Addiction and Bright-line Rules

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Abstract

To abstain from drugs or maintain low consumption, addicts often adopt *bright-line rules*, i.e., rules whereby a minor lapse triggers a full relapse. To explain this, I model a quasi-hyperbolic discounter consuming an addictive good, where past use increases current marginal utility. I find that heavy past consumption inevitably traps the consumer in excessive future use. But with low past use, the consumer can—and must—adopt a bright-line rule to avoid this trap. Moreover, any Markov equilibrium without such a rule is Pareto-dominated by one that includes it. These results explain the lapse-activated consumption patterns observed in addiction.

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A single cookie may violate a weight-loss plan, but the impact of that cookie on the diet is probably minimal.

Baumeister and Heatherton (1996, p. 11)

1 Introduction

Individuals struggling with addiction—whether to tobacco, alcohol, food, or another substance—often display a bimodal pattern of consumption: they either engage in excessive use or maintain low or no intake. This low state is typically fragile; even a minor lapse can trigger a full relapse. Psychologist Ainslie (1992) attributed this pattern to individuals' adoption of bright-line personal rules—rigid boundaries intended to preserve self-control. A drawback of such rules is that once a line is crossed, the individual may abandon self-regulation entirely. This leads to what psychologists call lapse-activated responses, wherein a single violation undermines the individual's entire effort at self-control.

In practice, addicts commonly seek to limit their consumption of the addictive good by enrolling in residential rehabilitation programs or taking medications like naltrexone that reduce the utility of consumption.[1] These behaviors suggest that time-inconsistent preferences play a significant role in addiction. This paper demonstrates how such preferences can lead addicts to adopt bright-line rules to limit their consumption of an addictive substance. The main result is that bright-line rules are not only necessary to avoid excessive consumption but also sufficient, when past consumption is low, to sustain lower use. Moreover, bright-line rules are more efficient within the class of equilibria I study.

To model addiction, I follow Becker and Murphy (1988) and assume that a consumer chooses her consumption of a harmful addictive good every period. The good is addictive in the sense that an increase in past consumption, summarized by a "consumption stock," raises the marginal utility of current consumption. The good is harmful in that a higher consumption stock reduces utility. The former assumption captures the "reinforcement" effect associated with addiction, while the latter reflects the "tolerance" effect.[2] To capture time inconsistency, I assume the consumer has hyperbolic discounting preferences and model her decision problem as an intrapersonal game. I analyze Markov perfect equilibria (MPE) using the consumption stock as the state variable.[3]

[1] Schilbach (2019) provides evidence that low-income individuals exhibit a demand for commitment to sobriety, consistent with time-inconsistent preferences and sophistication.

[2] For expositional purposes, I focus on harmful addiction. However, the main results also hold with beneficial addiction, such as exercising, saving, etc.

[3] Most of the hyperbolic discounting literature analyzes Markov perfect equilibria to avoid reliance on self-punishment strategies. Notable exceptions include Bénabou and Tirole (2004), Laibson (1994),

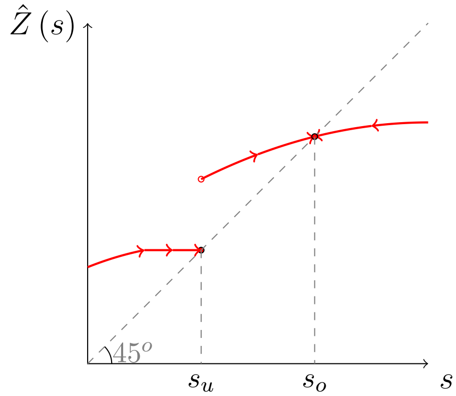


Figure 1: An MPE with a bright-line rule

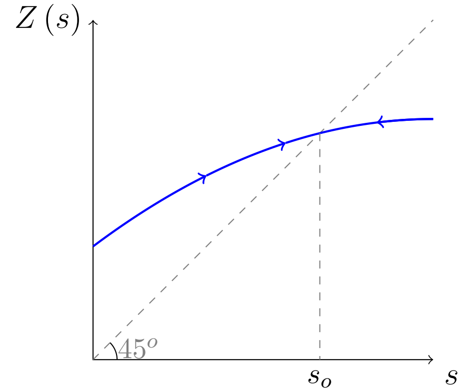


Figure 2: An MPE without a bright-line rule

I identify a class of equilibria, which I call equilibria with *bright-line rules*, in which the consumer sustains lower consumption than they do in any other equilibrium. Figure 1 illustrates such an equilibrium. The horizontal axis denotes the current stock, and the vertical axis shows the stock a current self chooses for tomorrow, $\hat{Z}(\cdot)$, which equals the (depreciated) current stock plus today's consumption. The threshold s_u is called a bright line because behavior changes sharply near it: when the stock is below s_u, the consumer maintains low consumption; once it exceeds s_u, consumption rises substantially and converges to the higher state s_o.

Is a bright-line rule appealing to the consumer? Yes, because without it, she will inevitably be trapped in excessive consumption in any MPE, even if she starts from a low stock (see Figure 2 for an illustration). Intuitively, in such equilibria, a current self recognizes that regardless of how much discipline she exercises today, her future selves will consume heavily. Given her present bias, she has little incentive to resist immediate gratification when long-term self-control appears unattainable.

How does a bright-line rule help an individual to sustain low consumption? As illustrated in Figure 1, as long as a current self chooses low consumption such that the resulting stock is below the bright line s_u, future selves will also do so. In this way, future selves cooperate to reward a current self's discipline by maintaining permanently low consumption and to punish her misbehavior by shifting to the excessive-consumption trap at s_o. This mechanism creates stronger incentives to maintain low consumption than equilibria without a bright-line rule do.

Under some mild conditions, if the consumer does not adopt a bright-line rule, there is an alternative equilibrium with a bright-line rule that every self strictly prefers to the original

Bernheim et al. (2015), etc. These three papers study personal rules using perfect Bayesian or subgame perfect equilibria.

equilibrium. Specifically, under these conditions, every equilibrium without a bright-line rule is Pareto-dominated by an equilibrium with a bright-line rule if the initial stock is around or above the bright line.

Can a bright-line rule sustain moderation? If the intertemporal conflict is sufficiently strong—that is, if the long-run discount factor is high relative to the degree of present bias—the consumer can sustain *moderation* or even *underconsume* in some equilibria, relative to the long-run consumption of a time-consistent consumer. Ainslie (1992) argues that while bright-line rules can prevent excessive consumption, they can also lead to compulsion: a consumer may consume too little if she is overly concerned with setting a precedent.

Finally, bright-line rules may appear fragile—an unexpected shock can tempt the consumer to cross the line and relapse into excessive consumption. I show that while this concern applies to non-abstinence bright lines, it does not hold for abstinence bright lines. This might explain why over 90% of addiction treatment programs in the United States adopt an abstinence-based approach and promote zero-tolerance policies.

1.1 Related Literature

In their pioneering work, Becker and Murphy (1988) emphasize the role of unstable steady states in understanding addiction. They define addiction as behavior such that a small increase in current consumption leads to a significant increase in future consumption. While their theory speaks to happy addicts, it does not explain lapse-activated consumption patterns. For example, under their framework, generically, a time-consistent consumer either does not find abstinence optimal or, if she does, a small lapse will not trigger a relapse.

A number of papers have studied how hyperbolic discounting preferences lead to overconsumption of addictive goods.[4] In a binary framework where an agent is either addicted or sober and chooses between smoking and not smoking, O'Donoghue and Rabin (2003) show that, in the long run, a time-inconsistent agent is more likely to become addicted than a time-consistent agent. Gruber and Koszegi (2001) introduce time-inconsistent preferences into the rational addiction theory. They focus on differentiable Markov strategies and show that time inconsistency leads to overconsumption. This paper contributes to the literature by analyzing how internal commitment devices—specifically, bright-line rules—can mitigate overconsumption. Such rules are nondifferentiable and thus ruled out by Gruber and Koszegi (2001).

