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Iterated Expectations under Rank-Dependent Expected Utility and Model Consistency

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Abstract

Under expected utility theory, compound lotteries can be valued by "iterating" expectations: the expected utility of a compound lottery is the expected value of a simple lottery over prizes that are certainty equivalents to follow-up lotteries. We derive necessary and sufficient conditions for a similar valuation technique in the framework of rank-dependent expected utility (RDU) when a decision maker has to choose between prospects that belong to a comonotonic class and his preferences satisfy consequentialism. The conditions are so restrictive that they can be viewed as an impossibility result. Our contribution thus identifies a challenge for future research. If we accept RDU as the model of behavior, we either need to find alternative valuation algorithms, or we need to relax the assumption of preference exogeneity.

Keywords: Iterated expectations, rank-dependent expected utility, model consistency, valuation methods, probability weighting function, conditioning, updating, dynamic consistency, consequentialism.
JEL classification: D80, D84

1 Motivation

In many areas of applied economics, it is standard procedure to determine the value of compound lotteries by means of iterative valuation techniques based on expected utility theory. For example, consider capital budgeting, where iterative valuation techniques are frequently used because many investment opportunities contain some option value of waiting for a resolution of uncertainty before an irreversible decision is taken. To determine the option value, binomial

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trees are commonly used as models for the value of an option's underlying asset.[1] At each node of a tree, an option's value is computed as the value of a simple lottery over certainty equivalent values of lotteries starting at successor nodes ("follow-up lotteries"). The certainty equivalent values are expected values of future payoffs of the option, given contingent plans for option exercise.[2] By focusing on successor nodes and using contingent plans, the option holder's behavior is assumed to be consequentialist and dynamically consistent. Consequentialism is the property that "preferences at a given decision node in the tree are fully determined by the residual uncertainty and the payoffs that may be obtained starting from the node under consideration" (Siniscalchi 2009). Dynamic consistency is defined by Hanany and Klibanoff (2007, 2009) as "the requirement that ex-ante contingent choices are respected by updated preferences."

For a simple example of standard iterative valuation, consider a real option that is based on a dividend-paying underlying asset, the ex-dividend value of which evolves according to the tree in Figure 1.[3] There are three dates that are instants apart (so that we can abstract from discounting): date 0 is the current date, date 1 is a future date, and date 2 is the maturity date of the option contract. Suppose that the option is the right to buy the asset for a price of K, i.e. a call option. At date 1, the option may be exercised "early" in order to collect a dividend D that the underlying asset pays before date 2. Figure 1 shows that one of two events can occur at date 1, i.e. the event A and its complement A^C. Suppose that, ex post, early exercise is not optimal conditional on event A, but it is optimal conditional on event A^C. Then, dynamic consistency can be invoked in order to derive the option value at date 0 given a contingent plan for exercising the option early if and only if event A^C occurs. The contingent plan specifies payoffs to be earned at date 2: a payoff of $\max[P_2 - K, 0]$ conditional on event A, and a payoff of $P_2 + D - K$ conditional on event A^C. The payoffs correspond to conditional option values for each of the two events at date 1, i.e. the certainty equivalent value of the option's payoffs at follow-up nodes of the tree. Iterative option valuation proceeds by taking the conditional option values as the prizes of a simple lottery defined by the events A and A^C, and computing the value of the simple lottery. The result is taken to be the value of the option at date 0. In this final step, consequentialism is invoked in defining the simple lottery. For example, the prize conditional on event A^C is defined as the certainty equivalent of the payoff $P_2 + D - K$ where P_2 can only take the two values associated with the two states that are consequences of event A^C: $P_2 \in \{u^2 P_0, udP_0\}$.

The common use and teaching of iterative valuation techniques such as that outlined above is potentially at odds with theories of decision making under risk in which decision makers' behavior

[1]See Chapter 17 of McDonald (2006) for a textbook treatment of real options valuation, i.e. the application of option pricing in the area of capital budgeting.

[2]To obtain certainty equivalent values, risk-adjusted ("risk-neutral") probabilities must be used in computing expected values of future option payoffs.

[3]The ex-dividend value of the asset is the prepaid forward price one would pay in order to receive the asset after the dividend, i.e. at the time corresponding to the final nodes of the tree. By defining the tree in Figure 1 as a model for the ex-dividend value, we avoid a technical problem. See Schroder (1988) or McDonald (2006), pages 361-364.

$$
\begin{tikzcd}
 & & P_2 = uP_1 = u^2 P_0 \\
 & \text{Event } A^C\text{: } P_1 = uP_0 \arrow[ur] \arrow[dr] & \\
 & & P_2 = dP_1 = udP_0 \\
P_0 \arrow[uur] \arrow[ddr] & & \\
 & & P_2 = uP_1 = duP_0 \\
 & \text{Event } A\text{: } P_1 = dP_0 \arrow[ur] \arrow[dr] & \\
 & & P_2 = dP_1 = d^2 P_0
\end{tikzcd}
$$

Figure 1: Binomial tree for our example

is in general not dynamically consistent and consequentialist.[4] Such theories have received increasing attention, both experimentally and theoretically, over the past decades. With the theories' increasing popularity, it has become important to establish the robustness of commonly known results derived under expected utility.

A prominent alternative to expected utility theory is rank-dependent expected utility (RDU), first proposed by Quiggin (1982) and Yaari (1994). In the present paper, we consider RDU and derive necessary and sufficient conditions for a law of iterated expectations when choice is restricted to a generic class of comonotonic prospects. As illustrated by our motivating example, this class of choices includes many choices that are commonly made in applied economics, and is therefore of great practical relevance. Dhaene, et al. (2002) survey applications of comonotonicity in finance and insurance.

