apply them to various economic contexts. First, we consider statistical discrimination and fairness in algorithm design. In legal contexts, the notion of *disparate impact* describes the situation where different groups experience different outcomes even if the underlying policy are not explicitly based on group identities. Our results lead to a characterization of optimal signals for a decision maker that do not create disparate impacts, which generalizes existing results in the algorithmic fairness literature. Furthermore, we lay out an optimal and detail-free procedure for regulating statistical discrimination.

The second application considers disclosure in ad auctions without violating users' privacy. A publisher runs a second price auction to sell targeted audiences to multiple advertisers. The publisher has rich information about the targeted audience, and can selectively disclose some information to the advertisers. However, some information about the audience cannot, or should not be disclosed, such as religious beliefs and sexual orientation. Theorem 3, together with the results of Bergemann, Heumann, Morris, Sorokin, and Winter (2022), provides a characterization of optimal privacy-preserving information disclosure for the auctioneer.

The third application considers a monopolistic pricing setting where both the buyer and the seller receive a signal about the buyer's value. For arbitrarily fixed signal of the seller, we characterize the buyer's signals that maximize their surplus. We prove that the buyer-optimal signal must be privacy-preserving with respect to the seller's signal, so that the seller cannot make any inferences about the buyer's posterior expected value. Theorem 3, together with the arguments developed by Roesler and Szentes (2017), then characterizes the buyer-optimal signal, as well as all feasible welfare outcomes.

The fourth application considers price discrimination and market segmentation in the spirit of Bergemann, Brooks and Morris (2015). In a setting where a monopolist is able to segment consumers based on their values, we consider a situation where consumers with different protected characteristics (e.g., gender and race), must face the same distribution of prices, even though the monopolist engages in third-degree price discrimination. Solving the

optimal transport problem derived in Proposition 2 allows us compute the market segmentation that maximizes the monopolist's revenue while preventing price discrimination based on consumers' protected characteristics.

Related Literature We follow the extremal approach of Blackwell (1953) and model information as signals about an underlying state. We generalize Blackwell's characterization of feasible signals, by characterizing all feasible signals that do not reveal information about a given collection of events. While Blackwell shows that a signal is feasible if and only if it is dominated by the signal which fully reveals the state, we show that a signal is feasible and privacy-preserving if and only if it is dominated by a "reordered quantile signal".

In the literature on algorithmic fairness, one of the most common criteria for fairness requires the decisions to be statistically independent of protected characteristics (see, e.g., Calders, Kamiran and Pechenizkiy 2009; Hardt, Price and Srebro 2016; Corbett-Davies, Pierson, Feller, Goel and Huq 2017). These papers characterize optimal fair algorithms for decision problems with binary actions, binary states, or specific payoff structures.

He, Sandomirskiy and Tamuz (2023) introduce a Blackwell approach, and use tools from mathematical tomography, to characterize the optimal privacy-preserving signals for all decision problems with a binary state and general payoffs that do not depend on protected characteristics.[1] Our results unify and generalize their findings and findings from the computer science literature to more than two actions, more than two states, and general payoffs.

Liang, Lu and Mu (2023) and Doval and Smolin (2023) also consider fairness and algorithm design. They adopt different notions of fairness and characterizes the Pareto frontier in terms of payoffs different groups. Eilat, Eliaz and Mu (2021) consider mechanism design problems under privacy constraints where a reduction in privacy is measured by the Kullback-Leibler divergence between the designer's prior and posterior belief.

Privacy-preserving signals also relate to the notion of *belief-invariant* Bayes correlated equilibria (Forges 1993, 2006; Liu 2015) in games with incomplete information, which requires the action recommendations received by each player not to affect the player's belief about the unknown state and other players' types. While both notions are stated in terms of statistical

[1] He et al. (2023) and our paper use different notions of Blackwell dominance, and we provide a detailed discussion in §6.1. Less directly related to our work, He et al. also introduce the multi-agent concept of "private private" signals, which are signals structures such that no agent's signal realization reveals information about the signal realization of other agents.

independence, we are interested in the informational content that can be provided subject to privacy constraints, rather than the strategic implications of correlating actions in games.

2 Model

States Throughout the paper, we fix the probability space $(\Omega \times [0,1], \mathcal{F} \times \mathcal{B}, \mathbb{P})$ where $(\Omega, \mathcal{F})$ is a standard Borel space, $\mathcal{B}$ is the Borel sigma algebra on $[0,1]$, and $\mathbb{P}$ is the product measure induced by some probability measure over Ω and the Lebesgue measure over $[0,1]$. We denote the outcome by

$$(\omega, r) \in \Omega \times [0,1].$$

The first component of the outcome ω is the state, which captures all economically relevant information, the second part of the outcome r is the source of additional randomization that is independent of the state. We impose no further restrictions on the state ω and it might be multi-dimensional to capture all relevant characteristics of an economic agent such as income, age, gender, race, address, etc.

Signals A signal is a random variable $s : \Omega \times [0,1] \to S$.[2] This definition of a signal is completely equivalent to the definition of experiments given in Blackwell (1953).[3] Intuitively, if the signal is random conditional on the state ω, this randomization is achieved by explicitly conditioning on the randomization device r. For any signal s, let $\mathbb{P}[\cdot \mid s]$ denote the conditional distribution given s.[4]

[2] If one focuses on the case of countably many signal realizations, then this definition of a signal is equivalent to the one in Green and Stokey (2022). To see this, consider the partition of $[0,1]$ induced by the signal realization $\{(\omega, r) \in \Omega \times [0,1] : s(\omega, r) = \hat{s}\}_{\hat{s} \in S}$. The posterior belief after observing signal realization $\hat{s}$ is then the conditional distribution of $\mathbb{P}$ over Ω given the event $\{(\omega, r) \in \Omega \times [0,1] : s(\omega, r) = \hat{s}\}$. Conversely, for any countable partition of $\Omega \times [0,1]$, one can define a random variable s as an indicator function of its partitional elements.

[3] To see this, note that for every signal we can define a (essentially) unique Blackwell experiment by $(S, \mathbb{P}[s \in \cdot | \omega]_{\omega \in \Omega})$. Conversely, as the probability space $(\Omega \times [0,1], \mathcal{F} \times \mathcal{B}, \mathbb{P})$ is standard, for every Blackwell experiment we can construct a random variable s with the correct joint distribution (see von Neumann 1932; Rokhlin 1952).

[4] More precisely, since $(\Omega \times [0,1], \mathcal{F} \times \mathcal{B})$ is a standard Borel space, a regular version of the conditional expectation given the σ-algebra generated by s exists, which we denote by $\mathbb{P}[\cdot \mid s]$ (Çinlar 2010, Theorem 2.7, pp. 151).

Definition 1 (Blackwell Dominance)**.** A signal s is Blackwell dominated by a signal $\tilde{s}$, denoted by $s \preceq \tilde{s}$, if for every measurable function $u : \Omega \times A \to \mathbb{R}$ and action set A

$$\mathbb{E}\left[\sup_{a \in A} \mathbb{E}\left[u(\omega, a) \mid s\right]\right] \leq \mathbb{E}\left[\sup_{a \in A} \mathbb{E}\left[u(\omega, a) \mid \tilde{s}\right]\right].$$

Privacy-Preserving Signals We are interested in signals about the state—or decisions that are made conditional on signal realizations—that do not reveal certain type of information. For example, in some economic context the signals might be required to preserve some notion of privacy; or actions taken by a firm might not be allowed to condition on protected characteristics, such as race and gender. For a collection of *privacy-sets*

$$\mathcal{P} \subseteq \mathcal{F}$$

that are closed under finite intersections,[5] each $P \in \mathcal{P}$ defines an event that has be to "kept private" and no information about it can be revealed. The number of privacy sets is allowed to be finite, countably infinite, or uncountable. Privacy sets encode what information (or characteristics) are protected in a given context, such as a person's race, gender, or age.

Definition 2 (Privacy-Preserving Signals)**.** A signal s is *privacy-preserving* (with respect to $\mathcal{P}$) if the prior and posterior probability of the state being in any privacy set coincide, i.e., for all $P \in \mathcal{P}$, a.s.

$$\mathbb{P}[\omega \in P \mid s] = \mathbb{P}[\omega \in P]. \tag{1}$$

Our main result will establish a characterization of all Blackwell undominated privacy-preserving signals.

Remark 1 (Privacy and Discrimination)**.** One can think of the signal s as a decision taken which affects the outcomes of an economic agent (e.g., whether or not to grant a loan, or what interest rate to charge). The privacy sets encode protected characteristics based on which discrimination is illegal, such as race, age or gender. The set of privacy-preserving signals then corresponds to the set of decision rules that are "non-discriminatory", in the sense that

[5]This requirement is equivalent to saying that, if two sets P_1 and P_2 are events that need to be kept private, then their intersection $P_1 \cap P_2$ has to be kept private too. For example, if a privacy-preserving signal is not allowed to reveal information about a person's race or gender, then it is also not allowed to reveal information about whether a person is a white male or a non-white female.

the decision is independent of the agent's protected characteristics. This connection between discrimination and privacy goes beyond the formal level: Often people want to keep certain information private, such as their sexual orientation, exactly because they fear discrimination.

3 Characterization of Privacy-Preserving Signals

We first note that the privacy sets $\mathcal{P}$ can always be encoded into a random variable θ, which will lead to more succinct notation.

Lemma 1. *There exists a random variable $\theta : \Omega \to \Theta$ such that the following are equivalent: (i) The signal s is privacy-preserving, (ii) s is independent of θ.*

For finite $\mathcal{P}$ we can define $\theta = (\theta_P)_{P \in \mathcal{P}} \in \Theta = \{0,1\}^{|\mathcal{P}|}$ to be a vector of indicators indicating which privacy sets the state belongs to, i.e., $\theta_P(\omega) = \mathbf{1}\{\omega \in P\}$. For example, if the privacy sets divide the population into female/male and white/non-white, then the realization of θ would correspond to non-white females; non-white males; white females; and white males. The proof for the infinite case is slightly more involved and provided in the Appendix.

We next establish that the set of privacy-preserving signals is closed with respect to the Blackwell order.

Lemma 2. *Every signal Blackwell-dominated by a privacy-preserving signal is privacy-preserving.*

Lemma 2 follows from the following observation. Consider the decision problem where the decision-maker bets on the probability of a privacy set $P \in \mathcal{P}$ and faces a quadratic penalty, i.e., $A = [0,1]$ and

$$u(\omega, a) := -\left(\mathbf{1}\{\omega \in P\} - a\right)^2 .$$

For each signal realization $\hat{s}$, the unique optimal action equals $\mathbb{P}[\omega \in P \mid \hat{s}]$, which by definition is not updated and equals $\mathbb{P}[\omega \in P]$. Thus, a signal is privacy-preserving if and only if it does not increase the value in any such problem. Since any Blackwell dominated signal leads to a (weakly) lower payoff in any, and thus, these specific decision problems, it follows that a signal dominated by a privacy-preserving signal is itself privacy-preserving.

Conditionally Revealing Signals We next consider signals that reveal the state to an outside observer who knows the characteristic θ.

Definition 3 (Conditionally Revealing Signals)**.** A signal s is *conditionally revealing* if observing s and θ reveals ω.[6]

The definition of a conditionally revealing signal does not have direct implications for the informativeness of the signals s, as the characteristic θ is unknown. Moreover, a conditionally revealing signal may not be privacy-preserving.[7] However, our next result establishes that every privacy-preserving signal is less informative than a conditionally revealing, privacy-preserving signal.

Proposition 1. *A signal is privacy-preserving if and only if it is Blackwell dominated by some conditionally revealing privacy-preserving signal.*

The "if" part follows immediately from Lemma 2. To get an intuition for the "only if" part, suppose that Θ is finite and consider an arbitrary privacy-preserving signal s. We explicitly construct a signal s' that Blackwell dominates s: First, take a vector of random variables $(t_{\hat{\theta}})_{\hat{\theta}\in\Theta} : \Omega \times [0,1] \to \Omega^{|\Theta|}$ such that

(i) $(t_{\hat{\theta}})_{\hat{\theta}\in\Theta}$ are independent;

(ii) $t_{\hat{\theta}}$ has the same distribution as ω conditional on $\theta = \hat{\theta}$ and s, i.e.,

$$\mathbb{P}[\omega \in A \mid \theta = \hat{\theta}, s] = \mathbb{P}[t_{\hat{\theta}} \in A \mid s],$$

for all $\hat{\theta} \in \Theta$ and for any measurable set $A \subseteq \Omega$.

Define the new signal $s' = (s, t')$ that reveals the original signal s and in addition a signal $(t'_{\hat{\theta}})_{\hat{\theta}\in\Theta}$ defined as

$$t'_{\hat{\theta}}(\omega, r) = \begin{cases} \omega, & \text{if } \theta(\omega) = \hat{\theta} \\ t_{\hat{\theta}}(\omega, r), & \text{otherwise} \end{cases}.$$

The signal $s' = (s, t')$ dominates the original signal s as it reveals more information. It is conditionally revealing as one can read off ω simply from the vector $(t'_{\hat{\theta}})_{\hat{\theta}\in\Theta}$ if one knows the realization of θ. Finally, it is privacy-preserving as the distribution of t' conditional on s is—by construction—the same for every realization of the characteristic θ.[8]

[6] Or more formally, for any $A \in \mathcal{F}$, $\mathbb{P}[\omega \in A \mid s, \theta] \in \{0, 1\}$ a.s..

[7] For instance, the fully-revealing signal $s(\omega, r) = \omega$ is conditionally revealing, but not privacy-preserving.

[8] This construction breaks down in the case where the Θ is uncountable as it is impossible to construct a random vector that involves uncountably many independent random variables. The general proof in Lemma A.2 the Appendix thus uses a different method.

Quantile Signals According to Proposition 1, a conditionally revealing privacy-preserving signal always exists, and any privacy-preserving signal is a garbling of some conditionally revealing privacy-preserving signal. We next explicitly construct a conditionally revealing privacy-preserving signal, and show that it can further generate *all* such signals.

Consider any random variable $\phi : \Omega \to \mathbb{R}$, which summarizes the state into a single dimensional statistic. Let

$$F_\phi(z \mid \hat{\theta}) := \mathbb{P}[\phi(\omega) \leq z \mid \hat{\theta}]$$

be the distribution of ϕ conditional on $\hat{\theta}$, $F_\phi^-(\cdot \mid \hat{\theta})$ its left limit, and $F_\phi^{-1}(\cdot \mid \hat{\theta})$ its inverse.[9]

Definition 4 (Quantile Signal)**.** Define the *ϕ-quantile signal* as

$$q_\phi := rF_\phi(\phi \mid \theta) + (1-r)F_\phi^-(\phi \mid \theta)\,. \tag{2}$$

If $F_\phi(\cdot \mid \hat{\theta})$ is continuous for all $\hat{\theta} \in \Theta$, then q_ϕ is the empirical quantile of the realization of ϕ. For example, $q_\phi = 0.3$ reveals that ϕ takes a lower value than its current realization 30% percent of the time for each realization of θ. In the continuous case, $q_\phi = F_\phi(\phi \mid \theta)$ is uniformly distributed, and thus is independent of θ. The randomization in q_ϕ ensures that this property generalizes to discontinuous F_ϕ, as stated below.

Lemma 3. *The ϕ-quantile signal q_ϕ is privacy-preserving.*

The signal q_ϕ allows one to identify the realization of ϕ if θ is known. While q_ϕ may not be conditionally revealing in general, whenever ϕ is invertible, with its inverse ϕ^{-1} being measurable,[10] the signal q_ϕ is conditionally revealing, since $\omega = \phi^{-1}(F_\phi^{-1}(q_\phi \mid \theta))$.

Reordered Quantile Signals A measurable function $M : [0,1] \to [0,1]$ is a (Lebesgue) measure-preserving transformation if $\int_0^1 \mathbf{1}\{M(z) \leq x\}\,\mathrm{d}z = x$ for all $x \in [0,1]$. Two signals s, s' are said to be Blackwell equivalent, denoted by $s \sim s'$, if $s \preceq s'$ and $s' \preceq s$.