This paper is also closely related to work on personal rules. Bénabou and Tirole (2004) introduce bounded memory into the hyperbolic discounting literature by assuming that agents

[4]"Overconsumption" is defined relative to a time-consistent consumer.

may forget whether they are strong- or weak-willed while in a "cool" state. They find that strong types might underconsume in "hot" states to signal their strong willpower to their future selves, so that those future selves are less likely to misbehave. Bernheim et al. (2015) study asset accumulation by a time-inconsistent agent and analyze how personal rules can prevent poverty traps. Their solution concept is renegotiation-proof subgame perfect equilibria. In particular, they analyze equilibria that feature the worst punishment, showing that in such equilibria, a deviation is punished by a binge lasting at most two periods. In contrast, my result features permanent punishment, which better fits the persistent effects of lapses observed in addiction.

This paper also relates to the literature on consumption-saving decisions with present-biased agents. Krusell and Smith (2003) study a Ramsey model with a time-inconsistent agent. They construct "step-function equilibria" over a restricted subset of states and show that there is indeterminacy—that is, a continuum of states can be sustained as steady states.[5] My results imply that each step-function equilibrium, which lacks bright-line rules, can be improved upon by an equilibrium with a bright-line rule. Cao and Werning (2018) characterize conditions under which a time-inconsistent agent either dis-saves in all Markov equilibria or saves infinitely. Their focus on robust prediction rules out internal commitment devices. This paper complements their analysis by focusing on situations in which moderation (or abstinence) is ideal, which is relevant when studying addiction. Moreover, my equilibrium existence proof differs from theirs and can be used for equilibrium construction. Richter (2020) introduces a new notion of efficiency in consumption-saving problems with time-inconsistent preferences and examines how equilibrium efficiency relates to the convexity of time preferences. The study focuses on stable steady states in MPE, demonstrating that all such states are inefficient. Again, focusing on stable steady states means that bright-line rules are not considered.

The result that a time-inconsistent agent may underconsume is related to the findings of several other studies, including that of Bénabou and Tirole (2004) mentioned above. Carrillo (2005) shows that if a time-inconsistent agent can learn her preference through consumption, she may choose abstinence preemptively, fearing that discovering a strong taste for the addictive good would cause her future selves to overindulge. O'Donoghue and Rabin (1999) examine a stylized model in which agents choose whether to complete a task now or later in a finite time horizon with nonstationary cost distribution. They find that a sophisticated agent, anticipating future procrastination, may complete a tedious task earlier than a time-consistent agent.

[5] These are equilibria over a restricted subset of the state space, but may not be equilibria over the entire state space.

Finally, Bernheim and Rangel (2004) propose a model of addiction in which a time-consistent agent may temporarily enter a "hot" mode triggered by environmental cues. In the hot mode, she consumes the addictive good impulsively, while in the cold mode, she chooses a lifestyle—such as joining Alcoholics Anonymous, entering rehab, or doing neither—to manage exposure to such cues. Because the agent is time-consistent, she does not need internal commitments.

2 The Model

2.1 Addiction

Time is discrete, indexed by $t = 1, ..., \infty$. In each period t, a consumer (she) receives a fixed income $y > 0$ and allocates it between an ordinary good and an addictive good. Her utility depends not only on her current consumption of the ordinary good c_t and the addictive good a_t but also on a consumption stock s_t, which captures the cumulative effects of past consumption of the addictive good. The stock evolves according to

$$s_{t+1} = \rho s_t + a_t, \tag{1}$$

where $0 < \rho < 1$ is the depreciation rate. Let $\bar{s} \equiv \frac{y}{1-\rho}$ denote an upper bound on the consumption stock and $S \equiv [0, \bar{s}]$ the set of stocks.

Let $u(c_t, a_t, s_t)$ be the consumer's utility in period t. I assume that preferences are nonsatiated in the ordinary good—$u_c(c, a, s) > 0$ for all nonnegative c, a, and s—and that saving is infeasible. These assumptions, standard in the rational addiction literature, allow the analysis to focus on addiction. They imply that in an optimal path or "equilibrium" (defined below), the consumer exhausts her income each period: $c_t + a_t = y$. Therefore, the utility function u can be expressed in a reduced form as a function $w : [0, y] \times S \to \mathbb{R}$ defined by $w(a_t, s_t) \equiv u(y - a_t, a_t, s_t)$. I impose the following assumptions: (1) w is twice continuously differentiable. (2) The ordinary good is a necessity good: $w_a(y, s) < 0$, for all $s \in S$. Except in Section 4, I focus on the situation in which (3) w is strongly concave[6] and (4) moderation is ideal: $\arg\max_{s \in S} w((1-\rho)s, s) \in (0, \bar{s})$. I will discuss how the main results carry over to the situation in which w is not strongly concave, or when abstinence is ideal, in Section 4. The following two assumptions from Becker and Murphy (1988) capture

[6] A twice-continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is *strongly concave with coefficient* α if the matrix $\nabla^2 f(x) + \alpha I$ is negative semidefinite for every $x \in \mathbb{R}^n$, where $\nabla^2 f(x)$ is the Hessian matrix of f and I is the identity matrix. See page 72 in Bertsekas et al. (2003) for reference.

A sufficient condition for w to be strongly concave is that the utility function u is strongly concave.

key features of addictive goods:

A. 1 (Complementarity): $w_{as}(a,s) > 0$.

This captures *reinforcement*: past consumption increases craving for the addictive good.

A. 2 (Harmful Addiction): $w_s(a,s) < 0$.

This captures *tolerance*: holding consumption fixed, utility falls with higher past use. A sufficient condition for A.1 to hold is that the complementarity in (a,s) is stronger than the complementarity in (c,s): $u_{as} > u_{cs}$. A sufficient condition for A.2 to hold is that $u_s(c,a,s) < 0$. While I focus on harmful addictions, the analysis extends to beneficial addictions—such as exercising and saving—with the directions of overconsumption and underconsumption reversed. See Section 4.4 for further discussion.

Example 1 (Quadratic example)**.** *Consider a utility function w given by*

$$w(a,s) = -\frac{\alpha}{2}a^2 + ba - \lambda s - \frac{\nu}{2}s^2 + \zeta as,$$

where all the parameters are positive. Parameter b captures the marginal utility of the first unit of the addictive good at zero stock, while α governs how fast marginal utility decreases in consumption. Parameter λ measures the harm from the first unit of the consumption stock, and ν regulates how rapidly the harm of an additional unit increases in the stock level. Parameter ζ captures the degree of habit formation or the addictiveness of the good. Since $\zeta > 0$, Assumption A.1 holds. An example of parameter values satisfying conditions (1) to (4) and Assumption A.2 is $y = b = 1$, $\alpha = 10$, $\nu = 4$, $\zeta = 6$, $\lambda = 0.5$, and $\rho = 0.3$. □

Notation. For any differentiable function f, I denote $[f_{x_1} + f_{x_2}](x_1, x_2) \equiv f_{x_1}(x_1, x_2) + f_{x_2}(x_1, x_2)$.

2.2 Time inconsistency

The consumer has hyperbolic discounting preferences. Given a sequence of consumption and stock levels $(a_\tau, s_\tau)_{\tau \geq 1}$, her discounted payoff at time t is

$$w(a_t, s_t) + \beta \sum_{\tau=1}^{\infty} \delta^\tau w(a_{t+\tau}, s_{t+\tau}), \tag{2}$$

where $\delta \in (0,1)$ is the long-run discount factor and $\beta \in (0,1)$ captures present bias.