Our analysis specifies a process of preference updating required for valuing a compound lottery by iterating expectations. The preference updating yields conditional RDU values that we take as the prizes of a simple lottery. Under our law of iterated expectations, the RDU value of the simple lottery is supposed to equal that of the compound lottery to be valued (see expression (2) below).

The preference updating process that we derive follows from two key features of our analysis. The first feature is part of the model that we pick to represent preferences, i.e. the rank-dependence of preferences. In weighting outcomes of a risky choice, the decision maker (DM) in our model uses weights that depend not only on the probability of an outcome, but also on how the outcome ranks relative to alternative outcomes in terms of the DM's utility. We derive conditions under which the occurrence of an event does not change a DM's ranking of

[4]With respect to option pricing, there exists evidence that decision makers' behavior is at odds with expected utility theory. Hao, Kalay and Mayhew (2010) estimate that, during their sample period 1996-2006, 40% of the call options that should have been exercised remain unexercised in the US exchange-traded equity option market. More general effects of rank-dependent expected utility on option pricing are the subject of a growing literature, with recent contributions by Kliger and Levy (2009), Polkovnichenko and Zhao (2012) and Dierkes (2013).

any outcomes that remain possible, given the event. It turns out that the latter restriction is a necessary and sufficient condition for valuing compound lotteries by iterating expectations while maintaining model consistency. Imposing model consistency on RDU preference updating is the second key feature of our analysis. Epstein and Le Breton (1993) define model consistency as the notion that axioms imposed on the initial preference ordering of a DM should be satisfied also by orderings that result from preference updating conditional on new information.

To highlight how our results depend on the two key features of our analysis, we state the results in two Theorems. Theorem 1 states conditions for model consistency under RDU. The conditions define a benchmark prospect that a DM must use in preference updating in order for the updated preferences to remain within the RDU class. Theorem 2 builds on the results in Theorem 1 and states our necessary and sufficient conditions for valuing a compound lottery by iterating expectations as discussed above. The link between the two Theorems is constituted by the conditional RDU values that represent the DM's updated preferences under Theorem 1. In Theorem 2, the conditional RDU values are taken as the prizes of a simple lottery, the RDU value of which is supposed to equal that of the compound lottery to be valued in the first place.

Our analysis yields an impossibility result in that the conditions in Theorem 2 are quite restrictive. The conditions will rarely hold under the standard assumption of preference exogeneity. Put differently, when DMs have RDU preferences, the iterative valuation techniques that are commonly used under expected utility will only work for rare pairings of DMs with compound lotteries, i.e. pairings which give rise to the preference updating process that our analysis specifies. Our impossibility result also applies to the positive and negative parts of Tversky and Kahneman's (1992) Cumulative Prospect Theory (CPT) representation, since these are two rank-dependent utility representations.[5] Our contribution thus identifies a challenge for future research. If we accept RDU as the model of behavior, we either need to find alternative valuation algorithms, or we need to relax the assumption of preference exogeneity.

The paper is structured as follows: Section 2.1 sets up the model and formalizes our research question, while Section 3 contains our theorems. Section 4 discusses our results and relates them to the literature.

2 Model and Research question

2.1 Setup

We consider a decision maker (DM) whose choices are characterized by a preference relation $\succsim$ over a class of prospects, H, where $\sim$ denotes indifference and $\succ$ denotes strict preference. The prospects are functions from a finite state space $S = \{1, \ldots, n\}$ to a compact set of outcomes X. In Quiggin's (1982) RDU theory, the outcomes in X are simply outcomes, while in CPT they describe gains or losses with respect to a reference point. Let $x_* \in X$, respectively $x^* \in X$, denote the worst, respectively best, possible outcome in X. The outcomes x_* and x^* exist since

[5] Wakker and Tversky (1993) provide an axiomatization for the CPT representation.

X is a compact set. The states $\{1, \dots, n\}$ occur with probabilities $(p_1, \dots, p_n)$, all assumed strictly positive. We will use lower case subscripts (e.g. p_s) to denote states and upper case subscripts (e.g. p_A) to denote events. For any prospect h, we will use h_s to denote the outcome that h returns in state s. For any outcome $x \in X$, x also denotes the constant prospect $(x, \dots, x)$. The preference relation $\succsim$ also ranks outcomes, i.e. for any $x, y \in X$, $x \succsim y$ if and only if $(x, \dots, x) \succsim (y, \dots, y)$.

Throughout our paper, we assume that the DM has to make a choice from a set of comonotonic prospects $\mathscr{H}$. Comonotonicity of the prospects in $\mathscr{H}$ means that for no pair of prospects $h, g \in \mathscr{H}$ and no pair of states $s, s' \in S$, it holds that $h_s \succ h_{s'}$ and $g_{s'} \succ g_s$. We can therefore label the states such that $h_n \succsim \dots \succsim h_1$ for all $h \in \mathscr{H}$. Choices between comonotonic prospects occur in many applications; in fact, one example is the early options exercise problem described in the introductory section.

We further assume that the DM's preferences are represented by a RDU value $V(h)$ corresponding to the negative (or "loss") part of the CPT representation in Prelec (1998), such that for a prospect h,[6]

$$V(h) = \sum_{s=1}^{n} \left[w\Big(\sum_{j=1}^{s} p_j\Big) - w\Big(\sum_{j=1}^{s-1} p_j\Big) \right] v(h_s), \tag{1}$$

where $w(\cdot)$ is a unique nondecreasing probability weighting function satisfying $w(0) = 0$ and $w(1) = 1$, $v(x)$ is a continuous and increasing utility index, and we define $\sum_{j=a}^{c} z_j \equiv 0$, whenever $c < a$. For each s, the expression in brackets is the decision weight associated with outcome h_s.