Definition 5 (Reordered Quantile Signal)**.** For any random variable $\phi : \Omega \to \mathbb{R}$, signal s is a reordered ϕ-quantile signal if there exists a family of measure-preserving transformations $\{M_{\hat{\theta}}\}_{\hat{\theta} \in \Theta}$ such that $M_\theta(s) \sim q_\phi$.

[9] Formally, for any CDF F, $F^-(z) := \lim_{\varepsilon \searrow 0} F(z-\varepsilon)$, for all $z \in \mathbb{R}$, while $F^{-1}(q) := \inf\{z \in \mathbb{R} : F(z) \geq q\}$, for all $q \in [0,1]$.

[10] Such a function always exists, since $(\Omega, \mathcal{F})$ is a standard Borel space and there is isomorphic to a subset of $\mathbb{R}$ with the Borel σ-algebra (see von Neumann 1932; Rokhlin 1952).

For every ϕ-quantile signal q_ϕ and every family $\{M_{\hat{\theta}}\}_{\hat{\theta}\in\Theta}$ of measure-preserving transformations, there exists (up to Blackwell equivalence) a unique reordering of q_ϕ, which can be explicitly constructed.[11] By definition, every reordered ϕ-quantile signal is privacy-preserving. Furthermore, as knowing θ and s reveals $q_\phi \sim M_\theta(s)$, it follows that s is conditionally revealing whenever q_ϕ is conditionally revealing (e.g., when ϕ is invertible).

We now present our main result, which characterizes the set of privacy-preserving signals.

Theorem 1 (Characterization of Privacy-Preserving Signals)**.** *Fix any conditionally revealing quantile signal $q^\star$.*

(i) A signal is privacy-preserving if and only if it is Blackwell dominated by some reordering of $q^\star$.

(ii) Every reordering of $q^\star$ is Blackwell undominated among privacy-preserving signals.

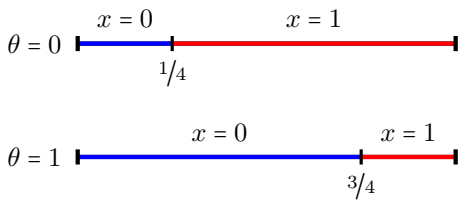
Part (i) of Theorem 1 establishes that the set of *all* privacy-preserving signals can be generated from a *single* conditionally revealing quantile signal $q^\star$, via reordering and garbling.[12] In particular, it is without loss to optimize only over reorderings of $q^\star$ instead of all privacy-preserving signals, as one can always ignore additional information. As we discuss in §4, this is a drastic simplification as it reduces the problem of optimizing over privacy-preserving signals to an optimization problem over measure-preserving transformations. Part (ii) establishes that every reordering of $q^\star$ is undominated. Thus, without imposing structures on the decision problem, no further restriction of the set of privacy-preserving signals is without loss.

Example 1 (The Single-Dimensional Case)**.** Suppose that $\Omega = X \times \Theta$, where $x \in X \subseteq \mathbb{R}$ denotes an economic outcome (e.g. insurance risk, income, etc), suppose that the privacy sets $\mathcal{P}$ equal the σ-algebra generated by the projection $(x, \theta) \mapsto \theta$, and suppose that the distribution $F(\cdot \mid \theta)$ of x conditional on θ is continuous. Then, the quantile signal

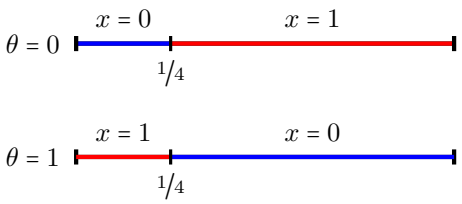
$$q = F(x \mid \theta)$$

[11] More details for the construction can be found in the Online Appendix.

[12] Although some quantile signals are conditionally revealing, not all conditionally revealing privacy-preserving signals are equivalent to a quantile signal (e.g., the signal described by Figure 3b in § 5.4 is conditionally revealing but is not equivalent to a ϕ-quantile signal for any ϕ). Nonetheless, part (i) of Theorem 1 ensures that any conditionally revealing privacy-preserving signal is a *reordering* of some conditionally revealing quantile signal $q^\star$. Intuitively, this is because a statistic ϕ does not allow for randomization but a reordering does (see the Online Appendix for more details about this randomization).



(a) Signal 1: $M_0(s) = M_1(s) = s$



(b) Signal 2: $M_0(s) = s; M_1(s) = 1 - s$

Figure 1: Two examples of reordered quantile signals. The signal realizations are uniformly distributed over $[0, 1]$ and correspond to different points on the line. The color indicates which state is revealed by a given signal realization conditional on θ.

is conditionally revealing, as it reveals (x, θ) to an outside observer who knows θ. By Theorem 1, every privacy-preserving signal is thus Blackwell dominated by a reordering of q.

Example 2 (Undominated Privacy-Preserving Signals)**.** We next present an example illustrating that a privacy-preserving signal might seem to reveal little information but is in fact undominated. Suppose that $\Omega = X \times \Theta = \{0, 1\}^2$ and the signal cannot reveal any information about θ.[13] Suppose that $\mathbb{P}[\theta = 1] = 1/2$ and that $\mathbb{P}[x = 1 \mid \theta = 0] = 3/4$ and $\mathbb{P}[x = 1 \mid \theta = 1] = 1/4$.

Let $\phi^\star(x, \theta) = x$. The $\phi^\star$-quantile signal can be represented by assigning to each point in the unit interval the value of x they reveal given θ, see Figure 1a. For example, a signal realization of 0.5 reveals that $x = 1$ if $\theta = 0$ and that $x = 0$ if $\theta = 1$. This signal thus perfectly reveals that $x = 0$ for realizations $\hat{s} \leq 1/4$, and that $x = 1$ for realizations $\hat{s} \geq 3/4$, while the posterior over X remains uniform and assigns equal probability to $x = 1$ and $x = 0$ for realizations $\hat{s} \in (1/4, 3/4)$.

Meanwhile, the reordered $\phi^\star$-quantile signal defined by the measure-preserving transformations $M_0(s) = s$ and $M_1(s) = 1 - s$ can be represented by Figure 1b. This signal induces a uniform posterior assigning equal probability to $x = 1$ and $x = 0$ for any realization of s.

According to Theorem 1, these two signals are both Blackwell undominated among all privacy-preserving signals. In particular, even though the reordered quantile signal given by Figure 1b does not reveal any information about x or θ separately, it is still Blackwell undominated, as it perfectly reveals the *correlation* between x and θ, so that a decision maker who cares only about whether $x = \theta$ can have their payoff maximized by observing this signal.

[13] That is the privacy sets are given by $\mathcal{P} = \{\{(1, 1), (0, 1)\}, \{(1, 0), (0, 0)\}\}$

3.1 Privacy-Preserving Signals for an Arbitrary Statistic

Theorem 1 follows from a more general result, as stated below. Consider any one-dimensional statistic $\phi : \Omega \to \mathbb{R}$ of the state ω. Say that a signal s is a signal for (ϕ, θ) if s is measurable with respect to (ϕ, θ, r). Intuitively, a signal for (ϕ, θ) only reveals information about (ϕ, θ) and nothing else.

Among all signals for (ϕ, θ), we may analogously define the Blackwell order. For any pair of signals $s, \tilde{s}$ for (ϕ, θ), s is said to be Blackwell dominated by $\tilde{s}$ *in terms of* (ϕ, θ) if for any action set A and any function $u : \Omega \times A \to \mathbb{R}$ that is measurable with respect to (ϕ, θ),

$$\mathbb{E}\left[\sup_{a \in A} \mathbb{E}[u(\omega, a) \mid s]\right] \leq \mathbb{E}\left[\sup_{a \in A} \mathbb{E}[u(\omega, s) \mid \tilde{s}]\right].$$

In other words, the Blackwell order can be restricted to signals for (ϕ, θ) by comparing the information about (ϕ, θ) given by these signals.

Theorem 2. *Consider any random variable $\phi : \Omega \to \mathbb{R}$.*

(i) A signal for (ϕ, θ) is privacy-preserving if and only if it is Blackwell dominated in terms of (ϕ, θ) by some reordered ϕ-quantile signal .

(ii) Every reordered ϕ-quantile signal is Blackwell undominated in terms of (ϕ, θ) among signals for (ϕ, θ) that are privacy-preserving.

Note that, for any statistic $\phi : \Omega \to \mathbb{R}$ such that the ϕ-quantile signal is conditionally revealing, the state ω can be recovered by the realization of (ϕ, θ). Therefore, every signal s is Blackwell equivalent to a signal for (ϕ, θ), and signal $\tilde{s}$ Blackwell dominates signal s in terms of (ϕ, θ) if and only if $\tilde{s}$ Blackwell dominates s. As a result, Theorem 1 follows from Theorem 2 by taking ϕ to be an invertible statistic whose inverse ϕ^{-1} is measurable.

3.2 Distributions of Posterior Means

While Theorem 1 established that privacy-preserving signals do not have a single Blackwell maximum, and Example 2 illustrates that undominated privacy-preserving signals could reveal no information about each component of the state, the characterization of privacy-preserving signals can be further sharpened if only the posterior mean of a statistic

$$\phi : \Omega \to \mathbb{R}$$

is payoff relevant. This assumption is natural in many economic settings. For example, in the context of a bank loan discussed in §1, it corresponds to the assumption that the bank's preference depends only on the default probability, but not the race or gender of an applicant.

It is well-known that a CDF G is a distribution of posterior means $\mathbb{E}[\phi \mid s]$ under some signal s if and only if G is a mean-preserving contraction of the prior distribution $F_\phi(z) \coloneqq \mathbb{P}[\phi(\omega) \leq z]$ (see e.g., Hardy, Littlewood and Pólya 1929; Strassen 1965). However, not every mean-preserving contraction of F_ϕ can be the distribution of posterior mean under a privacy-preserving signal. For example, if ϕ and θ are correlated, then F_ϕ could never be the distribution of posterior means.

Let $\overline{F}_\phi$ be of posterior means $\mathbb{E}[\phi \mid q_\phi]$ induced by the ϕ-quantile signal q_ϕ. Namely,

$$\overline{F}_\phi(z) \coloneqq \inf\left\{y \in [0,1] : \mathbb{E}\left[F_\phi^{-1}(y \mid \theta)\right] \geq z\right\},$$

for all $z \in \mathbb{R}$, where the expectation is taken over θ.

Theorem 3 (Distributions of Posterior Means)**.** *For any statistic $\phi : \Omega \to \mathbb{R}$, a CDF G is the distribution of posterior means $\mathbb{E}[\phi \mid s]$ induced by some privacy-preserving signal s if and only if G is a mean-preserving contraction of $\overline{F}_\phi$.*

According to Theorem 3, for any privacy-preserving signal s, $\mathbb{E}[\phi \mid s]$ must be less dispersed than $\mathbb{E}[\phi \mid q_\phi]$ under the convex order. Consequently, in decision problems where only posterior means of ϕ are relevant (e.g., when a decision maker's payoff is affine in some statistic ϕ), the ϕ-quantile signal gives the highest expected payoff.

Theorem 3 reduces the set of privacy-preserving signals to mean-preserving contractions of $\overline{F}_\phi$ in settings where only posterior means of a statistic ϕ are relevant. In particular, the privacy constraints, however complex they are, are entirely summarized by the mean-preserving-contraction upper bound $\overline{F}_\phi$. Moreover, the mean-preserving contraction structure, together with recent results in Kleiner, Moldovanu and Strack (2021) and Arieli, Babichenko, Smorodinsky and Yamashita (2023), who characterize the extreme points of this set, allows one to focus on signals that either fully reveal the q_ϕ-quantile signal, or pool its realizations into at most two mass points on an interval.

Remark 2 (Conditionally Privacy-Preserving Signals)**.** In many economic applications, a signal is only required to be privacy-preserving conditional on certain information. For

example, if some signal $y : \Omega \times [0,1] \to Y$ is already publicly available, then it would be natural to only restrict signals to not reveal *additional* information. This can be captured by defining a signal to be *conditionally privacy-preserving*, if for all privacy set $P \in \mathcal{P}$, a.s.,

$$\mathbb{P}[\omega \in P \mid s, y] = \mathbb{P}[\omega \in P \mid y]\,.$$

Mathematically, Theorem 1 and Theorem 3 immediately extend to this case by simply applying them for each realization of y (see the online Appendix).

4 Optimizing over Privacy-Preserving Signals

4.1 Decision Problems

In this section, we apply Theorem 1 through Theorem 3 to characterize optimal privacy-preserving signals for a decision-maker who takes an action after observing the signal. Proofs for this section and the next section can be found in the Online Appendix. For the ease of exposition, we assume finitely many privacy sets so that θ takes finitely many values: $\Theta = \{1, \ldots, J\}$. Consider a Bayesian decision problem (u, A): After observing the signal s, the decision-maker chooses an action $a \in A$ to maximize expected payoff

$$\mathbb{E}\left[u(\omega, a) \mid s\right]$$

where $u(\omega, a)$ denotes the decision-maker's ex-post payoff when the state is ω and the action is a. From Theorem 1 and Blackwell's theorem, it follows that there always exists an optimal privacy-preserving signal that is a reordering of some conditionally revealing quantile signal. For any reordering of a conditionally revealing quantile signal s, denote by

$$\tilde{\omega}(s) := (\tilde{\omega}_1(s), \ldots, \tilde{\omega}_J(s))$$

the vector of states revealed by the signal s, i.e., $\tilde{\omega}_j(\hat{s})$ is the state revealed by signal realization $\hat{s}$ if $\theta = j$. The marginals of $\tilde{\omega}$ are fixed by $\tilde{\omega}_j \sim \mathbb{P}\left[\cdot \mid \theta = j\right]$. Let $\mathcal{D}$ be the set of joint distributions ρ on Ω^J such that the marginal of the j-th coordinate is given by $\mathbb{P}[\cdot \mid \theta_j]$. Thus, any such signal corresponds to a coupling of states for different realizations of θ:

Lemma 4. *A distribution $\rho \in \Delta(\Omega^J)$ is the joint distribution of $\tilde{\omega}(s)$ for some reordering of a conditionally revealing quantile signal s if and only if $\rho \in \mathcal{D}$.*

With Lemma 4, we can now characterize the optimal privacy-preserving signals for the decision-maker. Let $V : \Omega^J \to \mathbb{R}$ be defined as

$$V(\omega_1, \ldots, \omega_J) := \sup_{a \in A} \left(\sum_{j=1}^{J} u(\omega_j, a) \mathbb{P}[\theta = j] \right), \tag{3}$$

for all $(\omega_j)_{j=1}^J \in \Omega^J$. The value of $V(\omega_1, \ldots, \omega_J)$ can be interpreted as the decision-maker's indirect utility after learning that the state equals ω_j when $\theta = \theta_j$, without any further information about θ. We then have the following characterization:

Proposition 2 (Optimal Privacy-Preserving Signal)**.** *The decision-maker's optimal value among all privacy-preserving signals is given by*

$$\sup_{s : s \perp \theta} \mathbb{E}\left[\sup_{a \in A} \mathbb{E}[u(\omega, a) \mid s] \right] = \sup_{\rho \in \mathcal{D}} \int_{\Omega^J} V(\omega_1, \ldots, \omega_J) \, \mathrm{d}\rho, \tag{4}$$

Moreover, fix any conditionally revealing quantile signal $q^\star$, every optimal privacy-preserving signal must be Blackwell-equivalent to a reordering s of $q^\star$ such that the distribution of $\tilde{\omega}(s)$ is a solution of (4).

The optimization problem (4) is a multi-marginal optimal transport problem. The existence of solutions can be guaranteed if Ω is compact and if V is upper-semicontinuous, which we assume henceforth. While the optimal transport problem (4) may still be complex, the structure of this problem leads to explicit characterizations of optimal privacy-preserving signals for a wide range of economically relevant decision problems. Indeed, as we show below, a closed form solution can be obtained within a certain class of decision problems.