2.3 Equilibrium concept: Markov perfect equilibrium

Because an optimal plan from today's perspective is generally not optimal from the perspective of future selves, dynamic programming does not yield a time-consistent solution. Instead, the consumption problem is modeled as an intrapersonal game among a sequence of temporal selves. Each self, when in control, chooses consumption and receives a payoff given by expression (2).

The solution concept is *stationary Markov perfect equilibrium* (henceforth, MPE), in which strategies depend only on the payoff-relevant state variables—here, the consumption stock s—and are identical across all selves. From the law of motion (1), choosing current consumption is equivalent to choosing the next period's consumption stock. Thus, a Markov strategy can be represented as a mapping $Z : S \to S$, where $Z(s)$ is next period's consumption stock at current stock s and satisfies the budget constraint $Z(s) \in [\rho s, \rho s + y]$. "Equilibria" hereafter refer to MPE if not otherwise specified. The terms "state" and "stock" are used interchangeably.

Each equilibrium Z is associated with two value functions, as shown by Harris and Laibson (2001) and Luttmer and Mariotti (2003). First, the *continuation value function* $V : S \to \mathbb{R}$ represents the current self's discounted payoff beginning tomorrow, assuming all future selves follow Z and tomorrow's stock is s:

$$V(s) = w(Z(s) - \rho s, s) + \delta V(Z(s)). \tag{3}$$

Second, the *current value function* $W : S \to \mathbb{R}$ captures the current self's discounted payoff, assuming that all selves follow Z:

$$W(s) = w(Z(s) - \rho s, s) + \beta\delta V(Z(s)). \tag{4}$$

For Z to be an equilibrium, the current self must find it optimal to follow Z:

$$Z(s) \in \arg\max_{z \in [\rho s, \rho s + y]} w(z - \rho s, s) + \beta\delta V(z). \tag{5}$$

These equations imply a simple relation between the value functions:

$$\beta V(s) = W(s) - (1 - \beta) w(Z(s) - \rho s, s). \tag{6}$$

This equation highlights the nature of intertemporal conflict. When $\beta = 1$, preferences are time-consistent and the two value functions coincide: $W = V$. When $\beta < 1$, the current self perceives tomorrow's self as overvaluing tomorrow's utility. The term $(1 - \beta)$ measures the degree of this misalignment.

2.4 Monotonicity and equilibrium existence

We now establish key properties of MPE that we use throughout the analysis.

Observation 1. *Let Z be an MPE. Then,*

1. *Z is weakly increasing; and*
2. *Z has at least one steady state. Every steady state is a stationary point.*[7]

Part 1 follows from a standard supermodularity argument. Specifically, since w is strongly concave (thus $w_{aa} < 0$) and exhibits complementarity between today's stock s and tomorrow's stock z (i.e., $w_{as} > 0$), a current self's objective function, $w(z - \rho s, s) + \beta\delta V(s)$, is supermodular in s and z. Thus, for the current self, the higher the current stock s, the higher the benefit of choosing a higher stock for tomorrow, implying that the equilibrium strategy Z is weakly increasing. Note that although Z is monotone, the corresponding consumption strategy $Z(s) - \rho s$ may not be monotone in s. Part 2 follows from Part 1 and Tarski's fixed-point theorem. Observation 1 parallels results in the literature on consumption-saving decisions and intergenerational altruism, including those of Leininger (1986) and Bernheim and Ray (1987).

Figures 1 and 2 in the introduction illustrate two possible equilibrium strategies. In Figure 1, the equilibrium has two steady states, exhibiting path dependence. In contrast, the equilibrium in Figure 2 features a unique steady state and no path dependence.

To ensure the analysis is not vacuous, we state the following existence result:

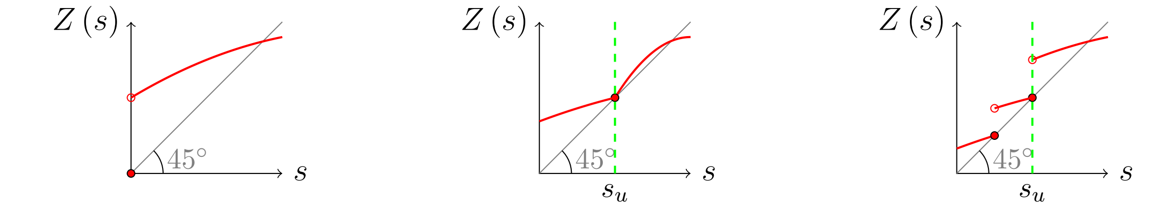
Proposition 1. *A Markov perfect equilibrium in pure strategies exists.*

The proof is close to Ray (1987) and can be found in the online appendix.

3 Main Results

To state the main results, we first define bright-line rules and Pareto dominance.

[7] If Z is not increasing, then a steady state might not be a fixed point. For example, if Z is strictly decreasing, then a steady state can be a cycle.



A bright line at abstinence   A continuous bright line   Multiple bright lines

Figure 3: Examples of Equilibria with Bright-Line Rules

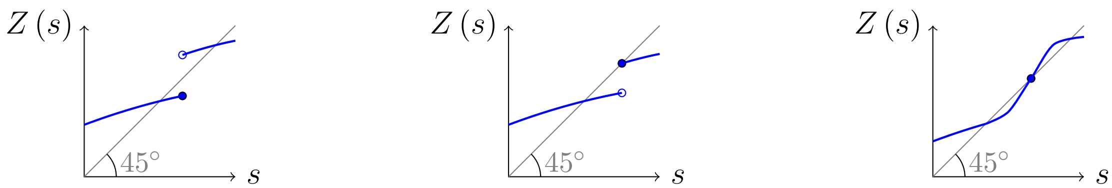


Figure 4: Examples of Equilibria Without Bright-Line Rules

3.1 Bright-line rules and Pareto dominance

Definition 1 (Bright-Line Rules)**.** An equilibrium Z features a *bright-line rule* if there exists a threshold $s_u \in S$, called a *bright line*, such that

(1) s_u is a steady state: $Z(s_u) = s_u$;

(2) At s_u, Z is attracting on the left and repelling on the right. That is, $Z(s) > s$ for all $s \in (s_u - \epsilon, s_u) \cup (s_u, s_u + \epsilon)$, for some $\epsilon > 0$.

Figure 1 illustrates an equilibrium with a bright-line rule. The threshold s_u is called a bright line because it delineates a stark behavioral contrast: deviations below it—seen as innocuous—are ultimately forgiven, while deviations above it—viewed as misbehavior—trigger a permanent shift to higher consumption. This contrast leads to path dependence: starting from a stock below s_u, the consumption path will never rise above the steady state s_u; however, starting above s_u, consumption will eventually converge to a strictly higher steady state. For comparison, Figure 2 depicts an equilibrium without a bright-line rule. Figure 3 shows additional examples of bright-line rules, including equilibria with a bright line at abstinence, with continuous bright lines, and with multiple bright lines. Figure 4 illustrates more equilibria without bright-line rules, including an equilibrium in which the point of discontinuity is not a steady state (the left graph) and equilibria in which the steady states are not repelling on the right (the middle and right graphs).

To evaluate whether bright-line rules improve efficiency, I introduce a notion of Pareto dominance. Let Z and $\hat{Z}$ be two equilibria with associated current value functions W and

$\hat{W}$, respectively. Given an initial stock s_1, let $(s_t)_{t\in\mathbb{N}}$ and $(\hat{s}_t)_{t\in\mathbb{N}}$ be the equilibrium stock paths under Z and $\hat{Z}$, respectively.

