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Hotelling's Exhaustible Resource Extraction Model as a Linear Program

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Abstract

We "translate" Hotelling's continuous-time, exhaustible resource extraction Model of 1931 into a linear program of present value extraction cost minimization subject to a stock endowment and period by period demand constraints. The appropriate form of the demand constraints allows for resource rent rising at the rate of interest in the dual program. A useful variant has the stock size endogenous.

- key words: resource extraction; linear program; rent and interest rate
- JEL: Q320; C61; D46

1. Introduction

We present a linear programming version of Hotelling's 1931 model of exhaustible resource extraction. The formulation turns on novel demand inequalities that incorporate present value terms, terms that were not present in Nordhaus's 1973 linear programming version of a multi-source, multi-site "Hotelling" extraction model.[1] We set out first a single location, multi-period case in the tradition of Hotelling [1931]. For our exposition below, there is only one type of exhaustible

[1] Nordhaus's demands are set out on page 546 of his article. Herfindahl [1967] originated the extension to multiple sources in Hotelling's model. Gaudet, Moreaux and Salant [2001] took up the Herfindahl model and extended the analysis to incorporate multiple distinct locations of demand and supply. Hence the Gaudet et. al. model is "parallel to" the Nordhaus model but is not in a linear programming form.

resource available at an exogenous unit cost of extraction. This classic formulation turns on the assumption of an exogenous stock of the exhaustible resource. The desired solution turns on special values of initial parameters (our knife-edged parameter choice problem). We turn then to a variant that has the size of the exhaustible stock endogenous. This model is free of the knife-edged value problem inherent in the first model. Both models are at odds with Nordhaus's approach, namely the "simulation" of energy futures, given initial stocks, using linear programming.

2. The Stock Constrained Model

Our primal linear program (LP) below has the present value of extraction costs as its objective function:

$$cq_0 + [\frac{1}{1+r}]cq_1 + ... + [\frac{1}{1+r}]^T cq_T, \tag{2.1}$$

for r, the interest or discount rate, c, the positive dollar cost of extracting one unit of the exhaustible resource, and q_t, the non-negative current quantity of the resource extracted. $T+1$, the number of periods is exogenous. A principal constraint is, the aggregate of demands for the exhaustible resource over the many specified periods cannot exceed the initial exogenous endowment, S, of stock of the exhaustible resource:

$$q_0 + q_1 + ... + q_T \leq S. \tag{2.2}$$

The other constraints indicate that exogenous demands for the resource specified by period, $\overline{q}_t$ must be satisfied in a solution. That is

$$[\frac{1}{1+r}]^t q_t \geq \overline{q}_t \ ; \quad t = 0, 1, ..., T; \ \ \overline{q}_0, \overline{q}_1, ..., \overline{q}_T \text{ positive.} \tag{2.3}$$

The $\overline{q}_t$ demand requirement parameters "connect to" flow variables, q_t, weighted by present value qualifiers (the $[\frac{1}{1+r}]^t$ terms).[2] Our primal problem is: select $q_0, q_1, ..., q_T$ to maximize (2.1), subject to (2.2) and the $T+1$ constraints in (2.3).

Observe that in (2.3), the left side can never solve as larger than the right side because such a solution involves "extra" costs, via an "inflated" value of a q_t, in

[2] These parameters, $[\frac{1}{1+r}]^t$, do not appear in Nordhaus's specification of demands for his model. Hence his linear program is not precisely in the Hotelling tradition. His dual problem will not have resource rent rising at the interest rate in a solution.

the objective function. Hence in the dual problem below, the shadow price, p_t associated with one of these constraints will in general not solve as zero (if the solution has $[\frac{1}{1+r}]^t q_t < \overline{q}_t$ then $p_t = 0$, (complementary slackness)). We note the other complementary slackness condition.

$$\text{if the solution has } q_0 + q_1 + ... + q_T < S, \text{ then } \lambda = 0,$$

for λ the shadow price of a unit of S.

It turns out that the "default solutions" involve $q_0 + q_1 + ... + q_T < S$ and $\lambda = 0$ or $q_0 + q_1 + ... + q_T > S$ and the problem is not feasible. This defaultness is occurring because the specification of demands, $\overline{q}_t$ is tightly connected to the solution values, $q_t's$, which in turn must be related to the initial exogenous value of S. Given exogenous demands, only a particular value of S is compatible with λ greater than zero (the knife-edged parameter choice problem). A Hotelling like solution would have $q_0 + q_1 + ... + q_T = S$ and $\lambda > 0$. Such a solution only obtains above if the exogenous demands and the value of S are carefully "calibrated" *ab initio*.

The dual problem can admit of a classic Hotelling valuation solution with current resource rent on a unit of resource stock rising at the exogenous rate of interest. The dual LP involves finding non-negative prices, p_t, $(t = 0, 1, ..., T)$ and a non-negative shadow price for the resource stock, λ, that maximize

$$p_0\overline{q}_0 + p_1\,\overline{q}_1 + ... + p_T\,\overline{q}_T - \lambda S \tag{2.4}$$

subject to

$$[\frac{1}{1+r}]^t p_t - \lambda \leq [\frac{1}{1+r}]^t c\ ; \qquad t = 0, 1, ...T; \tag{2.5}$$

Solutions, with λ positive, resemble a Hotelling formulation. Current rent, $[1 + r]^t\lambda$, is rising at the rate of interest between dates t and $t+1$. The other possible solution (one of our "default" solutions) solves with $\lambda = 0$ and $S > q_0 + q_1 + ... + q_T$ in (2.2). We do not see such solutions in the textbook versions of the Hotelling extraction problem.[3] The structure of exogenous demands here (scalar values) is quite different from that in the standard versions of a Hotelling problem with its given exogenous demand schedule. Given the exogenous demands, there is a unique value of S that can correspond with $\lambda > 0$. Only one value of S

[3] A textbook Hotelling extraction problem can exhibit rent of zero each period for the special case of the initial stock-size sufficiently large.