Supermodular Payoffs Many applications naturally admit supermodularity, which means that the objective depends on the state only through a single dimensional statistic ϕ and higher (or lower) actions are optimal for higher values of the statistic. For example, if $\phi(\omega)$ measures the probability of a borrower repaying a loan and a is the interest rate a bank requires from a borrower, then it would be natural to assume that the bank wants to charge a lower interest rate to those borrowers who are more likely to repay.

Proposition 3 (Supermodular Payoffs)**.** *Suppose that A is a totally ordered set, and that there exists a statistic $\phi : \Omega \to \mathbb{R}$ such that for all $\omega \in \Omega$ and for all $a \in A$,*

$$u(\omega, a) = h(\phi(\omega), \theta(\omega), a)$$

for some measurable $h : \mathbb{R} \times \Theta \times A \to \mathbb{R}$ that is supermodular in (ϕ, a). Then the ϕ-quantile signal is optimal.

According to Proposition 3, for any decision-maker who chooses an action a to maximize a payoff $h(\phi, \theta, a)$ that is supermodular in (ϕ, a), the ϕ-quantile signal is optimal. For example, for any prior, statistic ϕ, and collection of privacy sets $\mathcal{P}$, the loss $|\phi(\omega) - a|^p$ for $p > 1$ is minimized by revealing the ϕ-quantile signal.[14]

Binary Actions Another important special case is when the decision is only between two actions, e.g., a bank decides whether to extend a loan at an exogenously fixed interest rate.

Proposition 4 (Binary Actions)**.** *Suppose that $A = \{0, 1\}$, then, the ϕ-quantile signal is optimal, where $\phi(\omega) = u(\omega, 1) - u(\omega, 0)$.*

Proposition 4 follows immediately from Proposition 3, since the ϕ-quantile signal is optimal as $u(\omega, a) = \phi(\omega)\, a + u(\omega, 0)$ is supermodular in (ϕ, a). Proposition 4 thus completely solves the decision problem for binary actions, arbitrary payoffs and arbitrary privacy sets.

Separable Problems and Binary States In the meantime, Theorem 3 also implies that for a specific class of "separable" decision problems, the quantile signal is optimal.

Proposition 5 (Separable Problems and Binary States)**.** *Suppose there exists $\phi : \Omega \to \mathbb{R}$ and functions $f : \mathbb{R} \times A \to \mathbb{R}$, $g : A \to \mathbb{R}$, $h : A \times \Theta \to \mathbb{R}$ such that*

(i) $u(\omega, a) = \phi(\omega) g(a) + h(a, \theta(\omega))$ for all $\omega \in \Omega$, or

(ii) $u(\omega, a) = f(\phi(\omega), a)$ and $\phi(\omega) \in \{0, 1\}$ for all $\omega \in \Omega$.

Then the ϕ-quantile signal is optimal.

Part (ii) of Proposition 5 was previously obtained by He et al. (2023). Both part (i) and (ii) of Proposition 5 follow immediately from Theorem 3, which establishes that the

[14] The optimal action $a : [0, 1] \to \mathbb{R}$ in this case is given as the solution to $\mathbb{E}\left[(\phi(\omega) - a(z))^{p-1} \mid q_\phi = z\right] = 0$.

quantile signal is the signal that induces the most dispersion in the posterior means. The assumption that $u(\omega, a) = \phi(\omega)g(a) + h(a, \theta)$ is satisfied in many applications. In particular, a payoff function u that is constant in θ and affine in ϕ satisfies this condition. For instance, in the bank loan example, this corresponds to assuming that the bank's preference depends only on the expected amount repaid by the borrower, but not the race, gender, age, etc of the borrower. Meanwhile, since posterior distributions over ϕ is one-dimensional when $\phi(\omega) \in \{0, 1\}$ for all $\omega \in \Omega$, the second part of the corollary follows.

4.2 Information Design

Consider the Bayesian persuasion setting where a sender discloses information about $\omega \in \Omega$ to a receiver who chooses an action $a \in A$, and suppose that the sender is restricted to choose only signals that are privacy-preserving.[15] Let the sender's payoff be $u_S : \Omega \times A \to \mathbb{R}$ and the receiver's payoff be $u_R : \Omega \times A \to \mathbb{R}$. Let $V_S^\star$ be the sender's value from choosing the optimal privacy-preserving signal. For simplicity, suppose again that $|\Theta| = J < \infty$ and write Θ as $\{1, \ldots, J\}$. Let $V_R : \Omega^J \times A \to \mathbb{R}$ be defined as

$$V_R(\omega_1, \ldots, \omega_J, a) := \sum_{j=1}^{J} u_R(\omega_j, a)\mathbb{P}[\theta = j],$$

for any $(\omega_j)_{j=1}^J \in \Omega^J$. Moreover, for any $\rho \in \Delta(\Omega^J)$, let

$$V_S(\rho) := \mathbb{E}_\rho \left[\sum_{j=1}^{J} u_S(\omega_j, a^\star(\rho))\mathbb{P}[\theta = j] \right],$$

where $a^\star(\rho)$ is the (sender-preferred) optimal action of the receiver that maximizes V_R when the posterior over $(\omega_j)_{j=1}^J$ is ρ. To ensure the existence of optimal signals, we assume that Ω is compact and that V_S is upper-semicontinuous. The next proposition characterizes the sender's value $V_S^\star$.

[15] For example, the sender might be the prosecutor as in Kamenica and Gentzkow (2011), trying to convince the judge that the defendant should not be released on bail, but is restricted to not using any information related to the race of the defendant even though such information might be predictive about the probability of reoffense.

Proposition 6 (Value of Persuasion)**.** *Let $\overline{V}_S$ be the concave closure of V_S, the sender's value $V_S^\star$ is given by*

$$V_S^\star = \max_{\rho \in \mathcal{D}} \overline{V}_S(\rho)\,,$$

Proposition 6 states that the sender's value can be found by a two-step procedure: First, fix a joint distribution ρ and find the optimal garbling of it by computing $\overline{V}_S(\rho)$. Then, optimize across $\rho \in \mathcal{D}$. Just as in standard persuasion problems, the characterization of Proposition 6 requires computing the concave closure of the function V_S, which is typically computationally demanding. Nonetheless, when payoffs are such that the sender's indirect utility is measurable with respect to the posterior mean of some statistic ϕ, Theorem 3 provides a much more tractable way to characterize optimal signals. Specifically, suppose that there exist a statistic $\phi : \Omega \to \mathbb{R}$ such that the sender's indirect utility is a function only of the posterior mean $\mathbb{E}[\phi \mid s]$, which we denote by $U_S : \mathbb{R} \to \mathbb{R}$. Then the sender's payoff given a signal can be written as $\int_{\mathbb{R}} U_S(x)\,\mathrm{d}G$, where G is the CDF of posterior means induced by the signal. Theorem 3 implies the following characterization:

Proposition 7 (Value of Mean-Measurable Persuasion)**.** *Suppose that the sender's indirect utility is measurable with respect to posterior means and is denoted by $U_S : \mathbb{R} \to \mathbb{R}$. The sender's value $V_S^\star$ is given by*

$$V_S^\star = \sup_{G \leq_{\mathrm{MPS}} \overline{F}_\phi} \int_{\mathbb{R}} U_S(x)\,\mathrm{d}G\,, \tag{5}$$

The characterization given by Theorem 3 and Proposition 7 is particularly convenient for introducing privacy concerns to mean-measurable persuasion problems since the structure of the feasible privacy-preserving signals perfectly aligns with that of all feasible signals. In other words, the privacy constraints, however complex they are, are all summarized by the mean-preserving-spread upper bound $\overline{F}_\phi$. We further demonstrate the value of this characterization in §5.

5 Economic Applications

To illustrate the relevance of privacy-preserving signals and to demonstrate the implications of our main results, we discuss several economic examples.

5.1 Statistical Discrimination and Algorithmic Fairness

Statistical discrimination refers to discriminatory outcomes that arise simply because some protected characteristics are statistically correlated with the payoff-relevant variables—and thus are used in decision-making. Legal studies and the literature on algorithmic fairness in computer science aim to explore optimal ways to discipline statistical discrimination that arise from (algorithmic-assisted) decision-making.

Informational Environment We first introduce a general environment that can be used to study discrimination. There is an underlying outcome $\gamma \in \Gamma$. A decision-maker observes covariates $(\theta, y, z) \in \Theta \times Y \times Z$ that are correlated with the outcome γ, and has to take an action $a \in A$. Among the observable covariates (θ, y, z), θ denotes the "protected characteristics" (e.g., race and gender), y denotes the "materially relevant characteristics" and are deemed acceptable to be used despite their correlations with θ, (e.g., credit history or criminal history), and z denotes all other observable covariates (e.g., zipcode). The decision-maker's payoff equals $\hat{u}(\gamma, a) \in \mathbb{R}$ when the outcome is γ and when the action is a.

A statistical model (or an algorithm) takes the inputs (θ, y, z) and outputs a prediction $s \in S$ for the outcome γ. The decision-maker then chooses an action a to maximize $\mathbb{E}[\hat{u}(\gamma, a) \mid s]$. As protected characteristics θ might be correlated with outcome γ, so might the prediction s be. Therefore, the resulting outcome might exhibit statistical discrimination in the sense that a might be correlated with θ, even if $\hat{u}$ does not depend on θ. A common regulatory approach is to discipline the degree of *disparate impact*—a legal term that refers to correlations between a and θ, regardless of whether the decision is made explicitly based on θ.[16] This approach requires the action taken by the decision-maker to be independent of the protected characteristics θ, conditional on materially relevant characteristics y.[17] This approach also coincides with a commonly adopted notion of fairness in computer science,

[16] For instance, in the case of *Griggs v. Duke Power Company*, the U.S. Supreme Court held that requiring a high school degree and an aptitude test for jobs transfers to certain departments creates disparate impacts for Black employees due to the history of segregation (i.e., transfer decision a is correlated with race θ). The court thus found such a policy in violation of Title VII of the Civil Rights Act, even though these requirements do not explicitly refer to race.

[17] In practice, an essential argument employers use in defense of disparate impact claims is the "business necessity" criterion. Namely, the inputs upon which the alleged discriminatory policies or rules are essential for the success of the defendant's business. These necessities can be interpreted as the materially relevant characteristics y.

which is referred to as *demographic parity*.[18]

For example, suppose that a bank, who faces many loan applicants with observable characteristics (θ, y, z), needs to make loan decisions a. The relevant outcome is whether an applicant will default in the future, denoted by $\gamma \in \{0, 1\}$. The Equal Credit Opportunity Act (15 U.S.C. 1691 et seq.)"*prohibits creditors from discriminating against credit applicants on the basis of race, color, religion, national origin, sex, marital status, age [..].*" A concrete (although stringent) interpretation of this requirement taken in the literature is that the information about each individual's default probability the bank uses to make loan decisions must be independent of an individual's protected characteristics (potentially conditional on materially relevant information, such as income). This requirement avoids the problem that even when restricting the bank to not condition on race directly it might still do so indirectly through the use of covariates such as zip code, which is a well-known issue highlighted in the actuarial sciences and legal studies, for example Wiggins (2020) states:[19]

> "[...] race has become so highly correlated with other social statistics that actuarial science in general has developed a baked-in racial bias. Racial discrimination by proxy (e.g., zip code standing in for race) can be glimpsed in the disparate impact of data-driven decision-making in housing, healthcare, policing, sentencing, and more. Simply leaving out racial data in statistically aided decision-making distances institutions from claims of intentional discrimination, but a disparate, discriminatory impact lingers when other factors correlated with race power ac-

[18]See, e.g., Darlington (1971); Calders et al. (2009); Dwork, Hardt, Pitassi, Reingold and Zemel (2012); Calders and Verwer (2010); Kamishima, Akaho and Sakuma (2011); Corbett-Davies et al. (2017); Gillis, McLaughlin and Spiess (2021). Two other commonly adopted criteria are (i) *separation*, which requires balanced type-I and type-II errors (see, e.g., Hardt et al. 2016), or more generally, independence between a and θ conditional on the true outcome γ; and (ii) *sufficiency*, which requires the action a to be a sufficient statistics for γ, so that the outcome γ is independent of θ conditional on a. It is well-known that none of any pairs of these three common fairness criteria can be satisfied at the same time, and hence the choice of a fairness criteria is necessary (see Barocas, Hardt and Narayanan (2019) and Carey and Wu (2023) for a comprehensive review of these criteria). With appropriate projections, our results can also be applied when the notion of separation, instead of independence, is adopted. See the Online Appendix for more details.

[19]Former Attorney Eric Holder also made a similar remark in the context of sentencing: "[...] basing sentencing decisions on static factors and immutable characteristics—like the defendant's education level, socioeconomic background, or neighborhood—they may exacerbate unwarranted and unjust disparities that are already far too common in our criminal justice system and in our society." See https://www.justice.gov/opa/speech/attorney-general-eric-holder-speaks-national-association-criminal-defense-lawyers-57th.

tuarial analyses."

Existing results in the fairness literature (see, e.g., Calders et al. 2009; Hardt et al. 2016; Corbett-Davies et al. 2017) take the "recommendation approach" (i.e., by taking $S = A$ so that the prediction s is a recommended action) and solve for the optimal recommendation subject to the constraint that a has to be statistically independent of θ. This approach is convenient as it reduces the statistical problem of finding the optimal prediction s to a linear program, which is particularly tractable when the available actions are simple. The literature thus characterizes optimal algorithms in simple settings when the decision-maker's choice is binary, e.g., when the bank only decides whether to grant a loan, and when the payoff is given by $\hat{u}(\gamma, a) = a \cdot (1 - \gamma - c)$ for some $c \in (0, 1)$.[20]

However, in many applications, the decision-maker may need to make more than a binary choice. For example, a bank typically needs to decide—in addition to whether to grant the loan—how much to grant, what the interest rate should be, the amount of down payment, and the form of the collateral. Moreover, in regulatory contexts, a regulator typically faces a wide-range of decision problems that are very different in their natures. For these problems, using the recommendation approach might not be as fruitful, since the approach relies on the specific payoff structure and benefits the most from the simplicity of the action space.[21]

A General Solution Theorem 1 and Theorem 3 lead to characterizations of optimal algorithms for arbitrary decision problems. To apply our results while incorporating the fact that only the covariates (θ, y, z) are observable, one may define the state as $\omega = (x, \theta) \in X \times \Theta = \Delta(\Gamma) \times \Theta$, where x corresponds to the distribution of outcomes γ conditional on (θ, y, z).[22] The expected payoff is then given by

$$u(x, \theta, a) := \int_{\Gamma} \hat{u}(\gamma, a) \, \mathrm{d}x(\gamma) \, .$$

By Proposition 2, the optimal algorithms can be characterized by solving the optimal trans-

[20] The optimal algorithms adopt different thresholds for different groups θ (conditional on materially relevant characteristics), and chooses action $a = 1$, e.g., grants the loan to an applicant, if and only if the conditional expectation $\mathbb{E}[\gamma \mid \theta, y, z]$, e.g., expected default probability, is below their group-specific thresholds.

[21] In fact, we are not aware of any paper that explicitly solves a model with more than two actions, or that solves for an optimal algorithm across many decision problems.

[22] Formally, $x : \Theta \times Y \times Z \to \Delta(\Gamma)$ with $x(\theta, y, z) = \mathbb{P}[\gamma \in \cdot \mid \theta, y, z]$.

port problem (4). Moreover, Proposition 3, 4, and 5 lead to explicit characterizations of an optimal fair algorithm under a wide class of economically relevant environments.