Definition 2 (Pareto Dominance)**.** An equilibrium $\hat{Z}$ Pareto-dominates Z *over a set of stocks* $[s', s'']$, if

(1) $\hat{Z}$ is at least as good as Z for any initial stock $s_1 \in [s', s'']$ and every self: $\hat{W}(\hat{s}_t) \geq W(s_t)$ for all $t \in \mathbb{N}$;[8] and

(2) $\hat{Z}$ is strictly better than Z for some initial stock $s_1 \in [s', s'']$ and some self: $\hat{W}(\hat{s}_t) > W(s_t)$ for some t.

This definition is based on the preferences of *every* temporal self, not just that of the time-0 self. Therefore, if an equilibrium Pareto-dominates another equilibrium for some initial stock, then on path, every self prefers the former to the latter.

3.2 Necessity and sufficiency of bright-line rules for low consumption

The first main result demonstrates that bright-line rules are essential for time-inconsistent consumers to avoid excessive consumption. We introduce notation to define excessive consumption.

Since parameter β reflects both present bias and discounting, we aim to isolate the present-bias component. To this end, consider a time-consistent consumer with a lower discount factor, $\hat{\delta}$, defined by averaging β over the exponential discounting sequence δ, δ^2, δ^3,...; that is, $\hat{\delta}$ solves

$$\frac{\hat{\delta}}{1-\hat{\delta}} = \beta\delta + \beta\delta^2 + ... = \frac{\beta\delta}{1-\delta}. \tag{7}$$

This consumer, called the $\hat{\delta}$-*consumer* for short, has the same preference over sequences of steady-state stocks as the time-inconsistent consumer.[9]

Let $\gamma(s, \hat{\delta})$ denote this consumer's net benefit at a steady state s from consuming an additional unit today (and ρ units fewer tomorrow, so the stock on the day after tomorrow remains unchanged):

$$\gamma(s, \hat{\delta}) := \left[w_a - \hat{\delta}\left(\rho w_a - w_s\right)\right]\left((1-\rho)\, s, s\right), \tag{8}$$

where w_a is the benefit of consuming an additional unit today; $\hat{\delta}\rho w_a$ is the discounted loss of consuming ρ units fewer tomorrow; and $\hat{\delta} w_s$ is the discounted disutility from a higher stock

[8] To avoid repetition, the phrase "for any initial stock $s_1 \in [s', s'']$" is omitted.

[9] That is, at any initial stock s, if we let both consumers choose a stock level to remain at forever starting tomorrow, they will choose the same one.

tomorrow. For it to be optimal to stay at the steady state, the net benefit of the above option must be zero; that is, the Euler equation $\gamma(s, \hat{\delta}) = 0$ must hold.

Because w is strongly concave, there is a $\delta_\beta < 1$ such that for all $\delta \in (\delta_\beta, 1)$, there is a unique s, denoted as $s_{\delta,\beta}$, that solves the Euler equation $\gamma(s, \hat{\delta}) = 0$. Moreover, for all $s < s_{\delta,\beta}$, $\gamma(s, \hat{\delta}) > 0$; and for all $s > s_{\delta,\beta}$, $\gamma(s, \hat{\delta}) < 0$. For convenience, we write $s_{\delta,1}$ as s_δ. We have $s_{\delta,\beta} > s_\delta$ since the higher β is, the more patient the $\hat{\delta}$-consumer is, and hence the lower consumption she will choose.

We refer to s_δ as *moderation* since it is the steady-state stock of the time-consistent consumer with a discount factor δ. *Underconsumption* and *overconsumption* are defined relative to the moderation state s_δ. We define *excessive consumption* relative to $s_{\delta,\beta}$; that is, a steady state of the time-inconsistent consumer is called an *excessive-consumption trap* (or simply a "trap") if it is above $s_{\delta,\beta}$. In such a trap, the consumer not only overconsumes but also consumes more than the $\hat{\delta}$-consumer, i.e., a time-consistent consumer with the same effective discount factor.

We are ready to present the first result.

Theorem 1. *Assume that $\delta > \delta_\beta$.*

1. *If an equilibrium does not feature a bright-line rule, the consumer will fall into an excessive-consumption trap.*

2. *If an equilibrium features a bright-line rule, then, starting at any state below the bright line, the consumer will not be trapped in excessive consumption.*

Theorem 1 indicates that bright-line rules are not only necessary for a time-inconsistent consumer to avoid excessive consumption but also sufficient if her past consumption is low. Thus, incorporating bright-line rules into addiction treatment programs is crucial if no external commitment devices are available.

The intuition of Theorem 1 is that MPE with bright-line rules impose stronger penalties for overconsumption than MPE without them. Indeed, if an equilibrium lacks a bright-line rule, then its steady states are absorbing on the right, as seen in the steady state in Figure 2. Thus, small deviations are eventually forgiven, making self-discipline harder. In contrast, in an equilibrium with a bright-line rule, any upward deviation above the bright line results in permanently higher consumption, creating stronger deterrence and sustaining lower consumption.

I offer a heuristic argument. All formal proofs of this paper are in the appendix. Consider a steady state s of an MPE Z. The current self's net benefit from consuming an additional unit today is

$$[w_a - \beta\delta\left(\rho w_a - w_s\right)]\left(\left(1-\rho\right)s, s\right) - \left(1-\beta\right)\delta w_a\left(\left(1-\rho\right)s, s\right)\bar{D}_+ Z'(s), \tag{9}$$

where $\bar{D}_+ Z'(s) := \limsup_{h\downarrow 0} \frac{Z(s+h)-Z(s)}{h}$ measures how tomorrow's self reacts to today's deviation. The terms in the square brackets, $[w_a - \beta\delta\left(\rho w_a - w_s\right)]$, denote the net benefit enjoyed by a time-consistent consumer with discount factor $\beta\delta$, as explained in (8). The term $\left(1-\beta\right)\delta w_a \bar{D}_+ Z'(s)$ captures the strategic marginal cost borne only by a time-inconsistent consumer due to the misalignment between today's and tomorrow's selves: today's self perceives tomorrow's self as overvaluing tomorrow's utility by amount $(1-\beta)\delta w$, as (6) demonstrates. If the current self consumes an additional unit, tomorrow's stock will rise by one unit, and consequently, tomorrow's self will react by consuming $\bar{D}_+ Z'(s)$ unit more, thereby increasing the strategic cost by $(1-\beta)\delta w_a \bar{D}_+ Z'(s)$. The steeper strategy Z is (more precisely, the higher the term $\bar{D}_+ Z'(s)$ is), the more future selves punish today's consumption by overconsuming and thus the higher the cost to deviate. Using (7) and (8), we rewrite (9) as

$$(1-(1-\beta)\delta)\gamma(s,\hat{\delta}) - (1-\beta)\delta w_a((1-\rho)s, s)(\bar{D}_+ Z'(s) - 1), \tag{10}$$

where $\gamma(s,\hat{\delta})$ is the $\hat{\delta}$-consumer's net benefit of consuming an additional unit, as defined by (8). Recall $s_{\delta,\beta}$ is this consumer's steady-state stock; moreover, $\gamma(s,\hat{\delta}) > 0$ for any state $s \in [0, s_{\delta,\beta})$ and $\gamma(s,\hat{\delta}) < 0$ for any state $s \in (s_{\delta,\beta}, \bar{s}]$.

If a steady state s is not sustained by a bright-line rule (as in Figure 2), then it is absorbing on the right,[10] and thus $\bar{D}_+ Z'(s) \leq 1$. For the deviation of consuming an additional unit to be unprofitable, the net benefit (10) must be nonpositive, implying $\gamma(s,\hat{\delta}) \leq 0$, or equivalently, $s \geq s_{\delta,\beta}$. Thus, the consumer consumes excessively at s, by the definition of excessive consumption.