"works" with a value of $\lambda > 0$, given a reasonable selection of values for the initial exogenous quantities demanded.

Observe that the objective function in (2.4) can be interpreted as the present value of dollar surplus. Recall that each $\overline{q}_t$ is a standard quantity with an implicit discount factor "weight". In a solution to the primal and dual problems the values of the two distinct objective functions must be equal (a linear programming theorem). Hence surplus in (2.4) solves as equal to cost in (2.1).

We have left open the question of where the value of $T + 1$, number of periods, comes from. This is a somewhat complicated issue. So far here T is given exogenously. In Hotelling [1931], the price of an extracted unit rose period by period and was intended to reach a "choke price" at the instant that the initial stock in the ground reached exact depletion. Working backwards from the terminal date to the date of initial extraction defined the optimal number of periods of active extraction. Roughly speaking the presence of a given demand schedule in Hotelling [1931] allowed this scenario to occur. Hence to fit the correct number of periods above to the initial stock, S, is a finicky numerical problem.

Nordhaus [1973] was interested in the issue of how exogenous natural resource stocks related to energy supply, shaped the future of energy uses (demands) and energy prices. The above model captures the Nordhaus scenario, but only satisfactorily when λ is positive. And we emphasized that the solution with $\lambda > 0$ is a knife-edged solution. Hence the above model is not able to solve quite generally as a Nordhaus scenario. Only solutions with contrived initial parameters turn out as "Nordhaus scenarios". Now, we turn to a variant of the above model in which S is endogenous, a variant we refer to as demand driven. This alternative formulation is free of the knife-edged parameter-choice problem inherent in the model above.

3. A Demand Driven Variant

The central change is S is now endogenous, while the demands are the same as we have for the model above. The price of a unit of stock is now v, exogenous, and appears in the objective function. The new objective function is

$$cq_0 + [\frac{1}{1+r}]cq_1 + ... + [\frac{1}{1+r}]^T cq_T + vS.$$

This function is to be minimized. The constraints are as above:

$$[\frac{1}{1+r}]^t q_t \geq \overline{q}_t \ ; \quad t = 0, 1, ..., T; \ \overline{q}_0, \overline{q}_1, ..., \overline{q}_T \text{ positive}$$

and

$$q_0 + q_1 + ... + q_T - S \leq 0.$$

The complementary slackness consitions are

$$\text{if the solution has } [\frac{1}{1+r}]^t q_t < \overline{q}_t \text{ then } p_t = 0.$$

and

$$\text{if the solution has } q_0 + q_1 + ... + q_T < S, \text{ then } \lambda = 0,$$

The dual LP involves finding non-negative prices, p_t, $(t = 0, 1, ..., T)$ and a non-negative shadow price, λ for the resource stock that maximize

$$p_0 \overline{q}_0 + p_1 \, \overline{q}_1 + ... + p_T \, \overline{q}_T \tag{3.1}$$

subject to

$$[\frac{1}{1+r}]^t p_t - \lambda \leq [\frac{1}{1+r}]^t c \; ; \qquad t = 0, 1, ...T; \tag{3.2}$$

and

$$\lambda \;\leq\; v.$$

The complementary slackness conditions are of interest.

$$\text{for a solution with } q_0 + q_1 + ... + q_T - S < 0, \text{ then } \lambda = 0$$

and

$$\text{for a solution with } \lambda < v, \text{ then } S = 0.$$

This model is not burdened with the knife-edged property associated with the value of S that the first model above is burdended with. This model also exhibits the Hotelling property which we observed for the first model, namely rent rising at the rate of interest $(p_t - c = (1+r)^t \lambda)$. We repeat: this model has the stock size endogenous and the price of a unit of stock a parameter. The well-behavedness of this model makes it closer to the Hotelling extraction model. However, this model, like the one above, does not "create" an energy future in the sense of Nordhaus because here stock size is not exogenous. Exogenous stock sizes are not inducing a future energy scenario of the type Nordhaus was interested in.

REFERENCES

Gaudet, Gerard, Michel Moreaux, and Stephen W. Salant (2001) "Intertemporal Depletion of Resource Sites by Spatially Distributed Users", *American Economic Review*, 91, 4, Sept. pp. 1149-1159.

Herfindahl, Orris C. [1967] "Depletion and Economic Theory", in Mason Gaffney, editor, *Extractive Resources and Taxation*, Madison: University of Wisconsin Press, pp. 68-90.

Hotelling, Harold [1931] "The Economics of Exhaustible Resources", *Journal of Political Economy*, 39, 2, April, pp. 137-175.

Nordhaus, William D. [1973] "The Allocation of Energy Resources', *Brookings Papers on Economic Activity*, 3, pp. 529-670.