Perfectly Informative Covariates Suppose that $\Gamma \subseteq \mathbb{R}$ and that the covariates (θ, y, z) are perfectly informative, so that γ is a function of (θ, y, z). In this case, X can be equivalently defined as Γ. If $\hat{u}$ is supermodular in (γ, a), then by Proposition 3, the quantile signal is optimal. If $\hat{u}$ is separable so that $\hat{u}(\gamma, a) = f(\gamma)g(a) + h(a)$, for some real-valued measurable functions f, g, h, then, according to Proposition 5, the $\hat{\phi}$-quantile signal is optimal, where $\hat{\phi}(\gamma, \theta) := f(\gamma)$. For instance, a large corporate decides whether to hire a worker, and if so, which position to assign the worker to. Workers differ in their skill levels $\gamma \in [0, 1]$ and positions $a \in [0, 1]$ differ in their difficulties, with $a = 0$ being not hiring the worker. The employer's payoff is given by $v(\gamma, a) - w(a)$, where v is a supermodular production function and w is the market wage schedule for different positions.

Binary Outcomes Suppose that $\Gamma = \{0, 1\}$. Then, $X = \Delta(\Gamma) = [0, 1]$, with x being the probability that $\gamma = 1$. Moreover,

$$u(x, \theta, a) = x\hat{u}(1, a) + (1 - x)\hat{u}(0, a)$$

is affine in x. Proposition 5 (i) then implies that the quantile signal, which can simply be constructed by computing the quantile of the predicted score $x \in [0, 1]$ conditional on each protected characteristic, is optimal.[23] This set of assumptions is common in many relevant economic settings, such as loan decisions and bail decisions. For example, a lender makes loan decisions regarding an applicant. The applicant may either default ($\gamma = 0$) in the future or not ($\gamma = 1$). The actions of the bank could be highly complex, including deciding whether to give the loan, deciding interest rates, design payment schedules, and the choice of collaterals.

The Orthogonalization Procedure In the environments above, the quantile signals are optimal. In fact, the quantile signal can be generated by the following concrete and *detail-free* procedure that does not depend on the underlying decision making problems:

[23] This result generalizes the characterization of algorithmically fair optimal decisions obtained in Theorem 2 of He et al. (2023), which assumes the outcome γ to be binary, and the covariates to be *perfectly informative* about the outcome.

1. Use the most efficient algorithm and all covariates to generate a (one-dimensional) prediction score ϕ for the relevant outcome γ.
2. Adopt a "post-processing step" after generating the raw prediction and *orthogonalize* these predictions, by computing the quantile signal q_ϕ of the predicted scores for each group of protected characteristic, conditional on the materially relevant characteristics.
3. Decision-makers take an action based on the quantiles and the materially relevant characteristics.

If either the covariates are perfectly informative about a one-dimensional outcome $\gamma \in \mathbb{R}$, or the outcome γ is binary, as shown above, the quantile signal—and hence the output of the orthogonalization procedure—is always optimal. It is noteworthy that this procedure does not require *any* further details about the underlying decision problems. In particular, in the bank loan example, however complex a lender's decision problem is, and regardless of how many different decision-maker a regulator faces, adopting the orthogonalization procedure always leads to an optimal outcome.

This is of practical importance as it allows the reuse of existing econometric and statistical tools. The first step of the procedure can be carried out by an econometrician without any consideration for discrimination. The second step ensures that whatever procedure the econometrician uses, the final outcome will be non-discriminatory. While one might expect that such a procedure would lead to suboptimal decision rules, our result identifies conditions under which this two-step procedure will be optimal if the econometrician uses the best possible predictor of γ in the first step.[24]

This procedure also suggests a concrete approach to ensuring fairness. A regulator could legally require every decision maker to adopt a post-processing step and compute the quantiles after generating raw prediction scores, and enforce this requirement by audits or subpoena in legal proceedings.[25] This "output regulation" is distinct from the commonly adopted "input

[24] Kamiran, Žilobaitė and Calders (2013) and Feldman, Friedler, Moeller, Scheidegger and Venkatasubramanian (2015) propose a similar procedure to "repair" unfair algorithms. See also, Calders and Verwer (2010) and Kamishima et al. (2011), for earlier work on repairing and regularizing unfair algorithms. This literature focuses on transforming any (potentially unfair) algorithm into a fair one. Our results imply that, not only does a similar procedure lead to a fair algorithm, it is in fact the *optimal* way to transform any algorithm into a fair one.

[25] If the regulator requires the decision maker to store the prediction scores, they can at any point in time use these to check whether the procedure was applied correctly and apply appropriate penalties in case it

regulation" approach which limits the data available to the decision maker (which is often inefficient due to covariates that are correlate with protected characteristics). Nevertheless, the implementation and enforcement can be fulfilled by similar regulatory tools.

5.2 Optimal Privacy-Preserving Disclosure in Auctions

A publisher runs a second-price auction to sell targeted audiences to $N \geq 1$ advertisers with private values $\{v_i\}_{i=1}^N \in \mathbb{R}_+^N$ that are independently drawn from F. As in Bergemann et al. (2022) we assume that F is absolutely continuous. The publisher has control over how much information about the audience to give to the advertisers. Specifically, each advertiser observes a (conditionally independent) signal about their own value. Bergemann et al. (2022) study this problem and characterize the optimal symmetric signal to disclose to the advertisers. That is, they characterize the symmetric and independent signal $s : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ that maximizes the publisher's revenue, where each advertiser i observes a signal $s_i = s(v_i, r_i)$, with $\{r_i\}_{i=1}^n$ being the independent randomization device that are uniformly distributed on $[0,1]$. They characterize the optimal signal as follows:

Proposition 8 (Bergemann et al. 2022, Theorem 1)**.** *Any optimal symmetric information structure is equivalent, in terms of bidders' expected values* $\mathbb{E}[v_i \mid s_i]$*, to a partitional signal* $s : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$

$$s(v_i, r_i) = \begin{cases} v_i & \text{if } F(v_i) < q_N^* \\ \mathbb{E}\left[v_i \mid F(v_i) \geq q_N^*\right] & \text{if } F(v_i) \geq q_N^* \end{cases},$$

where q_N^* *is the unique root in* $(0,1)$ *of a* N*-degree polynomial and does not depend on* F*.*

In many economically relevant situations, disclosing some type of information about the audience is controversial. For example, Target was widely criticised for using pregnancy information in advertising.[26] Google and other advertising companies have been criticized for disclosing highly sensitive information to advertisers including religious beliefs, ethnicities, diseases, disabilities, sexual orientation, and whether a user informed themselves online about incest and sexual abuse.[27]

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was not.

[26] see https://www.forbes.com/sites/kashmirhill/2012/02/16/how-target-figured-out-a-teen-girl-was-pregnant-before-her-father-did/?sh=412314036668.

[27] See https://mashable.com/article/google-iab-gdpr-complaint.

To model such a situation within our framework, suppose that there is some protected information that should not be disclosed to advertisers. Such information is correlated with each advertiser's values v_i and is summarized by $\theta \in \Theta$. Assume that $\{v_i\}_{i=1}^N$ is independent and identically distributed conditional on θ.[28] Let $F(\cdot \mid \hat{\theta})$ be the distribution of v_i conditional on the realization $\hat{\theta}$ of θ. As before, we assume that $F(\cdot \mid \hat{\theta})$ is absolutely continuous for all $\hat{\theta} \in \Theta$.

The publisher is not allowed to disclose any information about θ and thus can only disclose a privacy-preserving signal to each advertiser. A symmetric privacy-preserving signal $s : \mathbb{R}_+ \times \Theta \times [0,1] \to \mathbb{R}$ discloses $s(\hat{v}_i, \hat{\theta}, \hat{r}_i)$ to advertiser i when their value is $\hat{v}_i \in \mathbb{R}_+$, protected characteristic is $\hat{\theta} \in \Theta$, and the randomization device is $\hat{r}_i$. A symmetric signal s is privacy-preserving if $s(v_i, \theta, r_i)$ is independent of θ for all i.

As the advertisers' preferences only depend on their expected value $\mathbb{E}[v_i \mid s_i]$ after observing a signal s_i, we can apply Theorem 3 to conclude that for any symmetric signal s, the distribution of $\mathbb{E}[v_i \mid s]$ must be a be a mean-preserving contraction of $\mathbb{E}[v_i \mid q_i]$, where q_i is the quantile signal

$$q_i := F(v_i \mid \theta),$$

whose distribution is given by $\overline{F}(z) := \inf\{y \in [0,1] : \mathbb{E}[F^{-1}(y \mid \theta)] \geq z\}$. Moreover, for any symmetric privacy-preserving signal, advertisers' conditional expected values are independently and identically distributed.[29] Together with the result from Bergemann et al. (2022), we obtain a complete characterization of optimal information disclosure in auctions subject to privacy constraints.

Proposition 9. *Any optimal symmetric information structure that does not reveal any information about θ is equivalent, in terms of bidder's expected values $\mathbb{E}[v_i \mid s_i]$, to a partitional*

[28] For instance, if θ denotes gender, then the values of each advertiser to target a female or a male audience are assumed independently and identically distributed due to, say, heterogeneity among advertisers—just as they were assumed to be independent and identically without conditioning in Bergemann et al. (2022).

[29] To see this, consider any symmetric privacy-preserving signal s. Let $s_i := s(v_i, \theta, r_i)$ for all i. Since $\{v_i\}_{i=1}^n$ is independent conditional on θ, $\mathbb{P}[s_1 \leq z_1, \ldots, s_N \leq z_N] = \mathbb{E}[\mathbb{P}[s_1 \leq z_1, \ldots, s_N \leq z_N] \mid \theta] = \mathbb{E}[\mathbb{P}[s_1 \leq z_1 \mid \theta] \cdots \mathbb{P}[s_N \leq z_N \mid \theta]]$. Moreover, since s is privacy-preserving, we have $\mathbb{E}[\mathbb{P}[s_1 \leq z_1 \mid \theta] \cdots \mathbb{P}[s_N \leq z_N \mid \theta]] = \mathbb{E}[\mathbb{P}[s_1 \leq z_1] \cdots \mathbb{P}[s_N \leq z_N]] = \mathbb{P}[s_1 \leq z_1] \cdots \mathbb{P}[s_N \leq z_N]$.

signal $s : \mathbb{R}_+ \times \Theta \times [0,1] \to \mathbb{R}_+$

$$s(v_i, \theta, r_i) = \begin{cases} \mathbb{E}\left[v_i \mid F(v_i \mid \theta)\right] & \text{if } F(v_i \mid \theta) < q_N^* \\ \mathbb{E}\left[v_i \mid F(v_i \mid \theta)) \geq q_N^*\right] & \text{if } F(v_i \mid \theta) \geq q_N^* \end{cases},$$

where q_N^* *is the same as in Proposition 8.*

Example 3. Suppose that the willingness to pay for a good or service is higher among people with certain sexual orientation, ethnic group, or among people with a specific disease. The auctioneer does not want to reveal any information about this sensitive information encoded in θ. For concreteness, suppose that the values are drawn from an exponential distribution conditional on protected information θ. Simple algebra shows that the optimal privacy-preserving signal is given as

$$s(v_i, \theta, r_i) = \begin{cases} v_i \times \frac{\mathbb{E}[v_i]}{\mathbb{E}[v_i|\theta]} & \text{if } F(v_i \mid \theta) < q_N^* \\ \mathbb{E}\left[v_i \mid F(v_i) \geq q_N^*\right] \times \frac{\mathbb{E}[v_i]}{\mathbb{E}[v_i|\theta]} & \text{if } F(v_i \mid \theta) \geq q_N^* \end{cases}$$

This signal pools agents from different groups with different values in a way not present in the optimal signal without privacy concerns (given in Proposition 8): Values of agents from a group θ with a higher expected valuation $\mathbb{E}[v_i \mid \theta]$ are multiplicatively scaled down, while values of agents from a group with an ex-ante lower valuation are multiplicatively scaled up.

5.3 Buyer Information with an Informed Seller

A monopolist sells a product to a buyer with unit demand and quasi-linear preference. The buyer's value is $v \in [0,1]$ and follows a distribution F. The seller observes a private signal $\theta \in \Theta$ that is correlated with the buyer's value v. The buyer also observes a private signal s that reveals information about v and θ. Given a joint distribution of (v, θ), we model the buyer's signal by considering a probability space where the outcomes (v, θ, r) consists of the buyer's value v, the seller's signal θ, as well as an independently uniformly distributed randomization device r, and define the buyer's signal as a random variable s on this probability space. The timing of the game is as follows:

1. The monopolist commits to a mechanism for selling the good, that specifies a probability

of receiving the good $x(\theta, m)$ and a payment made by the buyer $t(\theta, m)$.

2. The monopolist privately observes θ and the buyer privately observes s.
3. The buyer sends a message m to the mechanism, which determines together with the realization of θ, the probability of trade x and the transfer t.

Importantly, the seller commits to how his signal θ and the report of the buyer will be used in the mechanism. Given the distribution of (v, θ), which signal s maximizes the buyer's expected surplus? Roesler and Szentes (2017) studies a special case of this problem when θ is completely uninformative. They fully characterizes the buyer's optimal signal in terms of the distribution of posterior means, as well as the feasible welfare outcomes.

5.3.1 Signals that Maximize Buyer Surplus

When θ is informative about the buyer's value, the buyer's signal s might be correlated with θ. In this case, if the buyer's beliefs about the seller's signal θ (i.e., $\mathbb{P}[\theta \in \cdot \mid s]$), are linearly independent across all realizations of s, and the posterior expected value $\mathbb{E}[v \mid s]$ is a sufficient statistic for the buyer's belief (i.e, $\mathbb{P}[\theta \in \cdot \mid s] = \mathbb{P}[\theta \in \cdot \mid \mathbb{E}[v \mid s]]$), then the monopolist can (almost) fully extract the buyer's surplus (Crémer and McLean 1988; McAfee and Reny 1992). Clearly, any signal satisfying these conditions can not be optimal for the buyer as it leaves them with zero surplus.

While such (almost) full surplus extraction can be avoided if the posterior expected value $\mathbb{E}[v \mid s]$ is not a sufficient statistic for the buyer's belief about θ (e.g., when the buyer is fully informed about the seller's signal), the seller is still able to price discriminate, which sometimes would also leave the buyer with little information rent. One way to avoid both surplus extraction and price discrimination is to have a buyer signal s that is independent of the seller's signal θ. Using Theorem 3 and arguments from majorization theory, we prove in Strack and Yang (2024) that, perhaps surprisingly, having a signal that is independent of θ is always optimal for the buyer.

Proposition 10 (Strack and Yang 2024, Theorem 1)**.** *For any buyer signal $\tilde{s}$, there exists a buyer signal s that is independent of θ under which the buyer's expected surplus is weakly higher.*

According to Proposition 10, it is without loss to solve for the optimal privacy-preserving signal. Under any privacy-preserving signal s, the seller's private signal θ is not informative

about the buyers expected value $\mathbb{E}[v \mid s]$, and the optimal mechanism is a uniform posted price.[30] As only the expected value of the buyer matters for their purchase decision in posted price mechanism, Theorem 3 implies that the buyer's optimal signal is given by a distribution G that maximizes the buyer's surplus under the optimal posted price among all mean-preserving contractions of

$$\overline{F}(z) := \inf\left\{y \in [0,1] : \mathbb{E}\left[F^{-1}(y \mid \theta)\right] \geq z\right\}.$$

Therefore, the problem where the seller observes a private signal θ is equivalent to the the problem where the distribution of values F is replaced by its mean preserving contraction $\overline{F}$. Intuitively, the fact that the seller has some private information limits the amount of private information the buyer can learn without creating correlation to the sellers information (which would lead to surplus extraction). We can thus adopt the arguments of Roesler and Szentes (2017), to obtain the following result:

Proposition 11 (Strack and Yang 2024, Proposition 1)**.** *Consider any privacy-preserving signal s under which the distribution of the buyer's posterior expected value $\mathbb{E}[v \mid s]$ is*

$$G_{\pi^\star}^{b^\star}(z) = \begin{cases} 0, & \text{if } z < \pi^* \\ 1 - \frac{\pi^*}{z}, & \text{if } z \in [\pi^*, b^*) \\ 1, & \text{if } z > b^* \end{cases},$$

where π^* is the smallest π such that $G_\pi^b \leq_{\text{MPS}} \overline{F}$ for some $b \geq \pi$, and $b^\star$ is the unique b for which $G_{\pi^\star}^b \leq_{\text{MPS}} \overline{F}$. The signal s maximizes the buyer's surplus. Moreover, under this signal, the monopolist's optimal price equals $\pi^\star$, and trade occurs with probability 1.