2.2 Research question

We now formalize our research question. As discussed in Section 1, we analyze preference updating conditional on information that leads to a partial resolution of uncertainty. The information is modeled as a partitioning of the state-space into an event A and its complement event A^C. The preference updating takes place after the DM learns which of the two events occurred. To make the preference updating non-trivial, we assume that there are at least four states in the state space S, and that the events A and A^C both contain at least two states. We also assume that $w(\cdot)$ is a strictly increasing function, and that X contains at least four distinct outcomes (i.e., outcomes none of which are pairwise indifferent to each other).

Given the partitioning of the state space into the events A and A^C, a prospect $h \in \mathscr{H}$ defines a compound lottery: the DM first learns which one of the two events occurred, and then faces a lottery over the outcomes of the prospect h in the states associated with the event that occurred. In this paper, we seek to derive conditions so that the DM's RDU from the compound lottery can be represented by a "law of iterated expectations" which is formally defined as follows:

$$V(h) = w(p_A) V_A(h) + (1 - w(p_A)) V_{A^C}(h). \tag{2}$$

[6] Symmetric results to those presented here apply in the gains domain. The gains domain results and proofs are available from the authors upon request.

where the term on the right-hand side is the RDU value of a simple lottery over conditional RDU values given by value functions V_A and V_{A^C} that represent conditional (i.e., updated) preference relations $\succsim_A$ and $\succsim_{A^C}$ (defined in (3) below), respectively. The value $V_A(h)$ is assigned the weight $w(p_A)$, i.e. the weight that the DM assigns to event A, given the probability of this event.

We insist that the DM's conditional preferences remain within the RDU class, thus requiring model consistency. More specifically, the DM will consistently apply probability weighting when choosing between lotteries. The preference updating process will be defined in terms of a benchmark prospect b that the DM uses for updating her preferences. The benchmark can be interpreted as the DM's belief about outcomes in a counterfactual event. We define

$$h_A b = \begin{cases} h_s & \text{if } s \in A \\ b_s & \text{if } s \notin A \end{cases} \quad \text{and} \quad g_A b = \begin{cases} g_s & \text{if } s \in A \\ b_s & \text{if } s \notin A. \end{cases}$$

For example, suppose that event A occurs, and event A^C is the counterfactual event. Then, the DM's conditional preferences are given by the preference relation $\succsim_A$ which ranks a pair of prospects $g, h \in \mathscr{H}$ as follows:

$$h \succsim_A g \text{ iff } h_A b \succsim g_A b. \tag{3}$$

The preference relation $\succsim_{A^C}$ is defined similarly. In the lotteries in (3), the benchmark appears as the DM's payoff in states $s \notin A$ that cannot follow event A. The conditional preference $\succsim_A$ thus ranks the prospects g and h as if the DM would have obtained the outcomes specified by the benchmark b conditional on the counterfactual event A^C. The benchmark b is part of the specification of the DM's preferences. It can be interpreted as a belief of the DM about outcomes that were not obtained.

3 Results

Theorem 1 characterizes model consistency of preference updating under RDU: it specifies a class of benchmark prospects for which the DM's conditional preferences (defined in expression (3) are represented by a conditional RDU function $V_A(h)$ for any conditioning event A and for any prospect h in our class $\mathscr{H}$ of comonotonic prospects. The event B in Theorem 1 can be any event, and different Bs will correspond to different benchmarks and different conditional probability weighting functions. Below, o is the function defined on $\{1, \ldots, |A|\}$ that orders the states in A such that $h_{o(|A|)} \succsim \cdots \succsim h_{o(1)}$ for any act h in the comonotonic class $\mathscr{H}$.

Theorem 1 *The following two statements are equivalent:*

(a) For any increasing weighting function $w(\cdot)$ satisfying $w(0) = 0$ and $w(1) = 1$, for any continuous increasing utility index $v(x)$, for any prospects in $\mathscr{H}$, and any event A, preferences

conditional on arrival of information that $s \in A$ are represented by $V_A(h)$, for

$$V_A(h) \equiv \sum_{i=1}^{|A|} \Big(w_A\big(\sum_{j=1}^{i} p_{o(j)}\big) - w_A\big(\sum_{j=1}^{i-1} p_{o(j)}\big)\Big) v(h_{o(i)}), \tag{4}$$

where w_A is the conditional probability weighting function:

$$w_A\big(\sum_{j=1}^{i} p_{o(j)}\big) \equiv \frac{w\big(\sum_{j=1}^{i} p_{o(j)} + p_{B^C \setminus A}\big)}{w(p_{B^C \cup A}) - w(p_{B^C \setminus A})} \tag{5}$$

(b) The benchmark prospect used for updating is $b_* \equiv x_B^* x_*$.

Proof: See Appendix A.

Theorem 1 specifies the values $V_A(h)$ and $V_{A^C}(h)$ that are the prizes of the simple lottery, the value of which appears on the right-hand side of our law of iterated expectations (2). If the DM would receive information that $s \in A^C$, A^C would replace A in (4) and (5). It is easy to show that the conditional probability weighting functions $w_A(\cdot)$ and $w_{A^C}(\cdot)$ inherit the properties of the unconditional probability weighting function $w(\cdot)$.

Our next result builds on Theorem 1 and further specifies the benchmark prospect that the DM uses for preference updating. Theorem 2 shows that, under RDU and model consistency, the law of iterated expectations (2) holds only for a very specific benchmark prospect and only if the conditioning events are such that one dominates the other. The event A^C dominates the event A if for all $s \in A$ and for all $s' \in A^C$, $h_{s'} \succsim h_s$, for all $h \in \mathscr{H}$.

Theorem 2 *A law of iterated expectations holds, that is,*

$$V(h) = w(p_A)V_A(h) + (1 - w(p_A))V_{A^C}(h) \tag{6}$$

if and only if the benchmark prospect is b_* defined in Theorem 1, event B defining the benchmark prospect satisfies $B = A^C$, and A^C dominates A.

Proof: See Appendix B.