In contrast, if a steady state is sustained by a bright-line rule (as is steady state s_u in Figure 1), then it is repelling on the right, implying $\bar{D}_+ Z'(s) \geq 1$. Thus, even if a bright line is below $s_{\delta,\beta}$, as long as $\bar{D}_+ Z'(s)$ is high, i.e., as long as tomorrow's self sufficiently punishes the current self for deviating, the above deviation will be unprofitable.

To see why the consumer never consumes excessively at *any* bright line, suppose instead that she consumes excessively at some bright line s_u in some equilibrium Z. Then, this bright-line rule must be supported by an even more excessive consumption trap, denoted as s_o. See Figure 1 for an illustration. We show in the appendix that this is impossible.

[10] One can show that if $\delta > \delta_\beta$, then there are no steady states that are repelling on both sides. This implies that any steady state that is not a bright line is absorbing on the right.

Intuitively, at any stock $s \in (s_u, s_o)$, the consumer strictly prefers to remain at s forever rather than follow the equilibrium path that ultimately leads to the excessive consumption trap s_o.[11] Consequently, if a previous self slightly crosses above the bright line s_u, today's self will rather forgive the misbehavior and choose the bright-line stock s_u to get all future selves back on track.

In a sense, in MPE without a bright-line rule, the consumer consumes excessively because future selves do not cooperate to punish bad behavior and reward good behavior. The differentiable equilibria that Gruber and Koszegi (2001) focus on and the step-function equilibria constructed by Krusell and Smith (2003) are equilibria without bright-line rules. In contrast, a bright-line rule is a mechanism through which future selves cooperate to punish the current self for crossing the line, which strongly disciplines the current self.

Remark 3.1. *The threshold $\hat{\delta}$ also appears in the consumption-saving models studied by Bernheim et al. (2015) and Cao and Werning (2018). In their models, the $\hat{\delta}$-consumer accumulates infinite wealth if and only if the interest rate r exceeds $1/\hat{\delta}$. Cao and Werning (2018) find that if $r > 1/\hat{\delta}$, the time-inconsistent consumer also accumulates infinite wealth in all MPE. When $r \in (\frac{1}{\delta}, \frac{1}{\hat{\delta}})$, and thus the $\hat{\delta}$-consumer does not accumulate infinite wealth, the time-inconsistent consumer also fails to accumulate infinite wealth in every MPE. In this intermediate region, however, Bernheim et al. (2015) demonstrates through a numerical example that bright-line rules may help a consumer sustain positive wealth (see their Figure 5 for reference).*

Unlike in their models, in my model, the steady-state stock of a time-consistent consumer is interior, rather than bang-bang (i.e., not simply zero or infinity).[12] This allows for a richer comparison between the long-run behavior of the time-inconsistent consumer and that of the $\hat{\delta}$-consumer. Theorem 1 shows that bright-line rules are crucial in determining this relationship: without bright-line rules, the time-inconsistent consumer consumes more than the $\hat{\delta}$-consumer; but with them, the time-inconsistent consumer consumes strictly less if the initial stock is low. Moreover, as Section 4.2 shows, the time-inconsistent consumer can consume even less than the time-consistent consumer with discount factor δ.

3.3 Efficiency of bright-line rules

Theorem 1 shows that bright-line rules are both necessary and sufficient for a time-inconsistent consumer to avoid excessive-consumption traps. But would the consumer prefer to avoid such traps? Theorem 2 demonstrates that, under some mild conditions, every temporal self

[11] This result is also used in Step 1 of the proof sketch of Theorem 2, where we provide more explanation.

[12] This is because in my model, the consumption stock directly affects utility, allowing me to introduce an additional source of concavity, i.e., utility w is strictly concave in s.

prefers to be in equilibria with bright-line rules. Thus, equilibria with bright-line rules are more efficient than equilibria without such rules.

Theorem 2. *Assume that $\delta > \delta_\beta$ and that the complementarity at $s_{\delta,\beta}$ is not too low (i.e, $w_{as}((1-\rho)s_{\delta,\beta}, s_{\delta,\beta}) > \underline{K}$ for some $\underline{K} < \infty$). Then, for any MPE Z without a bright-line rule, there are stocks s' and s_u, where $s' \leq s_u$, and an MPE $\hat{Z}$ with a bright-line rule at s_u, such that $\hat{Z}$ Pareto-dominates Z for all stocks above s', i.e., over the interval $[s', \bar{s}]$.*

Theorem 2 indicates that for any equilibrium Z without a bright-line rule, there is an equilibrium $\hat{Z}$ with a bright-line rule, such that every self prefers equilibrium $\hat{Z}$ to Z over all states near and above the bright line (possibly over the entire state space). Thus, addiction treatment programs that emphasize bright-line rules—such as Alcoholics Anonymous—are likely to be more successful than those that do not, because time-inconsistent individuals will indeed adopt such rules. This might explain why most alcohol treatment programs promote all-or-nothing thinking, which resembles bright-line rules.

Before presenting a proof sketch, we briefly discuss whether MPE with bright-line rules are renegotiation-proof. Following Bernheim et al. (2015), we examine this question using the notion of weak renegotiation proofness introduced by Bernheim and Ray (1989) and Farrell and Maskin (1989). Essentially, an equilibrium is weakly renegotiation-proof if, at any state (or stock), the set of continuation equilibria is not strictly Pareto-ranked. Since each MPE defines a unique continuation equilibrium for each stock, MPE are weakly renegotiation-proof.

A proof sketch. Theorem 2 is proved by construction. Specifically, for any equilibrium Z without a bright-line rule, let s_o be its steady state. We construct an equilibrium $\hat{Z}$ with a bright-line rule that Pareto-dominates Z over stocks near or above the bright line. The left panel of Figure 5 illustrates equilibrium Z.

First, we show that compared with playing equilibrium Z, at any state $s \in (s_{\delta,\beta}, s_o)$, the consumer strictly prefers to remain at s forever—if an external commitment device could enforce this. Formally, let W be the current value function of equilibrium Z. Let $U(s)$ denote the discounted payoff from staying at s forever: $U(s) \equiv \left(1 + \frac{\beta\delta}{1-\delta}\right) w\left((1-\rho)s, s\right)$. In the right panel of Figure 5, functions W and U are represented by the solid curve and the dash-dot curve, respectively. As the figure shows, $W(s) < U(s)$ for all $s \in [s_{\delta,\beta}, s_o)$; that is, staying at s forever is more desirable than following equilibrium Z.[13]

Thus, by the continuity of both W and U over $[0, s_{\delta,\beta}]$, either there exists some $s_u < s_{\delta,\beta}$ such that $W(s_u) = U(s_u)$, as the right panel of Figure 5 shows, or $W(s) < U(s)$ for all

[13] To get this inequality, we first use the equation $W'(s+) = [-\rho w_a + w_s]((1-\rho)s, s)$ (proved in Lemma 8 in the Online Appendix) and the inequality $\gamma(s, \hat{\delta}) < 0$ to arrive at $W'(s+) > U'(s)$ for $s \in [s_{\delta,\beta}, s_o)$. Then, we use the fact that s_o is a steady state, that is, $W(s_o) = U(s_o)$, to obtain the desired inequality.