5.3.2 Feasible Welfare Outcome

Proposition 10 and Proposition 11 further lead to a characterization of welfare outcomes that can be induced by buyer's signal s, for any given seller signal θ, which is stated below.

Proposition 12 (Strack and Yang 2024, Proposition 2)**.** *For any $(\sigma, \pi) \in [0,1]^2$, there exists a signal s such that the buyer's surplus is σ and the seller's profit is π if and only if $\pi \geq \pi^\star$*

[30] See Lemma 1 of Strack and Yang (2024).

and $\sigma + \pi \leq \mathbb{E}[v]$.

Example 4. Suppose that the buyer's value v is uniformly distributed on $[0,1]$, and that the seller's signal θ equals the buyers value with probability p, and equals a random variable independently drawn from F with probability $1-p$. It then follows that $\overline{F}$ is given by

$$\overline{F}(z) = \begin{cases} (1-p)(1-\sqrt{1-2z}), & \text{if } z \in [0, 1/2) \\ p + (1-p)\sqrt{2z-1}, & \text{if } z \in [1/2, 1] \end{cases},$$

if $p > 1/2$, and

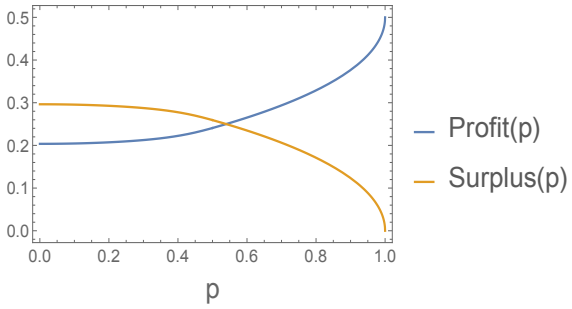
$$\overline{F}(z) = \begin{cases} (1-p)(1-\sqrt{1-2z}), & \text{if } z \in \left[0, \frac{1}{2}\left(1 - \frac{(1-2p)^2}{(1-p)^2}\right)\right) \\ \frac{(1-p)^2 z - \frac{1}{2}p^2}{1-2p}, & \text{if } z \in \left[\frac{1}{2}\left(1 - \frac{(1-2p)^2}{(1-p)^2}\right), \frac{1}{2}\left(1 + \frac{(1-2p)^2}{(1-p)^2}\right)\right) \\ p + (1-p)\sqrt{2z-1}, & \text{if } z \in \left[\frac{1}{2}\left(1 + \frac{(1-2p)^2}{(1-p)^2}\right), 1\right] \end{cases},$$

if $p < 1/2$. If $p = 1$, the seller knows the buyer's value and $\overline{F}$ is a Dirac measure at $\mathbb{E}[v] = 1/2$ and thus the buyer can only observe an uninformative signal; while if $p = 0$, then the seller's signal θ is completely uninformative and any mean-preserving contraction of F is feasible for the buyer, as in Roesler and Szentes (2017).

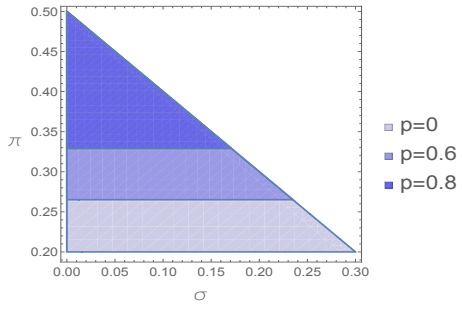
Figure 2a plots the seller's profit and the buyer's surplus as a function of p. As $p \to 1$, the seller's profit converges to $\mathbb{E}[v] = 1/2$ and the buyer's surplus converges to zero; while as $p \to 0$, the seller's profit is approximately 0.2 and the buyer's surplus is approximately 0.3, as in Roesler and Szentes (2017). Figure 2b depicts the sets of possible welfare outcomes for different values of p. When $p = 0$, the feasible welfare outcomes coincide with that in Roesler and Szentes (2017), when $p = 1$, the seller fully extracting the surplus is the only possible welfare outcome. Welfare outcomes for $p = 0.6$, $p = 0.8$ are as shown in Figure 2b.

5.4 Price Discrimination

Our results can also be applied to settings of price discrimination in the spirit of Bergemann et al. (2015). Consider a monopolist who uses consumer data to price-discriminate consumers. The monopolist sells a single product to a unit mass of consumers. Each consumer



(a) Profit and Surplus



(b) Feasible Welfare Outcomes

Figure 2: Panel (a) plots the seller's profit and the buyer's surplus under the buyer-optimal signal as a function of p. Panel (b) plots the feasible welfare outcomes for $p \in \{0, 0.6, 0.8\}$.

demands a single unit and has value $x \in X := [\underline{x}, \overline{x}] \subset \mathbb{R}_+$ for the product. Moreover, each consumer belongs to one of the protected groups $\theta \in \Theta = \{\theta_1, \ldots, \theta_J\}$. These characteristics are correlated with consumers' values. For each $j \in \{1, \ldots, J\}$, let $F(\cdot \mid \theta_j)$ be the distribution of values of consumers with characteristic $\theta = \theta_j$. With different combinations of consumer data, the monopolist is able to charge different prices to different groups of consumers and engage in third-degree price discrimination.

While consumer data enables the monopolist to engage in price discrimination, it is often required by law or regulations that consumers cannot be price-discriminated based upon their protected characteristics. For example, a recent legislation (AB1287) in California specifically prohibits businesses from price-discriminating based on gender. Given such legal constraints, it is natural to ask: What market segmentations allow the monopolist to price-discriminate, but are not based on protected characteristics?

Clearly, the market segmentation that fully segments consumers by their values allows the monopolist to extract all the surplus. This, however, would typically lead to price discrimination based on protected characteristics, in the sense that consumer of different characteristics would face a different distribution of prices. Moreover, simply prohibiting the monopolist from using protected characteristics to segment consumers would not be privacy-preserving either, since the monopolist may have access to close proxies of these characteristics. For example, as noted by The White House (2015):

> "Big data naturally raises concerns among groups that have historically been victims of discrimination. Given hundreds of variables to choose from, it is easy

to imagine that statistical models could be used to hide more explicit forms of discrimination by generating customer segments that are closely correlated with race, gender, ethnicity, or religion [...], even if the profit motive is different from, and in many cases fundamentally inconsistent with, the sort of prejudice that our antidiscrimination laws seek to prohibit."

Our results lead to a characterization of all market segmentations that prohibit the monopolist from price-discriminating consumers based on their protected characteristics, in the sense that consumers of different protected characteristics face the *same* distribution of prices.

Seller-Optimal Segmentations A natural question is what non-discriminatory market segmentation maximizes the seller's profit. Proposition 2 shows that this question reduces to a multi-marginal optimal transport problem. Specifically, let the state space be $\Omega := X \times \Theta$, and suppose that no information about θ can be revealed. For any vector of of consumer values $(x_1, \dots, x_J) \in X^J$, let $V(x_1, \dots, x_J)$ be the maximal profit of the monopolist when knowing that characteristic θ_j has value x_j, without knowing the characteristics:

$$V(x_1, \dots, x_n) = \max_{p \geq 0} \sum_{j=1}^{J} p \mathbf{1}\{x_j \geq p\} \mathbb{P}[\theta = \theta_j] = \max_{j \in \{1,\dots,J\}} \left\{ \sum_{i=j}^{J} \mathbb{P}[\theta = \theta_{(j)}] x_{(j)} \right\},$$

where $x_{(j)}$ is the j-th smallest element of $(x_1, \dots, x_n)$, and (j) is the index of the j-the smallest element $x_{(j)}$.

Proposition 2 implies that the profit-maximizing market segmentation can be identified by finding the joint distribution ρ of $(x_1, \dots, x_J)$ that solves the optimal transport problem

$$\sup_{\rho \in \mathcal{D}} \int_{X^J} V(x_1, \dots, x_J) \, \mathrm{d}\rho \,. \tag{6}$$

Suppose that $F(x \mid \theta_1) \geq \cdots \geq F(x \mid \theta_J)$, so that $\{F(\cdot \mid \theta_j)\}_{j=1}^{J}$ are ranked by first-order stochastic dominance, and suppose that $(1 - \mathbb{P}[\theta = \theta_1]) \cdot \overline{x} \leq \underline{x}$ so that there are enough mass of the "lowest-type" consumers. Under this sufficient condition, we can obtain a closed-form solution of the transport problem (6), which in turn leads to a characterization of a seller-optimal segmentation.

Proposition 13. *Let q be the quantile signal. That is,*

$$q := rF(x \mid \theta) + (1-r)F^{-}(x \mid \theta).$$

The market segmentation corresponding to quantile signal q maximizes the seller's revenue. Moreover, under any optimal market segmentation:

(i) The outcome is efficient and every consumer purchases the good.

(ii) Consumers with characteristic θ_1 always retain zero surplus, while consumers with characteristic $\theta_j \neq \theta_1$ retain positive surplus whenever $F(x \mid \theta_j) > F(x \mid \theta_1)$ for some $x \in X$.

(iii) The seller's profit equals $\mathbb{E}[x \mid \theta_1]$. In particular, increases in consumers' values with characteristics $\theta \neq \theta_1$ in the sense of FOSD do not affect the seller's profit, while any increase in consumers' values with characteristic θ_1 in the sense of FOSD is completely captured by the seller.

Under the seller-optimal segmentation, consumers with different characteristics are pooled assortatively into the same segment. Since consumers' values are FOSD-ranked based on their characteristics, the lowest value in each segment must be of characteristic $\theta = \theta_1$. The assumption that $(1 - \mathbb{P}[\theta = \theta_1]) \cdot \overline{x} \leq \underline{x}$ then ensures that it is optimal for the seller to sell to every consumers in each segment by charging a price that equals the lowest value. Note that, however, the price distribution faced by each consumer characteristic θ is the same, which equals $F(\cdot \mid \theta_1)$.

This observation may serve as a cautionary tale, as in practice the legislation imposing privacy constraints typically mention explicitly that they are meant to protect groups that plausibly have lower willingness to pay. However, according to Proposition 13, this could mean that the group of consumers who statistically have lower willingness to pay would have their surplus extracted, while groups of consumers with higher willingness to pay would enjoy lower prices. In fact, when compared to uniform pricing (i.e., banning any forms of price discrimination), the group of consumers with the lowest willingness to pay must be worse-off compared to uniform pricing, while other groups of consumers might be better-off.[31]

When the assumptions in Proposition 13 are violated, the quantile signal may not be optimal. In this case, non-trivial reorderings of the quantile signal would be necessary for optimality, as shown in the next example.

[31] See the Online Appendix for a concrete numerical example.

(a) Quantile Signal

(b) Optimal Signal

Figure 3: Panel (a) depicts the quantile signal that pools two groups of consumers assortatively. Panel (b) depicts the optimal signal, which pools as many consumers with the same values as possible, and then pools the remaining consumers negatively assortatively.

Example 5. Suppose that $X = \{1,2,3\}$, and $\Theta = \{\theta_1, \theta_2\}$, with $\mathbb{P}[\theta = \theta_1] = 1/2$. Suppose that the conditional distribution of x given $\theta = \theta_1$ is $(1/2, 1/3, 1/6)$; while the conditional distribution of x given $\theta = \theta_2$ equals $(1/6, 1/3, 1/2)$. One can show that the solution to the optimal transport problem (6) is given by the joint distribution $\rho^\star$, where $\rho^*(1,1) = \rho^*(3,3) = 1/6$, $\rho^*(2,2) = \rho^*(1,3) = 1/3$,[32] corresponds to the reordered quantile signal generated by the measure-preserving transformations:

$$M_{\theta_1}(s) = \begin{cases} s, & \text{if } s \in [0, 1/6] \cup (5/6, 1] \\ 1/2 + (s - 1/6), & \text{if } s \in (1/6, 1/2] \\ 1/6 + (s - 1/2), & \text{if } s \in (1/2, 5/6] \end{cases} ; \qquad M_{\theta_2}(s) = s\,.$$

The quantile signal and the optimal reordered quantile signal can be represented in the same way as in Example 2 by Figure 3.

6 Discussion

6.1 Relation to "Private Private Information"

He et al. (2023) study what information about a state taking finitely many values can be revealed to a group of n agents if their signals $s_1, \ldots, s_n$ are restricted to be independent, in which case the information structure is called *private private*. The concept of private

[32] See the Online Appendix for detailed arguments.

private information structure relates to privacy-preserving signals as an information structure $s = (s_1, \dots s_n)$ is private private if and only if s_i is privacy-preserving with respect to the σ-algebra generated by s_{-i} for all i. An important difference between the concepts is that the information which needs to be kept private is exogenous in our model while it is endogenous in the case of private private information.

Section 4 of He et al. discusses how the results they obtain for private private information structures can be used to gain insights into privacy-preserving signals. In our notation, consider a state space $\Omega = X \times \Theta$ an associated probability measure $\mathbb{P}$. By treating θ as the signal received by a dummy agent, a signal s is privacy-prserving if and only if (s, θ) is private private. They then establish the following result: For any (sufficiently nice) decision problem (u, A), where $u : X \times A \to \mathbb{R}$ depends only on x,

Theorem (Theorem 2 in He et al. 2023)**.** *Whenever $X = \{0, 1\}$ is binary, there exists an optimal privacy-preserving signal $s^\star$. The optimal signal is unique up to equivalence: the distribution of $\mathbb{P}[x = 1 \mid s^\star]$ is $\overline{F}_x$.*[33] *Furthermore, every privacy-preserving signal s reveals less information about x than $s^\star$.*

Our Proposition 5 (ii) reproduces this insight. Our other results differ from Theorem 2 in He et al. along two dimensions:

(i) We do not restrict attention to binary states. In fact, our Theorem 3 generalizes Theorem 2 of He et al. from binary states to an arbitrary one-dimensional state space X for u affine in x.

(ii) Theorem 2 of He et al. considers decision problems in which the payoff does *not* depend on θ. Most of our results do not impose this restriction on the decision problem.[34] This restriction has meaningful implications, as (even when restricting to binary states) there does *not* exist a signal that is optimal for all decision problems (see Example 2) and thus Theorem 2 in He et al. does not apply for the wider class of decision problem we study.[35] In other words, the notion of Blackwell dominance is different since the underlying state spaces ($X \times \Theta$ versus X) are different.

[33] Note that $\overline{F}_x^{-1}(q) := \mathbb{E}[F^{-1}(x \mid \theta)] = \mathbb{E}[\mathbf{1}\{\mathbb{P}[x = 1 \mid \theta] > 1 - q\}]$ for all $q \in [0, 1]$, and hence $\overline{F}_x$ is exactly the conjugate of the distribution of the random variable $\bar{x}(\theta) = \mathbb{P}[x = 1 \mid \theta]$.

[34] Theorem 1, and Proposition 2,3,4,5 (i) all allow for the utility to depend on θ. As the approach in He et al. is based on a characterization of feasible posterior beliefs about x, it is not clear how it can be adapted to this wider class of decision problems where these beliefs are no longer a sufficient statistic for payoffs.

[35] In Appendix C, He et al. demonstrate that even when the utility does not depend on θ, the uniqueness

Finally, He et al. (2023) and the present paper use quite different mathematical methodologies, while He et al. use tools from tomography, our proofs are based upon majorization theory and optimal transport.