The result in Theorem 2 is an intuitive consequence of rank-dependence. As stated in expression (6), a DM's unconditional RDU value from any prospect $h \in \mathscr{H}$ should correspond to the value of a simple lottery. The lottery will only give the DM the same RDU value as the prospect h if conditioning on the events A and A^C does not change the DM's weighting of any outcomes that the prospect h may yield. Under RDU, the latter requirement will not be satisfied if the conditioning changes the way outcomes are ranked. To avoid changes in the ranking, the conditioning must be based on the benchmark prospect specified in Theorem 1 for $B = A^C$.

In order to interpret the result in Theorem 2, it is important to keep in mind that the benchmark a decision maker uses for preference updating is *part of* the decision maker's preferences. The benchmark is therefore taken to be exogenous, just like the utility index $v(\cdot)$ and the probability weighting function $w(\cdot)$ are. Against this backdrop, the result in Theorem 2 can be interpreted as an impossibility result since it hinges on quite restrictive conditions.

To illustrate the restrictions imposed by Theorem 2, suppose that the DM actually uses a benchmark as that defined in Theorem 1(b), i.e. $b_* \equiv x^*_B x_*$. Then, Theorem 2 states that a law of iterated expectations holds only when the conditioning is exactly on the events B and B^C defining the benchmark, and only when evaluating the subset of prospects for which the outcomes associated with the states in B^C are dominated by outcomes associated with the states in B. Put differently, the RDU value of a prospect h will only satisfy the law of iterated expectations (2) for a rare case of a DM. As a consequence, iterative valuation techniques that are commonly used under expected utility theory are not generally applicable under RDU. On a more positive note, the result in Theorem 2 presents a challenge for future research. If we accept RDU as the model of behavior, we either need to find alternative valuation algorithms, or to relax the assumption of preference exogeneity.

4 Discussion

Our analysis can be compared to a recent paper by Barberis (2012) who points out that CPT is well-suited as a theory of casino gambling. In Barberis' paper there is no actual updating of the probability weighting function. Instead, the gambler's preferences are always represented by the same CPT utility function and probability weighting function, but the probability weighting is applied to probabilities that are updated based on Bayes' rule. Updating of the probability weighting function also does not occur in a follow-up paper by Ebert and Strack (2014) which analyzes the dynamic gambling behavior of a naive time-inconsistent CPT agent and shows that, due to probability weighting, the agent never stops gambling. In the present paper, we specify a preference updating process which does involve updating of the probability weighting function according to expression (5).

Our analysis yields necessary and sufficient conditions for preference updating such that the DM's choices will be dynamically consistent.[7] We acknowledge that dynamic *in*consistency is an appealing feature of RDU and CPT preferences in many applications, including Barberis' (2012) paper. As discussed in the Introduction, our quest for dynamic consistency is motivated by the desire to analyze the applicability of commonly used valuation techniques in a RDU framework while maintaining model consistency.

Besides ensuring dynamic consistency, our conditions for preference updating will also ensure consquentialist choices, as defined in Karni and Schmeidler (1991): the DM's conditional evaluation of a prospect (given a conditioning event) will not depend on what would have resulted

[7] Recall: dynamic consistency is "the requirement that ex ante contingent choices are respected by updated preferences" (Hanany and Klibanoff 2007, 2009).

from the prospect outside of the conditioning event. Siniscalchi (2009) discusses that when conditioning can be on any event, maintaining both dynamic consistency and consequentialism requires restrictions on the set of acts that constitutes the domain of preferences. Otherwise, these two properties imply Savage's (1954) Sure-Thing principle (or independence), which is not satisfied by RDU.[8] We show that for given preferences, one can maintain the two properties by restricting the events that can be conditioned on. Dominiak (2013) has a related result about iterated Choquet expectations when the conditioning event is unambiguous.

Our result can also be put in perspective based on results in the literature on preference updating. For Choquet expected utility, Gilboa and Schmeidler (GS, 1993) define a set of f-Bayesian updating rules, where f can be interpreted as "what does the decision maker implicitly assume would have resulted if the event on which conditioning takes place had not occurred." It is interesting to compare these f-Bayesian updating rules with the conditional probability weighting function in Theorems 1 and 2. It turns out that, if event A represents "bad news" (in the sense of being dominated by event A^C) and $B = A^C$ as in Theorem 2, then conditioning on A yields the conditional probability weighting function (5), which is consistent with the updated capacity that GS obtain using their "optimistic" rule (which is a Bayesian rule). If the conditioning is instead on the dominating event A^C, we obtain an updating rule that is consistent with GS's "pessimistic" rule (which is a Dempster-Shafer rule), i.e.

$$w_{A^C}\big(\sum_{j=1}^{i} p_{c(j)}\big) \equiv \frac{w(\sum_{j=1}^{i} p_{c(j)} + p_A) - w(p_A))}{1 - w(p_A)},$$

where c defined on $\{1, \ldots, |A^C|\}$ orders the states in A^C such that $h_{c(|A^C|)} \succsim \cdots \succsim h_{c(1)}$.

If we were in the realm of cumulative prospect theory, the results in section 3 would apply to the loss domain. In CPT's gains domain, the correspondence between our updating rules and those of GS would be reversed. There, it would be the conditional probability weighting function for a "good news" event that would be consistent with the optimistic rule, while the pessimistic rule would be consistent with the conditional probability weighting function for a "bad news" event.[9] Across the two CPT domains, GS' updating rules therefore correspond to different types of "news" in our model. However, there is an alternative interpretation of the updating rules which applies to both domains. In both domains, the optimistic (pessimistic) rule is consistent with our conditional probability weighting functions for updating based on events which imply outcomes further away from (closer to) the status quo.[10]

Sarin and Wakker (1998) specify an updating rule for rank-dependent expected utility similar to that in Theorem 1. They note that the "relevance of forgone consequences is the price one has to pay for giving up the separability of disjoint events",[11] that is, they give up consequentialism.