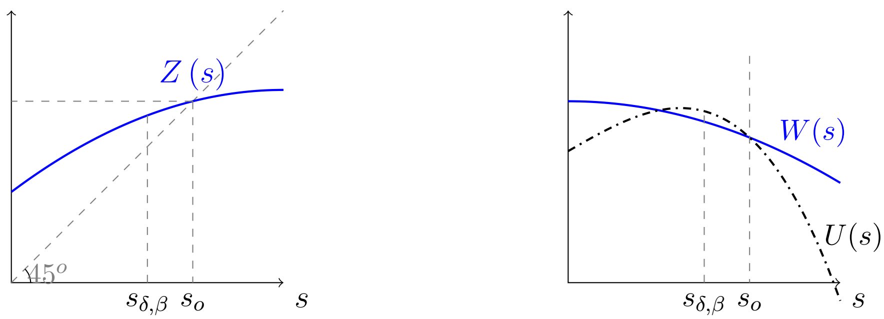


Figure 5: Left panel: equilibrium without a bright-line rule; Right panel: value functions

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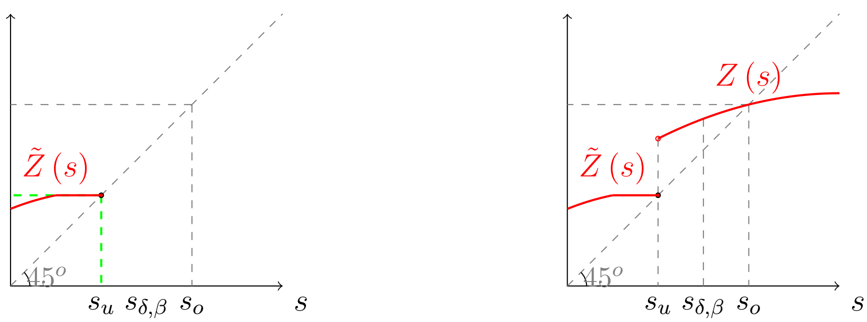

Figure 6: Left panel: constrained equilibrium; Right panel: MPE $\hat{Z}$ with a bright-line rule

$s \leq s_{\delta,\beta}$. In the latter case, we set $s_u = 0$.

Next, we construct an equilibrium $\hat{Z}$ with a bright-line rule at s_u by replacing Z over $[0, s_u]$ with a "constrained equilibrium" $\tilde{Z}$, which we now define. Consider a constrained game in which the set of stock and the consumer's choice set are restricted to $[0, s_u]$. That is, at every state below the bright line, $s \in [0, s_u]$, the consumer's choice (for tomorrow's stock) is restricted to stocks below the bright line, i.e., a stock in $[\rho s, \rho s + y] \cap [0, s_u]$. See the left panel of Figure 6 for an illustration. By arguments analogous to those in the proof of Proposition 1, there exists a constrained MPE $\tilde{Z}$; moreover, s_u is a steady state of $\tilde{Z}$.[14]

We define strategy $\hat{Z} : S \to S$ of the original, unconstrained game by replacing Z with $\tilde{Z}$ over $[0, s_u]$:

$$\hat{Z}(s) \equiv \begin{cases} \tilde{Z}(s), & \text{if } s \in [0, s_u] \\ Z(s), & \text{if } s \in (s_u, \bar{s}]. \end{cases}$$

The right panel of Figure 6 depicts this new strategy. By construction, $\hat{Z}$ features a bright-line rule at s_u. Supermodular arguments (detailed in the appendix) imply that $\hat{Z}$ is an

[14]To show that $\tilde{Z}(s_u) = s_u$, note that by restriction, $\tilde{Z}(s_u) \leq s_u$ holds. Suppose by negation that $\tilde{Z}(s_u) < s_u$. Then $\tilde{Z}$ has a stable steady state lower than s_u, contradicting the fact that the consumer overconsumes at every stable steady state.

equilibrium.

Finally, we show the new equilibrium $\hat{Z}$ (with a bright-line rule at s_u) Pareto-dominates the original equilibrium Z (that lacks a bright-line rule) over stocks $[s_u - \epsilon, \bar{s}]$ for some $\epsilon > 0$. Intuitively, a bright-line rule acts as an internal commitment device whereby future selves cooperate to reward good behavior and punish misbehavior. In equilibrium $\hat{Z}$, at states near the bright-line rule, a current self can opt out of the commitment device by choosing the action prescribed by the original equilibrium Z, thereby obtaining the original equilibrium payoff. Since she does not opt out, she must be at least as well off. Moreover, her future selves strictly benefit from the commitment device.

We now discuss over what set of stocks equilibrium $\hat{Z}$ Pareto-dominates equilibrium Z. Let $\tilde{s}$ be such that $\hat{Z}(s) < Z(s)$ on $(\tilde{s}, s_u)$; then equilibrium $\hat{Z}$ Pareto-dominates equilibrium Z on $[\tilde{s}, \bar{s}]$. Thus, if the bright-line equilibrium $\hat{Z}$ is below the original equilibrium Z for all states below the bright line, i.e., over $[0, s_u]$, then $\hat{Z}$ Pareto-dominates Z over the whole set of stocks S. A special case is when the bright line is at abstinence. In such cases, at any initial stock, all selves agree that equilibrium $\hat{Z}$ is superior to equilibrium Z. Thus, we expect the consumer to play equilibrium $\hat{Z}$ instead of equilibrium Z.

Remark 3.2. *Theorem 2 shows that MPE without a bright-line rule are Pareto-dominated. But are all MPE with bright-line rules Pareto-dominant? This is a challenging question because an equilibrium with a lower bright line might Pareto-dominate another equilibrium with a higher bright line at low stocks but not at higher stocks. In general, the ranking of different MPE with bright-line rules is complex, and it is difficult to characterize the set of Pareto-dominant MPE with bright-line rules.*

Remark 3.3. *Although we assume that 0 stock is not ideal, Theorems 1 and 2 do not rely on this assumption. Specifically, if the steady state of the $\hat{\delta}$-consumer, $s_{\delta,\beta}$, is strictly positive, then both theorems continue to hold. If instead $s_{\delta,\beta} = 0$, then there are MPE without bright-line rules whose steady state is 0. However, for any MPE whose steady state is strictly positive, there is an MPE with a bright-line rule at abstinence that Pareto-dominates it.*

4 Discussion

4.1 Can bright-line rules survive unexpected shocks?

Are bright-line rules fragile? One might worry that an unexpected shock—such as a sudden increase in cravings—can prompt the consumer to cross the line and fall into excessive consumption. We show that this concern applies to nonabstinence bright lines but, perhaps surprisingly, not to abstinence bright lines. An implication is that abstinence-based treatment programs, such as the twelve-step program of Alcoholics Anonymous, may be more robust to shocks than moderation-based ones. This perhaps explains why such abstinence-based programs are widely adopted.

We model shocks as follows. Suppose that in period t, an unexpected, one-time shock temporarily increases the consumer's marginal utility. Specifically, her utility in period t changes from $w(a, s)$ to $w(a, s + \epsilon)$, where $\epsilon > 0$, given her current stock s and consumption a. Her utility in all future periods remains unchanged.

Assume, first, that before the shock occurs, the consumer was playing an equilibrium Z with a bright-line rule at s_u, and her stock was at the steady state s_u. Second, assume that after the shock, she believes that her future selves will continue playing equilibrium Z. Will she cross the bright line in response to the shock?

As shown in the appendix, if $s_u > 0$, she will indeed cross it. For intuition, suppose that equilibrium Z is discontinuous at s_u, as in Figure 1. Absent the shock, the consumer is indifferent between remaining at the bright line and bingeing, i.e., choosing $Z(s_u+)$. This indifference must hold for two reasons. First, the consumer must weakly prefer to stay at the bright line; otherwise, she would cross it, which contradicts the assumption that Z is an equilibrium. Second, if she strictly preferred to remain at the bright line, then, if she slightly crossed the line, her future self would rather undo her deviation by reducing consumption, making the threat of excessive consumption incredible. Thus, the indifference must hold. The shock then increases her marginal utility, breaking this indifference in favor of bingeing.