6.2 Relation to Differential Privacy

While we define privacy-preserving signals through an abstract collection of privacy sets, another notion of privacy is *differential privacy*, proposed by Dwork, McSherry, Nissim and Smith (2006).[36] Specifically, suppose that Ω is a finite product set $\Omega_1 \times \cdots \times \Omega_n$, where each dimension Ω_i represents characteristics of a different agent. A signal s satisfies *ε-differential privacy* for $\varepsilon > 0$ if for any ω, ω' that differ only in the characteristic of a single agent i (i.e., $\omega_{-i} = \omega'_{-i}$), a.s.,

$$\left| \log \frac{\mathbb{P}[\omega \mid s]}{\mathbb{P}_\pi[\omega' \mid s]} - \log \frac{\mathbb{P}[\omega]}{\mathbb{P}[\omega']} \right| \leq \varepsilon \,.$$

Intuitively, the log-likelihood induced by the signal cannot be influenced by more than ε by each individual agent. Our notion of privacy considers signals only depends on the characteristics of a single individual, and are restricted to not not reveal certain information. In contrast, differential privacy considers signals which depend on a whole population of agents, but who are only influenced to a limited extent by each individual agent. Mathematically, these notions are unrelated and aim to capture different aspects of privacy.

7 Conclusion

We provide a characterization of signals which do not reveal certain information, and among others presented application to statistical discrimination, product information, auction, and price discrimination. An interesting avenue for future research is to use the mathematical characterization presented in this paper to understand the consequences of different notion of privacy and fairness.

is a only feature for binary states, by showing that there is a continuum of privacy-preserving signals that can be optimal for some decision problem in an example with three states and binary θ.

[36] Relatedly, Schmutte and Yoder (2022) characterizes the distribution of posteriors that can be induced by signals satisfying ε-differential privacy and studies an information design problem subject to differential privacy.

Appendix

Lemma A.1. *A signal s is privacy-preserving with respect to $\mathcal{P} \subseteq \mathcal{F}$ if and only if it is privacy-preserving with respect to the σ-algebra generated by $\mathcal{P}$.*

Proof. Fix any nonempty collection $\mathcal{P} \subseteq \mathcal{F}$ that is closed under finite intersections. Consider any signal s that is privacy-preserving with respect to the σ-algebra generated by $\mathcal{P}$, denoted by $\sigma(\mathcal{P})$. Since $\mathcal{P} \subseteq \sigma(\mathcal{P})$, s is privacy-preserving with respect to $\mathcal{P}$. Conversely, consider any signal s that is privacy-preserving with respect to $\mathcal{P}$. Let $\mathcal{P}^s \subseteq \mathcal{F}$ be the collection of events for which (1) holds. Clearly, $\mathcal{P}^s$ is nonempty since s is privacy-preserving with respect to $\mathcal{P}$. Moreover, from the fact that $\mathbb{P}$ and $\mathbb{P}[\cdot \mid \hat{s}]$ is a probability measure for all realization $\hat{s}$ of s, it follows that $\mathcal{P}^s$ is a λ-system. Therefore, by Dynkin's $\pi - \lambda$ theorem, since the π-system $\mathcal{P}$ is contained in the λ-system $\mathcal{P}^s$, the σ-algebra $\sigma(\mathcal{P})$ generated by $\mathcal{P}$ must also be contained in $\mathcal{P}^s$. Therefore, s is privacy-preserving with respect to $\sigma(\mathcal{P})$. □

Proof of Lemma 1. By Lemma A.1, it is without loss to assume that $\mathcal{P}$ is a σ-algebra. Since $(\Omega, \mathcal{F})$ is standard Borel and since $\mathcal{P} \subseteq \mathcal{F}$, $\mathcal{P}$ is countably generated. This completes the proof, as any countably generated σ-algebra can be generated by a random variable (see, e.g., Preston 2008, Proposition 3.2). □

Proof of Lemma 3. Fix any $\hat{\theta} \in \Theta$, we will show that $\mathbb{P}[q_\phi \leq z \mid \hat{\theta}] = z$ for all $z \in [0,1]$. To see this, first note that since $F_\phi(\cdot \mid \hat{\theta})$ has at most countably many jumps, enumerated by $\{x_n\} \subseteq \mathbb{R}$, $F_\phi(\cdot \mid \hat{\theta})$ can be written as

$$F_\phi(x \mid \hat{\theta}) = H(x) + \sum_{n=1}^{\infty} \mathbf{1}\{x_n \leq x\}(F_\phi(x_n \mid \hat{\theta}) - F_\phi^-(x_n \mid \hat{\theta}))\,,$$

for all $x \in \mathbb{R}$, where H is a nondecreasing and continuous function. For any $z \in [0,1]$ and for any $x \in \mathbb{R}$, let

$$\Gamma_z(x) := \begin{cases} 0, & \text{if } z < F_\phi^-(x \mid \hat{\theta}) \\ \frac{z - F_\phi^-(x|\hat{\theta})}{F_\phi(x|\hat{\theta}) - F_\phi^-(x|\hat{\theta})}, & \text{if } z \in [F_\phi^-(x \mid \hat{\theta}), F_\phi(x \mid \hat{\theta})) \\ 1, & \text{if } z \geq F_\phi(x \mid \hat{\theta}) \end{cases},$$

and note that

$$\begin{aligned}\int_{\mathbb{R}} \Gamma_z(x)\,\mathrm{d}F_\phi(x \mid \hat{\theta}) &= \int_{\mathbb{R}} \Gamma_z(x)\,\mathrm{d}H(x) + \sum_{n=1}^{\infty} \Gamma_z(x_n)(F_\phi(x_n \mid \hat{\theta}) - F_\phi^-(x_n \mid \hat{\theta})) \\ &= \int_{\mathbb{R}} \mathbf{1}\{x \le F_\phi^{-1}(z \mid \hat{\theta})\}\,\mathrm{d}H(x) + \sum_{n=1}^{\infty} \Gamma_z(x_n)(F_\phi(x_n \mid \hat{\theta}) - F_\phi^-(x_n \mid \hat{\theta})) \\ &= H(F_\phi^{-1}(z \mid \hat{\theta})) + \sum_{n=1}^{\infty} \Gamma_z(x_n)(F_\phi(x_n \mid \hat{\theta}) - F_\phi^-(x_n \mid \hat{\theta})).\end{aligned}$$

Therefore, if $F_\phi(\cdot \mid \hat{\theta})$ is continuous at $F_\phi^{-1}(z \mid \hat{\theta})$, then $F_\phi(F_\phi^{-1}(z \mid \hat{\theta}) \mid \hat{\theta} = z$ and $\Gamma_z(x_n) = \mathbf{1}\{z \ge F_\phi(x_n \mid \hat{\theta})\}$, and hence $\int_{\mathbb{R}} \Gamma_z(x)\,\mathrm{d}F_\phi(x \mid \hat{\theta}) = z$. Meanwhile, if $F_\phi(\cdot \mid \hat{\theta})$ is discontinuous at $F_\phi^{-1}(z \mid \hat{\theta})$, then $z = F_\phi(x_k \mid \hat{\theta})$ for some $k \in \mathbb{N}$, $F_\phi(F_\phi^{-1}(z \mid \hat{\theta}) \mid \hat{\theta}) = F_\phi(x_k \mid \hat{\theta})$, and $\Gamma_z(x_k) = (z - F_\phi^-(x_k \mid \hat{\theta}))/(F_\phi(x_k \mid \hat{\theta}) - F_\phi^-(x_k \mid \hat{\theta}))$ while $\Gamma_z(x_n) = \mathbf{1}\{z \ge F_\phi(x_n \mid \hat{\theta})\}$ for $n \ne k$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}} \Gamma_z(x)\,\mathrm{d}F_\phi(x \mid \hat{\theta}) =& F_\phi(F_\phi^{-1}(z \mid \hat{\theta}) \mid \hat{\theta}) + z - F_\phi^-(x_k \mid \hat{\theta}) \\ &+ \sum_{n \ne k}^{\infty} \mathbf{1}\{x_n \le F_\phi^{-1}(z \mid \hat{\theta})\}(F_\phi(x_n \mid \hat{\theta}) - F^-(x_n \mid \hat{\theta})) - \sum_{n=1}^{\infty} \mathbf{1}\{z \ge F_\phi(x_n \mid \hat{\theta})\} \\ =& F_\phi(x_k \mid \hat{\theta}) + z - F_\phi^-(x_k \mid \hat{\theta}) - (F_\phi(x_k \mid \hat{\theta}) - F_\phi^-(x_k \mid \hat{\theta})) = z\,.\end{aligned}$$

Consequently, for any $z \in [0,1]$,

$$\mathbb{P}[q_\phi \le z \mid \hat{\theta}] = \mathbb{P}[rF_\phi(\phi \mid \hat{\theta}) + (1-r)F_\phi^-(\phi \mid \hat{\theta}) \mid \hat{\theta}] = \int_{\mathbb{R}} \Gamma_z(x)\,\mathrm{d}F_\phi(x \mid \hat{\theta}) = z\,,$$

as desired. Therefore, q_ϕ is independent of θ and thus, by Lemma 1, is privacy-preserving. □

Definition A.1. *For any statistic $\phi : \Omega \to \mathbb{R}$. A signal s for (ϕ, θ)* conditionally reveals *ϕ if for all (Borel) measurable $A \subseteq \mathbb{R}$, $\mathbb{P}[\phi \in A \mid s, \theta] \in \{0,1\}$ a.s..*

Lemma A.2. *A signal for (ϕ, θ) is privacy-preserving if and only if it is Blackwell dominated in terms of (ϕ, θ) by a privacy-preserving signal that conditionally reveals ϕ.*

Proof. Sufficiency follows from the same arguments that prove Lemma 2, with ω replaced by (ϕ, θ). For necessity, consider any privacy-preserving signal $\tilde{s}$ for (ϕ, θ). Let $\xi, \zeta : [0,1] \to [0,1]$ be two independently and uniformly distributed random variables. Then there exists

another privacy-preserving signal $s \sim \tilde{s}$ that is measurable with respect to (ω, ξ) and is independent of (ζ, θ). Let $F_\phi(\cdot \mid \theta, s)$ be the conditional distribution of ϕ given θ and s, and let $t := \zeta F_\phi(\phi \mid \theta, s) + (1-\zeta)F_\phi^-(\phi \mid \theta, s)$. Then, since s and θ are independent of ζ, by the same arguments as the proof of Lemma 3, t is uniformly distributed conditional on θ and s. Therefore, the signal (t, s) is privacy-preserving. Moreover, by construction, (t, s) is measurable with respect to (ϕ, θ, r) and thus is a signal for (ϕ, θ). Meanwhile, since (t, s) reveals more information than s, $\tilde{s} \sim s \preceq (t, s)$. Lastly, by construction $F_\phi^{-1}(t \mid \theta, s) = \phi$ almost surely. Thus, (t, s) conditionally reveals ϕ. This completes the proof. □

Lemma A.3. *Any privacy-preserving signal for (ϕ, θ) that conditionally reveals ϕ is Blackwell undominated in terms of (ϕ, θ) among all privacy-preserving signals for (ϕ, θ).*

Proof. Let Γ be the set of probability measures whose marginal over Θ equals $\nu(\cdot) := \mathbb{P}[\theta \in \cdot]$ (i.e., for all $\mu \in \Gamma$, $\mu(\mathbb{R} \times B) = \mathbb{P}[\theta \in B] =: \nu(B)$). Then for any $\mu \in \Gamma$, since $\mathbb{R}$ is standard-Borel, there exists a transition probability $T[\mu] : \mathbb{R} \times \Theta \to \Delta(\mathbb{R} \times \Theta)$ (see, e.g., Çinlar 2010, Theorem 2.18, pp. 154),[37] such that

$$\mu((-\infty, x] \times B) = \int_B T[\mu](x \mid \theta)\, \mathrm{d}\nu(\theta),$$

for all $x \in \mathbb{R}$ and for all measurable $B \subseteq \Theta$.

Note that for any privacy-preserving signal s for (ϕ, θ), let λ^s be the induced distribution of posterior beliefs. Then $\lambda^s(\Gamma) = 1$. If, furthermore, s conditionally reveals ϕ, then for λ^s-almost all $\mu \in \Delta(\mathbb{R} \times \Theta)$, $T[\mu](\cdot \mid \theta)$ is a step function with two steps for ν-almost all $\theta \in \Theta$.

Suppose now that a privacy-preserving signal s for (ϕ, θ) which conditionally reveals ϕ is Blackwell dominated by a privacy-preserving signal $\tilde{s}$ for (ϕ, θ). Suppose also that s is not Blackwell equivalent to $\tilde{s}$. Then, by Theorem 2 of Strassen (1965), there exists a dilation $K : \Delta(\mathbb{R} \times \Theta) \to \Delta\Delta(\mathbb{R} \times \Theta)$ such that $\lambda_{\tilde{s}}(E) = \int_{\Delta(\mathbb{R} \times \Theta)} K(E \mid \mu)\, \mathrm{d}\lambda_s(\mu)$, for all measurable $E \subseteq \Delta(\mathbb{R} \times \Theta)$. Since s is not Blackwell equivalent to $\tilde{s}$, there exists a measurable set $E_0 \subseteq \Delta(\mathbb{R} \times \Theta)$ with positive λ^s-measure, such that for all $\mu \in E_0$, $K(\cdot \mid \mu)$ is not degenerate.

[37] By the proof of Lemma 1, the measurable Θ can be taken as a subset of $\mathbb{R}$. Therefore, $\Delta(\mathbb{R} \times \Theta)$ can be endowed with the weak-* topology and the associated Borel σ-algebra, which makes $\Delta(\mathbb{R} \times \Theta)$ a Polish space and hence is standard Borel. Therefore, $T[\cdot]$ can be regarded as a measurable function from $\Delta(\mathbb{R} \times \Theta)$ to $\mathcal{G}^\Theta$, where $\mathcal{G}$ is the space of CDFs on $\mathbb{R}$, with the Borel σ-algebra generated by the weak-* topology, and the σ-algebra over $\mathcal{G}^\Theta$ is the product σ-algebra. Moreover, since $T[\mu]$ is a transition probability, for all $\mu, \tilde{\mu} \in \Gamma$, $\mu \neq \tilde{\mu}$ implies $T[\mu] \neq T[\tilde{\mu}]$.

Moreover, since $\tilde{s}$ is privacy-preserving, $\lambda^{\tilde{s}}(\Gamma) = 1$, and hence $K(\Gamma \mid \mu) = 1$ for λ^s-almost all $\mu \in \Delta\Delta(\mathbb{R} \times \Theta)$. Lastly, since K is a dilation, for any $\mu \in E_0$, $\int_{\Delta(\mathbb{R}\times\Theta)} \tilde{\mu}(A)\,\mathrm{d}K(\tilde{\mu} \mid \mu) = \mu(A)$, for all $A \subseteq \mathbb{R} \times \Theta$.