[8] See also Hammond (1988) and Machina (1989).

[9] The results for the gains domain are available from the authors upon request.

[10] In the gains domain, "good news" means outcomes further away from the status quo. The same is true for "bad news" in the loss domain.

[11] Sarin and Wakker (1998, p. 241)

Zimper (2011) and Lapied and Toquebeuf (2013) also give up consequentialism when they show that a law of iterated expectations can be obtained for Choquet expected utility if the update rule is allowed to be act-dependent. Our contribution differs because we do impose consequentialism, given that our research question concerns the robustness of standard - consequentialist - valuation techniques.

In summary, we obtain an impossibility result that is stated in terms of necessary and sufficient conditions for a process of consequentialist RDU preference updating that is consistent with valuing compound lotteries by iterating expectations. Our analysis thus defines an agenda for future research: to either find alternative valuation techniques or to show that the standard techniques can be used if preference exogeneity can be relaxed in a suitable and realistic manner.

Appendix A Proof of Theorem 1

Sufficiency: We first prove that having $b_* = x^*_B x_*$ as the benchmark is sufficient for the representation of conditional preferences. So, consider conditioning on an event $A \subseteq S$. Recall that we consider choice between comonotonic acts. Let o be the function defined on $\{1, \dots, |A|\}$ ordering the states in A such that $h_{o(|A|)} \succsim \cdots \succsim h_{o(1)}$ for any act h in the comonotonic class $\mathscr{H}$. With the benchmark prospect b_*, we have, for acts h and g in $\mathscr{H}$, that

$$\begin{aligned} V(h_A b_*) &= w(P_{B^C \setminus A}) v(x_*) \\ &+ \sum_{i=1}^{|A|} \left(w\big(\sum_{j=1}^{i} p_{o(j)} + P_{B^C \setminus A}\big) - w\big(\sum_{j=1}^{i-1} p_{o(j)} + P_{B^C \setminus A}\big) \right) v(h_{o(i)}) + \Big(1 - w\big(P_{B^C \cup A}\big)\Big) v(x^*) \end{aligned}$$

and

$$\begin{aligned} V(g_A b_*) &= w(P_{B^C \setminus A}) v(x_*) \\ &+ \sum_{i=1}^{|A|} \left(w\big(\sum_{j=1}^{i} p_{o(j)} + P_{B^C \setminus A}\big) - w\big(\sum_{j=1}^{i-1} p_{o(j)} + P_{B^C \setminus A}\big) \right) v(g_{o(i)}) + \Big(1 - w\big(P_{B^C \cup A}\big)\Big) v(x^*). \end{aligned}$$

Thus,

$$
\begin{aligned}
h_A b_* \succsim g_A b_* \quad &\Leftrightarrow \quad V(h_A b_*) \geq V(g_A b_*) \\
&\Leftrightarrow \quad \sum_{i=1}^{|A|} \left(w\big(\sum_{j=1}^{i} p_{o(j)} + P_{B^C \setminus A}\big) - w\big(\sum_{j=1}^{i-1} p_{o(j)} + P_{B^C \setminus A}\big) \right) v(h_{o(i)}) \\
&\qquad \geq \sum_{i=1}^{|A|} \left(w\big(\sum_{j=1}^{i} p_{o(j)} + P_{B^C \setminus A}\big) - w\big(\sum_{j=1}^{i-1} p_{o(j)} + P_{B^C \setminus A}\big) \right) v(g_{o(i)}) \\
&\Leftrightarrow \quad \sum_{i=1}^{|A|} \frac{w\big(\sum_{j=1}^{i} p_{o(j)} + P_{B^C \setminus A}\big) - w\big(\sum_{j=1}^{i-1} p_{o(j)} + P_{B^C \setminus A}\big)}{w\big(P_{B^C \cup A} - w(P_{B^C \setminus A})\big)} v(h_{o(i)}) \\
&\qquad \geq \sum_{i=1}^{|A|} \frac{w\big(\sum_{j=1}^{i} p_{o(j)} + P_{B^C \setminus A}\big) - w\big(\sum_{j=1}^{i-1} p_{o(j)} + P_{B^C \setminus A}\big)}{w\big(P_{B^C \cup A} - w(P_{B^C \setminus A})\big)} v(g_{o(i)}),
\end{aligned}
$$

so preferences conditional on the information that $s \in A$ have a RDU representation with conditional probability weighting function

$$
w_A\big(\sum_{j=1}^{i} p_{o(j)}\big) = \frac{w\big(\sum_{j=1}^{i} p_{o(j)} + P_{B^C \setminus A}\big)}{w\big(P_{B^C \cup A} - w(P_{B^C \setminus A})\big)}
$$

Necessity: We proceed by showing that if the benchmark act is $f \neq b_*$, then we can choose $w(\cdot)$ and $v(\cdot)$ such that conditional preferences are not represented as stated in part (a) of the theorem for some $A \subseteq S$. That is, we can find functions $w(\cdot)$ and $v(\cdot)$ satisfying the stated conditions, such that $g_A f \succ h_A f$ but $V_A(h) \geq V_A(g)$, for some prospects $h, g \in \mathscr{H}$.

Consider a benchmark prospect f and suppose there exists an event $\hat{B} \subseteq S$, with $\hat{B} \neq \emptyset$, such that $x^* \succ f_i \succ x_*$ for all $i \in \hat{B}$. Let $B = \{i \, : \, f_i = x^*\}$.