In contrast, if the bright line is at abstinence, i.e., $s_u = 0$, and if the shock ϵ is sufficiently small, then, generically, the consumer will not cross the line. In this case, absent the shock, the consumer strictly prefers to remain abstinent rather than fall into excessive consumption. Unlike with nonabstinence rules, there is no need for indifference at the bright line because future selves cannot undo any deviation of earlier selves. Any violation of abstinence clearly sets a precedent and triggers a permanent shift to higher consumption. Consequently, a weak shock will not induce the consumer to abandon abstinence.

Finally, under what conditions does an MPE with an abstinence bright-line rule exist? Such an equilibrium exists if the threshold stock for excessive consumption (i.e., the steady-state stock level of the $\hat{\delta}$-consumer), $s_{\delta,\beta}$, is close to 0. This condition is met if (i) the consumer's present bias β is intermediate; and (ii) the addictive good's marginal utility of consumption at zero consumption and zero stock is positive, i.e., $w_a(0, 0) > 0$, but the good is sufficiently harmful that a time-consistent consumer with discount factor δ would choose 0 as the steady state. Substances such as cigarettes, alcohol (for some people), and other highly addictive substances likely satisfy this second condition. Thus, for these substances, if a consumer exhibits an intermediate level of present bias, then the threshold $s_{\delta,\beta}$ is close to 0, and thus the consumer may adopt a bright-line rule at abstinence.

4.2 Can bright-line rules lead to moderation or underconsumption?

The main results show that a bright-line rule reduces the tendency toward overconsumption. But can such a rule eliminate overconsumption entirely or even lead to underconsumption? Proposition 2 below shows that if the long-run discount factor is sufficiently high relative to present bias, moderation can be sustained in some equilibrium. Moreover, there exist equilibria in which the consumer even underconsumes. In such equilibria, the excessive-consumption trap is so severe that the consumer prefers to remain at a low state of consumption to avoid falling into it. Thus, as Ainslie (1992) notes, bright-line rules may induce compulsion (underconsumption), because the consumer is overly concerned with precedent. A notable example is anorexia, in which an individual becomes fixated on following a personal rule that results in dangerously low food consumption.

To state Proposition 2, recall that s_δ denotes the moderation level of stock—the steady state of a time-consistent consumer.

Proposition 2. *For every present bias $\beta < 1$, there exists some $\bar{\delta} < 1$ and some $\Delta > 0$ such that if $\delta > \bar{\delta}$, then for every $s_u \in (s_\delta - \Delta, s_\delta]$, there exists an MPE with a steady state s_u. Moreover, any MPE that has a steady state at s_u must be supported by a bright-line rule.*

4.3 Time consistency or naivety precludes bright-line rules

The consumer we have analyzed so far is time-inconsistent and sophisticated (i.e., aware of her self-control issue). Are both of these traits necessary for a consumer to adopt bright-line rules? I show that they are. These results indicate that bright-line rules are internal commitment devices that time-inconsistent consumers who are aware of their self-control issues adopt to mitigate excessive consumption. We now examine each factor in turn.

1. Time consistency precludes bright-line rules. Consider a time-consistent consumer. If she is sufficiently patient (i.e., $\delta > \delta_1$), then she never adopts a bright-line rule and her optimal policy has a unique steady state.[15] If instead she is impatient ($\delta < \delta_1$), or if the complementarity between consumption and stock (w_{as}) is sufficiently strong so that her utility function w is not strongly concave, then her optimal policy may feature multiple steady states, including some that are not absorbing. However, I show that the behavior

[15] This follows from the fact that when $\delta > \delta_1$, this consumer's Euler equation at steady state s, $\gamma(s, \delta) = 0$, has a unique solution (for fixed δ).

of a time-consistent consumer around such steady states differs qualitatively from that of a time-inconsistent consumer around bright-line rules.

Specifically, in the online appendix, I show that, generically, a time-consistent consumer never adopts a bright-line rule: her steady state is either absorbing on both sides or repelling on both sides. Figure 7 depicts a repelling steady state at $\tilde{s}$. Unlike the case of a bright-line rule, a small downward deviation leads to permanently lower consumption rather than a return to the prior steady state.

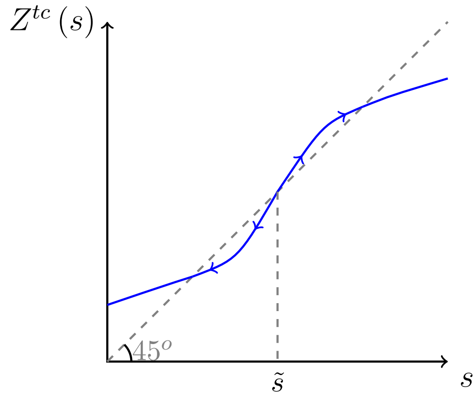

Figure 7: A two-sided unstable steady state at $\tilde{s}$ of a time-consistent consumer

2. Naivete and bright-line rules. We now examine how sophistication influences the adoption of bright-line rules. For simplicity, consider a fully naive consumer—one who incorrectly believes she will behave like a time-consistent consumer with discount factor δ in the future. Assume further that the consumer has quadratic utility, as in Example 1. As shown the online appendix, such a naive consumer never adopts a bright-line rule. Thus, sophistication is a requirement for bright-line rules to appeal to a time-inconsistent consumer.[16]

One can also show that the naive consumer overconsumes, i.e., consumes more than a time-consistent consumer would. Thus, in the absence of bright-line rules, naive consumers and sophisticated consumers behave similarly qualitatively: both fail to achieve the efficiency in the time-consistent benchmark. However, only sophisticated consumers, aware of their self-control issues, are capable of adopting bright-line rules to mitigate this behavior.

4.4 Beneficial addiction

Although the main results are stated for harmful addictions, they also apply to beneficial addictions—i.e., when Assumption A.2 is replaced with $w_s(a, s) > 0$—such as exercise and

[16] We leave for future research the question of how much sophistication is sufficient for a time-inconsistent consumer to adopt bright-line rules.

saving (as in Krusell and Smith (2003)). In this case, we reinterpret *overconsumption* as *underconsumption* and vice versa. Therefore, individuals can use bright-line rules to develop good exercise or saving habits.

Specifically, recall that $s_{\delta,\beta}$ and s_δ denote the steady-state stocks for time-consistent consumers with discount factors $\hat{\delta}$ (defined by (7)) and δ, respectively. Theorem 1 can be restated as follows: if a time-inconsistent consumer's discount factor δ is high, then in any equilibrium without a bright-line rule, her steady-state stock is at most $s_{\delta,\beta}$; in contrast, in any equilibrium with a bright-line rule, if her initial stock is high, her steady-state stock exceeds $s_{\delta,\beta}$.

Theorem 2 can be modified accordingly: Assume that a time-inconsistent consumer's discount factor δ is high and that the complementarity between consumption and stock at $s_{\delta,\beta}$ is high. Then, any equilibrium without a bright-line rule is Pareto-dominated by some equilibrium with a bright-line rule at some threshold s_u, for all initial stocks near or below the bright line—that is, for all $s_1 \in [0, s_u + \epsilon]$ for an $\epsilon > 0$.

4.5 Low long-run discount factors

Theorems 1 and 2 assume that the time-inconsistent consumer has a high long-run discount factor δ (i.e., $\delta > \delta_\beta$), ensuring a unique absorbing steady state in every MPE. However, if δ is low, an equilibrium may have multiple absorbing steady states.[17] In such cases, can bright-line rules still help individuals sustain lower consumption?