As a result, for any $\mu \in E_0$, for all $x \in \mathbb{R}$ and for all measurable $B \subseteq \Theta$,

$$\begin{aligned}\int_B T[\mu](x \mid \theta)\,\mathrm{d}\nu(\theta) = \mu((-\infty, x] \times B) &= \int_{\Delta(\mathbb{R}\times\Theta)} \tilde{\mu}((-\infty, B)\,\mathrm{d}K(\tilde{\mu} \mid \mu) \\ &= \int_{\Delta(\mathbb{R}\times\Theta)} \left(\int_B T[\tilde{\mu}](x \mid \theta)\,\mathrm{d}\nu(\theta)\right)\mathrm{d}K(\tilde{\mu} \mid \mu) \\ &= \int_B \left(\int_{\Delta(\mathbb{R}\times\Theta)} T[\tilde{\mu}](x \mid \theta)\,\mathrm{d}K(\tilde{\mu} \mid \mu)\right)\mathrm{d}\nu(\theta)\,,\end{aligned}$$

and hence for ν-almost all $\theta \in \Theta$,

$$T[\mu](x \mid \theta) = \int_{\Delta(\mathbb{R}\times\Theta)} T[\tilde{\mu}](x \mid \theta)\,\mathrm{d}K(\tilde{\mu} \mid \mu)\,,$$

a contradiction, since for ν-almost all θ, $T[\mu](\cdot \mid \theta)$ is a step function with two steps and hence cannot be written as a mixture of distinct CDFs. Therefore, s cannot be Blackwell dominated by $\tilde{s}$. □

Lemma A.4. *For any signal $\tilde{s}$, there exists a signal $s : \Omega \times [0,1] \to [0,1]$ such that $s \sim \tilde{s}$ and that $\mathbb{P}[s \le x] = x$ for all $x \in [0,1]$.*

Proof. Consider any signal $\tilde{s} : \Omega \to \mathbb{R}$. Let $\xi, \zeta : [0,1] \to [0,1]$ be two independently and uniformly distributed random variables. Then there exists another signal $\bar{s} \sim \tilde{s}$ that is measurable with respect to (ω, ξ) and is independent of ζ. Define another signal $s' : \Omega \times [0,1] \to \Delta(\Omega)$ as $s'(\omega, r) := \mathbb{P}[\omega \in \cdot \mid s](\omega, r)$. Since $\Delta(\Omega)$, with the Borel-σ algebra induced by the weak-* topology, is standard Borel, there exists an isomorphism $\psi : \Delta(\Omega) \to [0,1]$. Since ψ is invertible, $\psi(s') \sim \bar{s}$. Let F_ψ be the distribution of $\psi(s')$, so that $F_\psi(z) := \mathbb{P}[\psi(s') \le z]$ for all $z \in [0,1]$. Then, let $s := \zeta F_\psi(\psi(s')) + (1-\zeta)F_\psi^-(\psi(s'))$. By the same arguments as the proof of Lemma 3, s is uniformly distributed. Moreover, since $F_\psi^{-1}(s) = \psi(s')$ a.e., $s \sim \psi(s') \sim \bar{s} \sim \tilde{s}$. This completes the proof. □

Lemma A.5. *Every privacy-preserving signal for (ϕ, θ) that conditionally reveals ϕ is Blackwell equivalent to some reordered ϕ-quantile signal.*

Proof. Consider any privacy-preserving signal $s : \Omega \times [0,1] \to S$ for ϕ that conditionally reveals ϕ. By Lemma A.4, it is without loss to assume that $S = [0,1]$ and that the marginal distribution of s is uniform. Since s conditionally reveals ϕ, there exists a measurable function $\eta : [0,1] \times \Theta \to \mathbb{R}$ such that $\phi = \eta(s,\theta)$ almost surely. Therefore, for any $\hat{\theta} \in \Theta$,

$$\mathbb{P}[\eta(s,\theta) \leq x \mid \hat{\theta}] = \mathbb{P}[\phi \leq x \mid \hat{\theta}] = F_\phi(x \mid \hat{\theta}),$$

and hence, for any $\hat{\theta} \in \Theta$, the nondecreasing rearrangement of $\eta(\cdot, \hat{\theta})$ is $F_\phi^{-1}(\cdot \mid \hat{\theta})$. Thus, by Proposition 3 of Ryff (1970), there exists a family $\{M_{\hat{\theta}}\}_{\hat{\theta} \in \Theta}$ of measure-preserving transformations such that

$$\eta(s, \hat{\theta}) = F_\phi^{-1}(M_{\hat{\theta}}(s) \mid \hat{\theta}),$$

for all $\hat{\theta} \in \Theta$. Meanwhile, let t be the M-reordered ϕ-quantile signal, the posterior belief conditional on realization $\hat{t}$ and on $\hat{\theta}$ is $F_\phi^{-1}(M_{\hat{\theta}}(\hat{t}) \mid \hat{\theta})$. Together, $s \sim t$, as desired. □

Proof of Proposition 1. Take $\phi : \Omega \to \mathbb{R}$ to be an invertible statistic with ϕ^{-1} being measurable. Then any signal is a signal for (ϕ, θ) and signal $\tilde{s}$ Blackwell dominates signal s if and only if $\tilde{s}$ Blackwell dominates s in terms of (ϕ, θ). Proposition 1 then follows from Lemma 2 and Lemma A.2. □

Proof of Theorem 2. (i) follows from Lemma 2, Lemma A.2, and Lemma A.5. (ii) follows from Lemma A.3 and Lemma A.5. □

Proof of Theorem 1. For any statistic ϕ such that the ϕ-quantile signal q_ϕ is conditionally revealing, the σ-algebra generated by (ϕ, θ) must be $\mathcal{F}$. Therefore, every signal s is a signal for (ϕ, θ). Since Ω is a standard Borel space, there exists an invertible $\phi : \Omega \to \mathbb{R}$ such that ϕ^{-1} is measurable, and hence there exists a conditionally revealing ϕ-quantile signal. As a result, Theorem 1 follows from Theorem 2. □

Proof of Theorem 3. For sufficiency, since $\overline{F}_\phi$ is induced by the ϕ-quantile signal q_ϕ, every mean-preserving contraction G of $\overline{F}_\phi$ can be induced by a signal for (ϕ, θ) that is Blackwell dominated by q_ϕ by Strassen's theorem (Strassen 1965). Together with Theorem 2, G can be induced by a privacy-preserving signal. To prove necessity, by Theorem 2, it suffices to show that the distribution G of posterior means induced by any reordered ϕ-quantile signal is a mean-preserving contraction of $\overline{F}_\phi$. To see this, observe that for any family $\{M_{\hat{\theta}}\}_{\hat{\theta} \in \Theta}$, the

posterior mean conditional on a realization $\hat{s}$ of the M-reordered ϕ-quantile signal s is given by

$$\mathbb{E}[F_\phi^{-1}(M_\theta(\hat{s}) \mid \theta)]\,. \tag{A.7}$$

Let G be the distribution of the random variable $\mathbb{E}[F_\phi^{-1}(M_\theta(s) \mid \theta)]$, where the expectation is taken over θ. Since s is uniformly distributed, the quantile function G^{-1} is the nodecreasing arrangement of the function $\hat{s} \mapsto \mathbb{E}[F_\phi^{-1}(M_\theta(\hat{s}) \mid \theta)]$. Thus, by Proposition 3 of Ryff (1970), there exists a Lebesgue measure-preserving transformation $\psi : [0,1] \to [0,1]$ such that for almost all $\hat{s} \in [0,1]$,

$$G^{-1}(\psi(\hat{s})) = \mathbb{E}[F_\phi^{-1}(M_\theta(\hat{s}) \mid \theta)]\,.$$

As ψ and $M_{\hat{\theta}}$ are Lebesgue measure-preserving transformations for all $\hat{\theta} \in \Theta$, we have that for all $t \in [0,1]$,

$$\begin{aligned}\int_t^1 G^{-1}(y)\,\mathrm{d}y = \int_0^1 \mathbf{1}\{\psi(z) \geq t\} G^{-1}(\psi(z))\,\mathrm{d}z &= \int_0^1 \mathbf{1}\{\psi(z) \geq t\}\mathbb{E}[F_\phi^{-1}(M_\theta(z) \mid \theta)]\,\mathrm{d}z \\ &\leq \int_0^1 \mathbf{1}\{z \geq t\}\mathbb{E}[F_\phi^{-1}(z \mid \theta)]\,\mathrm{d}z \\ &= \int_t^1 \overline{F}_\phi^{-1}(z)\,\mathrm{d}z\,,\end{aligned}$$

where the first equality follows since ψ is Lebesgue measure-preserving, and the inequality follows from Fubini's theorem and the Hardy-Littlewood-Polya inequality (see, e.g., Hardy et al. 1929 and Puccetti and Wang 2015, Theorem 2.1). Thus, G^{-1} is majorized by $\overline{F}_\phi^{-1}$, which implies that G is a mean-preserving contraction of $\overline{F}_\phi$ (see, e.g., Shaked and Shanthikumar (2007), Section 3.A). □

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Online Appendix

Proof of Lemma 4

We prove Lemma 4 by proving a more general result, as stated below.

Lemma OA.1. *Consider any statistic $\phi : \Omega \to \mathbb{R}$. Let $\mathcal{D}_\phi$ be the collection of distributions $\rho \in \Delta(\mathbb{R}^J)$ such that the marginal of the j-th component equals $F_\phi(\cdot \mid \theta_j)$. Then, $\rho \in \Delta(\mathbb{R}^J)$ is the joint distribution of $(F_\phi^{-1}(M_{\theta_j}(s)) \mid \theta_j)_{j=1}^J$ for the M-reordered ϕ-quantile signal if and only if $\rho \in \mathcal{D}_\phi$.*

Proof. Consider any family $M = \{M_{\theta_j}\}_{j=1}^J$ of measure-preserving transformations. Since s is uniformly distributed, the distribution of $F^{-1}(M_{\theta_j}(s) \mid \theta_j)$ is $F(\cdot \mid \theta_j)$ for all $j \in \{1, \dots, J\}$. Therefore, the joint distribution ρ of $(F^{-1}(M_{\theta_j}(s) \mid \theta_j))_{j=1}^J$ is in $\mathcal{D}_\phi$.

Conversely, consider any $\rho \in \mathcal{D}$. Since Ω^J is standard Borel, there exists measurable functions $\{\eta_j\}_{j=1}^J$ such that the joint distribution of $(\eta_j(s))_{j=1}^J$ is ρ, where s is a uniform random variable on $[0,1]$. Since for all $j \in \{1, \dots, J\}$, $\eta_j(s)$ and $F^{-1}(s \mid \theta_j)$ have the same distribution, there exists a measure-preserving transformation $M_{\theta_j} : [0,1] \to [0,1]$ such that $\eta_j(\hat{s}) = F^{-1}(\Phi_{\theta_j}(\hat{s}) \mid \theta_j)$ for all $\hat{s} \in [0,1]$ and for all $j \in \{1, \dots, J\}$ by Proposition 3 of Ryff (1970), as desired. □

With Lemma OA.1, Lemma 4 follows immediately by taking ϕ to be a Borel isomorphism.

Proof of Proposition 2

We prove Proposition 2 by proving a more general result, stated as follows.

Proposition OA.1. *For any statistic $\phi : \Omega \to \mathbb{R}$, and for any decision problem (u, A) with $u(\omega, a) = h(\phi(\omega), \theta(\omega), a)$, let*

$$\widetilde{V}(x_1, \dots, x_J) := \sup_{a \in A} \left(\sum_{j=1}^J h(x_j, \theta_j, a) \mathbb{P}[\theta = \theta_j] \right), \tag{A.8}$$

for all $(x_j)_{j=1}^J \in \mathbb{R}^J$. Then, the decision-maker's optimal value $V^\star$ among all privacy-preserving

signals for (ϕ, θ) *is given by*

$$V^\star = \sup_{\rho \in \mathcal{D}_\phi} \int_{\mathbb{R}^J} \widetilde{V}(x_1, \dots, x_J) \, \mathrm{d}\rho \,. \tag{A.9}$$

Moreover, any optimal privacy-preserving signal must be Blackwell-equivalent to some M*-reordered* ϕ*-quantile signal such that the distribution of* $(F_\phi^{-1}(M_{\theta_j}(s) \mid \theta_j))_{j=1}^J$ *is a solution of* (A.9).

Proof. By Theorem 2 and Blackwell's theorem, any privacy-preserving signal for (ϕ, θ) yields a (weakly) lower payoff to the decision-maker than some reordered ϕ-quantile signal. Together with Lemma OA.1, it then follows that

$$V^\star = \sup_{\rho \in \mathcal{D}_\phi} \int_{\mathbb{R}^J} \widetilde{V}(x_1, \dots, x_J) \, \mathrm{d}\rho \,.$$

Moreover, by Theorem 2, any privacy-preserving for (ϕ, θ) that yields $V^\star$ must be the M-reordered ϕ-quantile signal s, for some family $M = \{M_{\theta_j}\}_{j=1}^J$ of measure-preserving transformations. Thus, by Lemma OA.1, the joint distribution of $(F_\phi^{-1}(M_{\theta_j}(s), \theta_j))_{j=1}^J$ must be a solution of (A.9). □

With Proposition OA.1, Proposition 2 follows immediately by taking ϕ as a Borel isomoriphism.

Proof of Proposition 3

Let $\widehat{V} : \mathbb{R}^J \times A \to \mathbb{R}$ be defined as

$$\widehat{V}(x_1, \dots, x_J, a) := \sum_{j=1}^J h(x_j, \theta_j, a) \mathbb{P}[\theta = \theta_j].$$

We first show that $\widehat{V}$ has increasing difference in $(x_1, \dots, x_J)$ and a, and is supermodular in $(x_1, \dots, x_J)$. Indeed, for any $a, a' \in A$ and $\mathbf{x} = (x_j)_{j=1}^J, \mathbf{x}' = (x'_j)_{j=1}^J \in \Omega^J$ such that $a \geq a'$ and

$x_j \geq x'_j$ for all j,

$$\begin{aligned}\widehat{V}(\mathbf{x}, a) - \widehat{V}(\mathbf{x}, a') =& \sum_{j=1}^{J} [h(x_j, \theta_j, a) - h(x'_j, \theta_j, a')]\mathbb{P}[\theta = \theta_j] \\ \geq& \sum_{j=1}^{J} [h(x'_j, \theta_j, a) - h(x'_j, \theta_j, a')]\mathbb{P}[\theta = \theta_j] \\ =& \widehat{V}(\mathbf{x}', a) - \widehat{V}(\mathbf{x}', a'),\end{aligned}$$

where the inequality follows from the supermodularity of h. Furthermore, for any $\mathbf{x} = (x_j)_{j=1}^J, \mathbf{x}' = (x'_j)_{j=1}^J \in \Omega^J$ and for all $a \in A$,

$$\begin{aligned}\widehat{V}(\mathbf{x} \vee \mathbf{x}', a) + \widehat{V}(\mathbf{x} \wedge \mathbf{x}', a) =& \sum_{j=1}^{J} [h(\max\{x_j, x'_j\}, \theta_j, a) + h(\min\{x_j, x'_j\}, \theta_j, a)]\mathbb{P}[\theta = \theta_j] \\ =& \sum_{j=1}^{J} [h(x_j, \theta_j, a) + u(x'_j, \theta_j, a)]\mathbb{P}[\theta = \theta_j] = \widehat{V}(\mathbf{x}, a) + \widehat{V}(\mathbf{x}', a),\end{aligned}$$

We next show that $\widetilde{V} : \Omega^J \to \mathbb{R}$ defined in (A.8) is supermodular. Since $\operatorname{argmax}_{a \in A} \widehat{V}(\mathbf{x}, a)$ is nonempty for all $\mathbf{x} \in \Omega^J$, for any $a^\star(\mathbf{x}) \in \operatorname{argmax}_{a \in A} \widehat{V}(\mathbf{x}, a)$ and for any $\mathbf{x} = (x_j)_{j=1}^J \in \mathbb{R}^J$, $\widetilde{V}(\mathbf{x}) = \widehat{V}(\mathbf{x}, a^\star(\mathbf{x}))$. Therefore, it suffices to show that

$$\widehat{V}(\mathbf{x} \vee \mathbf{x}', a^\star(\mathbf{x} \vee \mathbf{x}')) + \widehat{V}(\mathbf{x} \wedge \mathbf{x}', a^\star(\mathbf{x} \wedge \mathbf{x}')) \geq \widehat{V}(\mathbf{x}, a^\star(\mathbf{x})) + \widehat{V}(\mathbf{x}', a^\star(\mathbf{x}')),$$

for all $\mathbf{x}, \mathbf{x}' \in \Omega^J$, and for any selection $a^\star(\cdot)$ for the argmax correspondence. To see this, consider any $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^J$ and any selection $a^\star(\cdot)$. Since A is totally ordered, it is without loss

to assume that $a^\star(\mathbf{x}) \geq a^\star(\mathbf{x}')$. As a result,

$$
\begin{aligned}
&\widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}\vee\mathbf{x}')) + \widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}\wedge\mathbf{x}')))\\
=&\widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x})) + \widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}))\\
&+ [\widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}\vee\mathbf{x}')) - \widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}))] + [\widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}\wedge\mathbf{x}')) - \widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}))]\\
\geq&\widehat{V}(\mathbf{x}, a^\star(\mathbf{x})) + \widehat{V}(\mathbf{x}', a^\star(\mathbf{x}))\\
&+ [\widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}\vee\mathbf{x}')) - \widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}))] + [\widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}\wedge\mathbf{x}')) - \widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}))]\\
=&\widehat{V}(\mathbf{x}, a^\star(\mathbf{x})) + \widehat{V}(\mathbf{x}', a^\star(\mathbf{x}')) + \widehat{V}(\mathbf{x}', a^\star(\mathbf{x})) - \widehat{V}(\mathbf{x}', a^\star(\mathbf{x}'))\\
&+ [\widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}\vee\mathbf{x}')) - \widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}))] + [\widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}\wedge\mathbf{x}')) - \widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}))]\\
\geq&\widehat{V}(\mathbf{x}, a^\star(\mathbf{x})) + \widehat{V}(\mathbf{x}', a^\star(\mathbf{x}')) + \widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x})) - \widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}'))\\
&+ [\widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}\vee\mathbf{x}')) - \widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}))] + [\widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}\wedge\mathbf{x}')) - \widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}))]\\
=&\widehat{V}(\mathbf{x}, a^\star(\mathbf{x})) + \widehat{V}(\mathbf{x}', a^\star(\mathbf{x}'))\\
&+ [\widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}\vee\mathbf{x}')) - \widehat{V}(\mathbf{x}\vee\mathbf{x}', a^\star(\mathbf{x}))] + [\widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}\wedge\mathbf{x}')) - \widehat{V}(\mathbf{x}\wedge\mathbf{x}', a^\star(\mathbf{x}'))]\\
\geq&\widehat{V}(\mathbf{x}, a^\star(\mathbf{x})) + \widehat{V}(\mathbf{x}', a^\star(\mathbf{x}'))\,,
\end{aligned}
$$

where the first inequality follows from supermodularity of $\widehat{V}$, the second inequality follows from the increasing difference property of $\widehat{V}$ and from $a^\star(\mathbf{x}) \geq a^\star(\mathbf{x}')$, and the third inequality follows from optimality of $a^\star$.