Recall our assumption that $|S| \geq 4$, and that X contains at least four distinct outcomes (i.e., outcomes none of which are pairwise indifferent to each other). Consider an event A for which $|A| \geq 2$ and for which $\hat{B} \setminus A \neq \emptyset$. For such event, we can choose prospects $h, g \in \mathscr{H}$ such that

$$
x^* \succsim h_{o(2)} \succ g_{o(2)} \succ g_{o(1)} \succ h_{o(1)} \succsim x_*
$$

and $h_{o(i)} = g_{o(i)} = h_{o(2)}$ for all $i \in \{3, \ldots, |A|\}$.

Let β be the function defined on $\{1, \ldots, |\hat{B} \setminus A|\}$ ordering the states in $\hat{B} \setminus A$ such that $f_{\beta(|\hat{B} \setminus A|)} \succsim \cdots \succsim f_{\beta(1)}$.

Depending on f, there are different cases we need to consider. The different cases arise because the details of the proof depend on how f relates to the outcomes in X (recall that all we require is that there are at least four outcomes in X). The procedure is the same in each case: we calculate the utility of $h_A f$ and $g_A f$ and derive the conditions for $g_A f \succ h_A f$ and for $V_A(h) \geq V_A(g)$. We then show existence of a $w(\cdot)$ and $v(\cdot)$ such that both conditions are satisfied. Since such $w(\cdot)$ and $v(\cdot)$ exist, preferences cannot be generally represented as in statement (a) of Theorem 1.

Define

$$\eta \equiv \begin{cases} \min\big\{j \in \{1,\ldots,|\hat{B}\setminus A|\} \ : \ f_{\beta(j)} \succ f_{\beta(1)}\big\} & \text{if there exists } s \in \hat{B}\setminus A \text{ such that } f_s \succ f_{\beta(1)} \\ |\hat{B}\setminus A|+1 & \text{if } f_s \sim f_{\beta(1)} \ \text{ for all } s \in \hat{B}\setminus A, \end{cases}$$

and

$$\rho \equiv \begin{cases} \min\big\{j \in \{1,\ldots,|\hat{B}\setminus A|\} \ : \ f_{\beta(j)} \succ f_{\beta(\eta)}\big\} & \text{if there exists } s \in \hat{B}\setminus A \text{ such that } f_s \succ f_{\beta(\eta)} \\ |\hat{B}\setminus A|+1 & \text{if } f_{\beta(\eta)} \succsim f_s \ \text{ for all } s \in \hat{B}\setminus A. \end{cases}$$

The cases we need to consider are:

Case 1: There does not exist $x \in X$ such that $f_{\beta(1)} \succ x \succ x_*$. Then, since there are at least four distinct outcomes in X, there exists $y \in X$ such that $x^* \succ y \succ f_{\beta(1)}$.

Case 2: There does not exist $x \in X$ such that $x^* \succ x \succ f_{\beta(1)}$. Then, since there are at least four distinct outcomes in X, there exists $y \in X$ such that $f_{\beta(1)} \succ y \succ x_*$.

Case 3: There exist $x, y \in X$ such that $x^* \succ x \succ f_{\beta(1)} \succ y \succ x_*$.

Case 1: There are two sub-cases: **(1.a)** in which $\eta > |\hat{B}\setminus A|$ and **(1.b)** in which $\eta \leq |\hat{B}\setminus A|$.

(1.a) When $\eta > |\hat{B}\setminus A|$, $f_s \sim f_{\beta(1)}$ for all $s \in \hat{B}$. We can pick prospects $h, g \in \mathscr{H}$ such that $x^* \succsim h_{o(2)} \succ g_{o(2)} \succ f_{\beta(1)} = g_{o(1)} \succ h_{o(1)} = x_*$ and $h_{o(i)} = g_{o(i)} = x^*$ for all $i \in \{3,\ldots,,|A|\}$. Then

$$\begin{aligned} V[h_A f] \;\; = \;\; & w\big(p_{(B\cup\hat{B}\cup A)^C}\big)v(x_*) + \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)\Big)v(h_{o(1)}) \\ & +\Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big)\Big)v(f_{\beta(1)}) \\ & +\Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A} + p_{o(2)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A}\big)\Big)v(h_{o(2)}) \\ & +\Big(1 - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A} + p_{o(2)}\big)\Big)v(x^*) \end{aligned} \tag{7}$$

and

$$\begin{aligned} V[g_A f] \;\; = \;\; & w\big(p_{(B\cup\hat{B}\cup A)^C}\big)v(x_*) + \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)\Big)v(g_{o(1)}) \\ & +\Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big)\Big)v(f_{\beta(1)}) \\ & +\Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A} + p_{o(2)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A}\big)\Big)v(g_{o(2)}) \\ & +\Big(1 - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A} + p_{o(2)}\big)\Big)v(x^*). \end{aligned} \tag{8}$$

Therefore,

$$
\begin{aligned}
& V[g_A f] > V[h_A f] \\
\Leftrightarrow \quad & \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A} + p_{o(2)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A}\big)\Big)\big(v(g_{o(2)}) - v(h_{o(2)})\big) \\
& + \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)\Big)\big(v(g_{o(1)}) - v(h_{o(1)})\big) > 0.
\end{aligned}
$$

Conditional preferences are not represented by (4) if we also have that

$$
\begin{aligned}
& \Big(w\big(p_{o(1)} + p_{B^C\setminus A}\big) - w\big(p_{B^C\setminus A}\big)\Big)\big(v(h_{o(1)}) - v(g_{o(1)})\big) \\
+ \quad & \Big(w\big(p_{o(1)} + p_{o(2)} + p_{B^C\setminus A}\big) - w\big(p_{o(1)} + p_{B^C\setminus A}\big)\Big)\big(v(h_{o(2)}) - v(g_{o(2)})\big) \geq 0. \qquad (9)
\end{aligned}
$$

Hence, conditional preferences are not represented by (4) if

$$
\begin{aligned}
& \frac{w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A} + p_{o(2)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + p_{\hat{B}\setminus A}\big)}{w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)} \\
< \quad & \frac{v(g_{o(1)}) - v(h_{o(1)})}{v(h_{o(2)}) - v(g_{o(2)})} \leq \frac{w\big(p_{o(1)} + p_{o(2)} + p_{B^C\setminus A}\big) - w\big(p_{o(1)} + p_{B^C\setminus A}\big)}{w\big(p_{o(1)} + p_{B^C\setminus A}\big) - w\big(p_{B^C\setminus A}\big)}. \qquad (10)
\end{aligned}
$$

The functions $w(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (10) are satisfied. Therefore, preferences are not represented as in statement (a) of the theorem.

(1.b) When $\eta \leq |\hat{B} \setminus A|$, we can pick prospects $h, g \in \mathscr{H}$ such that $h_{o(2)} \succ g_{o(2)} = f_{\beta(\eta)} \succ f_{\beta(1)} = g_{o(1)} \succ h_{o(1)} = x_*$, such that there does not exist $z \in X$ such that $h_{o(2)} \succ z \succ f_{\beta(\eta)}$, and $h_{o(i)} = g_{o(i)} = x^*$ for all $i \in \{3, \ldots, , |A|\}$. Then

$$
\begin{aligned}
V[h_A f] \; = \; & w\big(p_{(B\cup\hat{B}\cup A)^C}\big)v(x_*) + \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)\Big)v(h_{o(1)}) \\
+ \; & \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big)\Big)v(f_{\beta(1)}) \\
+ \; & \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big)\Big)v(f_{\beta(\eta)}) \\
+ \; & \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big)\Big)v(h_{o(2)}) \\
+ \; & \sum_{i=\rho}^{|\hat{B}\setminus A|} \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}\big)\Big)v(f_{\beta(i)}) \\
+ \; & \Big(1 - w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + p_{\hat{B}\setminus A}\big)\Big)v(x^*) \qquad (11)
\end{aligned}
$$

and

$$\begin{aligned}
V[g_A f] &= w\big(p_{(B\cup\hat{B}\cup A)^C}\big)v(x_*) + \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)\Big)v(g_{o(1)}) \\
&+ \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big)\Big)v(f_{\beta(1)}) \\
&+ \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big)\Big)v(f_{\beta(\eta)}) \\
&+ \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big)\Big)v(g_{o(2)}) \\
&+ \sum_{i=\rho}^{|\hat{B}\setminus A|}\Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}\big)\Big)v(f_{\beta(i)}) \\
&+ \Big(1 - w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + p_{\hat{B}\setminus A}\big)\Big)v(x^*) \qquad (12)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& V[g_A f] > V[h_A f] \\
\Leftrightarrow\ & \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big)\Big)\big(v(g_{o(2)}) - v(h_{o(2)})\big) \\
& +\Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)\Big)\big(v(g_{o(1)}) - v(h_{o(1)})\big) > 0.
\end{aligned}$$

Conditional preferences are not represented by (4) if (9) also holds. Hence, conditional preferences are not represented by (4) if

$$\begin{aligned}
& \frac{w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}\big)}{w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)} \\
< \ & \frac{v(g_{o(1)}) - v(h_{o(1)})}{v(h_{o(2)}) - v(g_{o(2)})} \le \frac{w\big(p_{o(1)} + p_{o(2)} + p_{B^C\setminus A}\big) - w\big(p_{o(1)} + p_{B^C\setminus A}\big)}{w\big(p_{o(1)} + p_{B^C\setminus A}\big) - w\big(p_{B^C\setminus A}\big)}. \qquad (13)
\end{aligned}$$

The functions $w(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (16) are satisfied. Therefore, preferences are not represented as in statement (a) of the theorem.

Case 2: In this case we can pick prospects $h, g \in \mathscr{H}$ such that $x^* = h_{o(2)} \succ g_{o(2)} = f_{\beta(1)} \succ g_{o(1)} \succ h_{o(1)} \succsim x_*$ and $h_{o(i)} = g_{o(i)} = x^*$ for all $i \in \{3, \ldots, , |A|\}$. Then the calculations coincide with those in case (1.a), so conditional preferences are not represented by (4) if (10) holds. Since the functions $w(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (10) are satisfied, preferences are not represented as in statement (a) of the theorem.

Case 3: In this case we can pick prospects $h, g \in \mathscr{H}$ such that $h_{o(2)} \succ g_{o(2)} = f_{\beta(1)} \succ g_{o(1)} \succ h_{o(1)} \succsim x_*$, such that there does not exist $z \in X$ such that $h_{o(2)} \succ z \succ f_{\beta(1)}$, and $h_{o(i)} = g_{o(i)} = x^*$ for all $i \in \{3, \dots, , |A|\}$. Then

$$\begin{aligned}
V[h_A f] &= w\big(p_{(B\cup\hat{B}\cup A)^C}\big)v(x_*) + \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)\Big)v(h_{o(1)}) \\
&+ \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big)\Big)v(f_{\beta(1)}) \\
&+ \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big)\Big)v(h_{o(2)}) \\
&+ \sum_{i=\eta}^{|\hat{B}\setminus A|} \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}\big)\Big)v(f_{\beta(i)}) \\
&+ \Big(1 - w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + p_{\hat{B}\setminus A}\big)\Big)v(x^*) \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
V[g_A f] &= w\big(p_{(B\cup\hat{B}\cup A)^C}\big)v(x_*) + \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)\Big)v(g_{o(1)}) \\
&+ \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big)\Big)v(f_{\beta(1)}) \\
&+ \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big)\Big)v(g_{o(2)}) \\
&+ \sum_{i=\eta}^{|\hat{B}\setminus A|} \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}\big)\Big)v(f_{\beta(i)}) \\
&+ \Big(1 - w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + p_{\hat{B}\setminus A}\big)\Big)v(x^*). \qquad (15)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& V[g_A f] > V[h_A f] \\
\Leftrightarrow\ & \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + \sum_{j=1}^{2} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}\big)\Big)\big(v(g_{o(2)}) - v(h_{o(2)})\big) \\
& + \Big(w\big(p_{(B\cup\hat{B}\cup A)^C} + p_{o(1)}\big) - w\big(p_{(B\cup\hat{B}\cup A)^C}\big)\Big)\big(v(g_{o(1)}) - v(h_{o(1)})\big) > 0.
\end{aligned}$$