Now let

$$\widetilde{V}(\mathbf{x}) \coloneqq \sup_{a\in A} \widehat{V}(\mathbf{x}, a) = \sup_{a\in A} \sum_{j=1}^{J} h(x_j, \theta_j, a)\mathbb{P}[\theta = \theta_j]\,,$$

for all $\mathbf{x} \in \mathbb{R}^J$. Note that by Lemma OA.1, (A.9) is equivalent to choosing a family $\{M_j\}_{j=1}^J$ of measure-preserving transformations to maximize

$$\int_0^1 \widetilde{V}(F_\phi^{-1}(M_1(q) \mid \theta_1), \ldots, F_\phi^{-1}(M_J(q) \mid \theta_J))\,\mathrm{d}q\,.$$

Since $\widetilde{V}$ is supermodular, Theorem 5 of Tchen (1980) (see also, Theorem 2.1 of Puccetti and Wang 2015) implies that

$$\int_0^1 \widetilde{V}(F_\phi^{-1}(M_1(q) \mid \theta_1), \ldots, F_\phi^{-1}(M_J(q) \mid \theta_J))\,\mathrm{d}q \leq \int_0^1 \widetilde{V}(F_\phi^{-1}(q \mid \theta_1), \ldots F_\phi^{-1}(q \mid \theta_J))\,\mathrm{d}q$$

for any family $\{M_j\}_{j=1}^J$ of measure-preserving transformations. Together with Proposition OA.1, $V^\star$ is attained by the ϕ-quantile signal, as desired. □

Proof of Proposition 6

For any $\rho \in \mathcal{D}$, Lemma 4 implies that there exists a family $M = \{M_{\hat{\theta}}\}_{\hat{\theta}\in\Theta}$ of measure-preserving transformations such that the joint distribution of $(\tilde{\omega}_j^M)_{j=1}^J$ is ρ. Consider the problem where the sender is restricted to choose garblings of the M-reordered ϕ-quantile signal, for some conditionally revealing ϕ-quantile signal. Standard arguments (Kamenica and Gentzkow 2011) implies that the sender's value in this restricted problem is $\overline{V}_S(\rho)$. By Theorem 1, since every privacy-preserving signal is a garbling of some reordered ϕ-quantile signal, the sender's value $V_S^\star$ in the original problem must be given by

$$\max_{\rho\in\mathcal{D}} \overline{V}_S(\rho)\,. \qquad \square$$

Proof of Proposition 13

Consider the market segmentation that corresponds to the quantile signal q. Under this segmentation, there is a continuum of segments $\hat{q} \in [0,1]$, and in each segment $\hat{q} \in [0,1]$, there are J possible consumer values $\{F^{-1}(\hat{q} \mid \theta_j)\}_{j=1}^J$. Let $\rho^\star \in \mathcal{D}$ be the joint distribution of $\{F^{-1}(q \mid \theta_j)\}_{j=1}^J$.

We now show that $\rho^\star$ solves the optimal transport problem (6). To this end, we construct the Lagrange multipliers such that weak duality holds under $\rho^\star$. Let $K_1(x_1) := x_1$, and let $K_j(x_j) := 0$ for all $j \in \{2,\dots,J\}$. Then, since $F(\cdot \mid \theta_1) \geq \dots \geq F(\cdot \mid \theta_J)$, $x_1 \leq \dots, x_J$ for all $(x_j)_{j=1}^J \in \operatorname{supp}(\rho^\star)$. Moreover, since $(1 - \mathbb{P}[\theta = \theta_1]) \cdot \overline{x} \leq \underline{x}$, $V(x_1,\dots,x_n) = x_1$ for all $(x_j)_{j=1}^J \in \operatorname{supp}(\rho^\star)$. Therefore,

$$\sum_{j=1}^J K_j(x_j) = x_1 = V(x_1,\dots,x_J)\,,$$

for all $(x_j)_{j=1}^J \in \operatorname{supp}(\rho^\star)$.

Meanwhile, for any $(x_j)_{j=1}^J \in [\underline{x},\overline{x}]^J$, if $x_1 = \min\{x_j\}_{j=1}^J$, then $V(x_1,\dots,x_J) = x_1$. If $x_1 > \min\{x_j\}_{j=1}^J$, let $x_{(j)}$ denotes the (j)-th smallest element of $(x_j)_{j=1}^J \in [\underline{x},\overline{x}]^J$ and let

$(j) \in \{1, \ldots, J\}$ be the index of that element. Then,

$$x_1 \geq \underline{x} \geq (1 - \mathbb{P}[\theta = \theta_1])\overline{x} \geq \sum_{j=i}^{J} \mathbb{P}[\theta = \theta_{(j)}]\overline{x} \geq \sum_{j=i}^{J} \mathbb{P}[\theta = \theta_{(j)}]x_{(j)}\,,$$

for all $i \in \{2, \ldots, J\}$. Thus,

$$\sum_{j=1}^{J} K_j(x_j) = x_1 \geq \max_{i \in \{1,\ldots,J\}} \sum_{j=i}^{J} \mathbb{P}[\theta = \theta_{(j)}]x_{(j)} = V(x_1, \ldots, x_J)\,.$$

As a result, $\{K_j\}_{j=1}^{J}$ are the Lagrange multipliers that warrant $\rho^\star$ as a solution. This proves (i). (ii) through (iv) then follows immediately from the fact that $F^{-1}(\hat{q} \mid \theta_1) \leq F^{-1}(\hat{q} \mid \theta_j)$ for all $\hat{q} \in [0,1]$ and for all $j \in \{1, \ldots, J\}$. This completes the proof. □

Constructing a Reordered Quantile Signal

Consider any ϕ-quantile signal q_ϕ. Let $\{M_{\hat{\theta}}\}_{\hat{\theta} \in \Theta}$ be a family of measure-preserving transformations such that $\hat{\theta} \mapsto M_{\hat{\theta}}(s)$ is measurable, for all $s \in [0,1]$. We now construct explicitly the M-reordered ϕ-quantile signal.

Let $\xi, \zeta : [0,1] \to [0,1]$ be two independent random variables that are uniformly distributed. Let

$$q := \xi F_\phi(\phi \mid \theta) + (1 - \xi)F_\phi^-(\phi \mid \theta)\,.$$

Then q is independent of ζ and is Blackwell equivalent to q_ϕ. Fix any $\hat{\theta} \in \Theta$, and let $C_{\hat{\theta}}$ be the joint distribution of $(\zeta, M_{\hat{\theta}}(\zeta))$, i.e.,

$$C_{\hat{\theta}}(u, v) := \mathbb{P}[\zeta \leq u, M_{\hat{\theta}}(\zeta) \leq v]\,,$$

for all $u, v \in [0,1]$. Note that by definition $C_{\hat{\theta}}$ assigns probability one to the set $\{(u,v) \in [0,1]^2 : u = M_{\hat{\theta}}(v)\}$

Let $K_{\hat{\theta}} : [0,1] \to [0,1]$ be the disintegration (see, e.g., Çinlar 2010, Theorem 2.18, pp. 154) of $C_{\hat{\theta}}$ with respect to the Lebesgue measure, so that

$$C_{\hat{\theta}}(u, v) := \int_0^u K_{\hat{\theta}}(v \mid z)\,\mathrm{d}z\,.$$

Thus, for any random variable s that is distributed according to $K_{\hat{\theta}}(\cdot \mid \hat{q})$ conditional on $\hat{\theta}$ and $\hat{q}$,

$$\mathbb{P}[M_{\hat{\theta}}(s) = \hat{q} \mid \hat{\theta}, \hat{q}] = 1\,.$$

Thus, let

$$s \coloneqq K_{\theta}^{-1}(\zeta \mid q)\,.$$

The distribution of s conditional on $\hat{\theta}$ and $\hat{q}$ is then $K_{\hat{\theta}}(\cdot \mid \hat{q})$. Therefore, almost surely,

$$M_{\theta}(s) = q\,,$$

which in turns imply that $M_{\theta}(s) \sim q_{\phi}$, as desired.

Furthermore, note that for any s such that $M_{\theta}(s) \sim q_{\phi}$, s must be Blackwell equivalent to a signal $\tilde{s}$ that is distributed according to $K_{\hat{\theta}}(\cdot \mid \hat{q})$ conditional on $\hat{\theta}$ and $\hat{q}$, for almost all $\hat{\theta}$ and $\hat{q}$. Since the disintegration of $C_{\hat{\theta}}$ is essentially unique, it follows that a reordered ϕ-quantile signal is unique up to Blackwell equivalence.

Belief-Based Characterization of Privacy-Preserving Signals

For completeness, we provide a characterization of privacy-preserving signals in terms of distributions over posterior beliefs that is equivalent to Theorem 1. Denote by $p_0(A) \coloneqq \mathbb{P}[\omega \in A]$ the probability of event $A \in \mathcal{F}$ under the prior.

Suppose that there are only finitely many states $|\Omega| < \infty$ and (without loss) that the privacy sets are disjoint: $P \cap P' = \varnothing$ for all $P, P' \in \mathcal{P}$. From Blackwell's theorem (Blackwell 1953), a signal s can be equivalently represented by a random variable $p : \Omega \to \Delta(\Omega)$ such that $\mathbb{E}[p] = p_0$. Therefore, a signal p is privacy-preserving if and only if

$$\begin{aligned} \mathbb{E}\left[p\right] &= p_0 \\ \mathbb{P}\left[\sum_{\omega \in P} p(\omega) = \sum_{\omega \in P} p_0(\omega)\right] &= 1 \qquad \forall P \in \mathcal{P}\,. \end{aligned} \tag{A.10}$$

Our results shows that the Blackwell frontier of the set of privacy-preserving signals is given

by those privacy-preserving signals which in addition satisfy

$$\mathbb{P}\left[|\text{supp}\,(p) \cap P| = 1\right] = 1 \qquad \forall P \in \mathcal{P}\,. \tag{A.11}$$

In other words, (A.10) holds if and only if p is a mean-preserving contraction of some $\tilde{p}$ which satisfies (A.10) and (A.11). For a general state space Ω, the characterization is similar.

Relaxing the Independence Criterion

An immediate implication of Remark 2 is a relaxation of the definition of privacy-preserving signals. While we define privacy-preserving signals by the notion of independence, conditionally privacy-preserving signals relaxes the independence requirement by allowing for correlations with the component y. In particular, one may consider a signal y that is independent of a statistic $\phi : \Omega \to \mathbb{R}$, but is not privacy-preserving.[38] Conditionally privacy-preserving signals in this environment can then be regarded as privacy-preserving signals with a less stringent requirement for posterior beliefs, as changes in posterior beliefs on privacy sets conditional on realizations of s would be allowed as long as it is through y.

Another Notion of Algorithmic Fairness

In §5.1, we show how our results can be applied to the literature of algorithmic fairness and demonstrate how privacy-preserving signals are related to the notion of fairness called independence. In the literature on algorithmic fairness, there are other notions of fairness that do not require statistical independence, as discussed in §5.1. One of the most commonly used alternatives to statistical independence is called *separation*. Separation requires the decisions to be independent of protected characteristics *conditional on the true state*.

Our results can also be applied to this setting. To see this, suppose that the underlying outcome, γ, is binary and takes values 0 or 1. Let x be the expected probability of the underlying state being $\gamma = 1$, conditional on all the observable covariates (including protected characteristics θ). A signal would satisfy the requirement of separation if its realization is independent of θ conditional on γ. Consider any conditionally privacy-preserving signal s. By

[38] In fact, we can fully characterize these signals, as they are equivalent to privacy-preserving signals for θ that are independent of ϕ.

definition, such a signal would be independent of θ conditional on γ. Moreover, a conditionally privacy-preserving signal is Blackwell-undominated if and only if it takes the form of (s, γ), where $\tilde{s}$ is some reordered quantile signal conditional on γ. Although the signal that reveals $(\tilde{s}, \gamma)$ may not be feasible, as the outcome γ is typically unknown, one can project this signal by computing the conditional expectation of $(\tilde{s}, \gamma)$ given x. This conditional expectation is thus, by construction, a garbling of x, and is conditionally independent of θ given γ. Furthermore, since taking the conditional expectation preserves the Blackwell order, this signal must remain Blackwell-undominated among all feasible signals.

Privacy-Preserving Segmentation and Uniform Pricing

Consider the following example demonstrating that high value consumers might be better-off under the seller-optimal privacy-preserving segmentation than under uniform pricing. Suppose that $X = [1/3, 1]$, $F(x \mid \theta_1) = 3/2(x - 1/3)$, $F(x \mid \theta_2) = 2(x - 1/2)^+$ for all $x \in X$, and $\mathbb{P}[\theta = \theta_1] = 1/3$, $\mathbb{P}[\theta = \theta_2] = 2/3$. The optimal uniform price is $4/5$, and the surplus of θ_2 consumers is $1/25$. Under the seller-optimal privacy-preserving segmentation, the surplus of θ_2 consumers is $1/12 > 1/25$.

Verifying the Optimality of $\rho^\star$

In Section 5.4, we claim that the joint distribution $\rho^\star$ is optimal in our example. To see this, recall that a joint distribution $\rho \in \mathcal{D}$ is a solution of the associated optimal transport problem if and only if there exists Lagrange multipliers $K_1, K_2 : \{1, 2, 3\} \to \mathbb{R}$ that satisfy the complementary slackness condition: $K_1(x_1) + K_2(x_2) \geq V(x_1, x_2)$, for all $(x_1, x_2) \in \{1, 2, 3\}^2$, with equality on the support of ρ. It can then be verified that the complementary slackness condition is satisfied under the Lagrange multipliers $(K_1(x))_{x \in \{1,2,3\}} = (1, 2, 5/2)$ and $(K_2(x))_{x \in \{1,2,3\}} = (0, 0, 1/2)$, and hence $\rho^\star$ is indeed a solution.