Conditional preferences are not represented by (4) if (9) also holds. Hence, conditional preferences are not represented by (4) if

$$
\begin{aligned}
&\frac{w\big(p_{(B\cup\hat{B}\cup A)^C}+\sum_{j=1}^{2}p_{o(j)}+\sum_{j=1}^{\eta-1}p_{\beta(j)}\big)-w\big(p_{(B\cup\hat{B}\cup A)^C}+p_{o(1)}+\sum_{j=1}^{\eta-1}p_{\beta(j)}\big)}{w\big(p_{(B\cup\hat{B}\cup A)^C}+p_{o(1)}\big)-w\big(p_{(B\cup\hat{B}\cup A)^C}\big)}\\
<\;&\frac{v(g_{o(1)})-v(h_{o(1)})}{v(h_{o(2)})-v(g_{o(2)})}\\
\leq\;&\frac{w\big(p_{o(1)}+p_{o(2)}+p_{B^C\setminus A}\big)-w\big(p_{o(1)}+p_{B^C\setminus A}\big)}{w\big(p_{o(1)}+p_{B^C\setminus A}\big)-w\big(p_{B^C\setminus A}\big)}.
\end{aligned} \tag{16}
$$

The functions $w(\cdot)$ and $v(\cdot)$ can again be chosen such that both inequalities in (16) are satisfied. Therefore, preferences are not represented as in statement (a) of the theorem.

Since cases 1 through 3 are exhaustive, this completes the proof. ■

Appendix B Proof of Theorem 2

Sufficiency: Consider an event A, which is dominated by its complement A^C. Notice that, in this case, the function o (defined above Theorem 1) is the identity function. Also, let $B = A^C$. Then, $B^C \setminus A = \emptyset$ and $B^C \cup A = A$. Thus, by Theorem 1,

$$V_A(h) = \sum_{s=1}^{|A|}\Big(\frac{w\big(\sum_{j=1}^{s}p_j\big)}{w(p_A)} - \frac{w\big(\sum_{j=1}^{s-1}p_j\big)}{w(p_A)}\Big)v(h_s). \tag{17}$$

Let c be the function defined on $\{1,\dots,|A^C|\}$ that orders the states in A^C such that $h_{c(|A^C|)} \succsim \cdots \succsim h_{c(1)}$ for any act h in the comonotonic class $\mathscr{H}$, and note that $B^C \setminus A^C = A$ and $B^C \cup A^C = S$. By Theorem 1,

$$V_{A^C}(h) = \sum_{s=1}^{|A^C|}\Big(\frac{w\big(\sum_{j=1}^{s}p_{c(j)}+p_A\big)}{1-w(p_A)} - \frac{w\big(\sum_{j=1}^{s-1}p_{c(j)}+p_A\big)}{1-w(p_A)}\Big)v(h_{c(s)}),$$

which, since A^C dominates A, can be rewritten as

$$V_{A^C}(h) = \sum_{s=|A|+1}^{n}\Big(\frac{w\big(\sum_{j=1}^{s}p_j\big)}{1-w(p_A)} - \frac{w\big(\sum_{j=1}^{s-1}p_j\big)}{1-w(p_A)}\Big)v(h_s). \tag{18}$$

Now, by (17) and (18) we have that

$$V(h) = w(p_A)V_A(h) + (1-w(p_A))V_{A^C}(h). \tag{19}$$

Necessity: The first condition follows directly from the proof of Theorem 1: the benchmark prospect must be $b_* = x_B^* x_*$ in order for conditional preferences to be represented by V_A and

V_{A^C}.

We next consider the condition regarding the event B. Suppose that $B \neq A^C$. Then either $B^C \setminus A \neq \emptyset \Leftrightarrow A \subsetneq A \cup B^C$ or $A \setminus B^C \neq \emptyset \Leftrightarrow B^C \subsetneq A \cup B^C$. In either case, we generally have that $w(p_{B^C \cup A}) - w(p_{B^C \setminus A}) \neq w(p_A)$, so that the law of iterated expectations does not hold. Therefore, we must set $B = A^C$.

Finally, consider the condition that A be dominated by A^C. Suppose that the condition is violated, i.e. that there exist states $\hat{s} \in A$ and $s' \in A^C$ such that $h_{\hat{s}} \succ h_{s'}$. In the unconditional representation (1) , the coefficient for $v(h_{\hat{s}})$ is

$$w\left(\sum_{s \in A: h_{\hat{s}} \succsim h_s} p_s + \sum_{s \in A^C: h_{\hat{s}} \succsim h_s} p_s\right) - w\left(\sum_{s \in A: h_{\hat{s}} \succ h_s} p_s + \sum_{s \in A^C: h_{\hat{s}} \succ h_s} p_s\right), \tag{20}$$

while in the conditional representation (4), the numerator of the coefficient for $v(h_{\hat{s}})$ is

$$w\left(\sum_{s \in A: h_{\hat{s}} \succsim h_s} p_s\right) - w\left(\sum_{s \in A: h_{\hat{s}} \succ h_s} p_s\right). \tag{21}$$

The coefficients in (20) and (21) differ when there exist states $\hat{s} \in A$ and $s' \in A^C$ such that $h_{\hat{s}} \succ h_{s'}$. Then (6) will not hold. Hence we must require that A is dominated by A^C. ■

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