In the online appendix, I show that the following modified version of Theorem 2 continues to hold if we compare MPE based on whether their lowest steady state is sustained by a bright-line rule. Specifically, if complementarity between consumption and stock is high,[18] for any equilibrium Z whose lowest steady state is not sustained by a bright-line rule, there is another MPE $\hat{Z}$ whose lowest steady state is sustained by a bright line that Pareto-dominates Z for all initial stock near or above the bright line. The formal statement and proof are in the online appendix. However, Theorem 1 may not hold because there can be bright lines between two absorbing steady states, and consequently, the steady states at bright lines may be higher than other steady states.

[17]Mathematically, if $\delta > \delta_\beta$, then there is a unique stock s solving the Euler equation of a time-consistent consumer with discount factor $\hat{\delta}$: $\gamma(s, \hat{\delta}) = 0$. If $\delta < \delta_\beta$, then multiple s may solve this equation.

[18]For the modified Theorem 2 to hold, it suffices to have high complementarity at the absorbing steady state of equilibria. Formally, let S_β denote the set of s that solves $\gamma(s, \hat{\delta}) = 0$. If the complementarity term $w_{as}((1-\rho)s, s)$ at all $s \in S_\beta$ is not too low, analogous to the sufficient condition in Theorem 2, then the modified Theorem 2 holds.

5 Conclusion

This paper studies addictive good consumption by a time-inconsistent consumer. I show that bright-line personal rules are not only necessary to avoid excessive consumption but also sufficient if the individual's past consumption is low. Moreover, MPE with bright-line rules are more efficient than those without them. Among bright-line rules, those based on abstinence are more stable, as they can survive preference shocks. These results help explain why most treatment programs for alcohol and other substance addictions are abstinence-based and encourage all-or-nothing thinking that mirrors bright-line rules.

However, non-abstinence bright-line rules are still relevant when considering addictive goods for which zero consumption is impractical, such as food addiction. Examples include low-carbohydrate diets or calorie-restricted diets, such as 1,200- and 1,500-calorie diets.

Of course, non-abstinence bright-line rules can be difficult to maintain, and even abstinence rules may fail under sufficiently strong shocks. Are addicts doomed to excessive consumption? While the risk of relapse is indeed high, individuals can adopt practical measures to mitigate this risk or to regain control if a relapse occurs.

First, individuals may proactively manage their environment to reduce exposure to temptations or exercise additional caution to avoid crossing their bright lines. For example, if an individual is invited to a New Year's party where alcohol will be served, she may choose to skip the event, inform others in advance about her struggle with addiction, or—if her bright line does not require abstinence—reduce her alcohol consumption in the days leading up to the party—so that even if she consumes a drink at the party, she does not cross her bright line. In practice, Alcoholics Anonymous actively educates members on how to anticipate and manage high-risk situations that could trigger lapses.

Second, even after a full relapse and a return to excessive consumption, individuals may seek help from rehabilitation programs aimed at treating addiction (i.e., reducing consumption stock). After successful treatment, if an individual's addiction level is sufficiently reduced (i.e., her consumption stock is low), she may again adopt a bright-line rule to sustain lower or zero consumption.

6 Appendix

6.1 Implication of Strong Concavity

We show that w being strongly concave implies that for all $\rho \in [0,1]$, $a \in (0,y)$, and $s \in S$, it holds that $(1-\rho)^2 w_{aa} + 2(1-\rho)w_{as} + w_{ss} < 0$. First, if w is strongly concave, then $w_{aa} < 0$, $w_{ss} < 0$, and $w_{aa}w_{ss} > w_{as}^2$, which implies that $\sqrt{|w_{aa}|}\sqrt{|w_{ss}|} > w_{as}$. Thus,

$$\begin{aligned}(1-\rho)^2 w_{aa} + 2(1-\rho)w_{as} + w_{ss} &< (1-\rho)^2 w_{aa} + 2(1-\rho)\sqrt{|w_{aa}|}\sqrt{|w_{ss}|} + w_{ss} \\ &= -\left((1-\rho)\sqrt{|w_{aa}|} - \sqrt{|w_{ss}|}\right)^2 \leq 0.\end{aligned}$$

An implication of inequality $(1-\rho)^2 w_{aa} + 2(1-\rho)w_{as} + w_{ss} < 0$ is that $[(1-\rho)w_a + w_s]((1-\rho)s, s)$ strictly decreases in s. Thus, if there is some $s' \in S$ such that $[(1-\rho)w_a + w_s]((1-\rho)s', s') < 0$, then for all $s'' > s'$, $[(1-\rho)w_a + w_s]((1-\rho)s'', s'') < 0$.

6.2 Properties of steady states

Here we recall a result implied by Lemma 8 in the Online Appendix. Let $S := [0, \bar{s}]$.

Lemma 1. *Consider an MPE Z with value functions W and V.*

1. *For every $s \in S$ such that $Z(s) > \rho s$ and $Z(s+) < \rho s + y$, there is an $\eta > 0$, such that W is Lipschitz continuous on $(s-\eta, s+\eta)$. Moreover, we have*

$$\begin{aligned}W'(s+) &= [-\rho w_a + w_s](Z(s+) - \rho s, s), &(11)\\ W'(s-) &= [-\rho w_a + w_s](Z(s) - \rho s, s). &(12)\end{aligned}$$

2. *For every $s \in S$ such that $Z(s) = \rho s$, we have*

$$\limsup_{s_1 \uparrow s} \frac{W(s) - W(s_1)}{s - s_1} \leq [-\rho w_a + w_s](Z(s) - \rho s, s), \tag{13}$$

where the left-hand side could be $-\infty$.

3. *For every $s \in S$ such that $Z(s+) = \rho s + y$, we have*

$$\liminf_{s_1 \downarrow s} \frac{W(s_1) - W(s)}{s_1 - s} \geq [-\rho w_a + w_s](Z(s+) - \rho s, s) \tag{14}$$

Next, we prove a useful inequality.

Lemma 2. *Consider an MPE Z with value functions W and V. If $s_o \in (0, \bar{s})$ is a steady state (i.e., $Z(s_o) = s_o$) and if $|\eta|$ is small enough, then*

$$W(s_o + \eta) - W(s_o) \quad \geq \quad w\left(s_o - \rho\left(s_o + \eta\right), s_o + \eta\right) - w\left(s_o - \rho\left(s_o\right), s_o\right). \tag{15}$$

Proof. *Since $s_o \in (0, \bar{s})$ and $\bar{s} \equiv y/(1-\rho)$, we have $s_o \in (\rho s_o, \rho s_o + y)$. Thus, if $|\eta|$ is small enough, then $s_o \in (\rho(s_o + \eta), \rho(s_o + \eta) + y)$; that is, s_o is feasible at state $s_o + \eta$. Since at $s_o + \eta$, $Z(s_o + \eta)$ is optimal and s_o is feasible, we have*

$$\begin{aligned} W(s_o + \eta) \quad &= \quad w\left(Z\left(s_o + \eta\right) - \rho\left(s_o + \eta\right), s_o + \eta\right) + \beta\delta V(Z\left(s_o + \eta\right)) \\ &\geq \quad w\left(s_o - \rho\left(s_o + \eta\right), s_o + \eta\right) + \beta\delta V(s_o) \\ &\geq \quad w\left(s_o - \rho\left(s_o + \eta\right), s_o + \eta\right) - w\left(s_o - \rho\left(s_o\right), s_o\right) + W(s_o), \end{aligned}$$

where the last inequality follows from the assumption that $Z(s_o) = s_o$. Rearranging terms, we obtain (15). ■

Lemma 3. *Let Z be an MPE and $s_o \in (0, \bar{s})$ be such that $Z(s_o) = s_o$. If there is a $\bar{\eta} > 0$ such that $Z(s_o + \eta) \leq s_o + \eta$ for all $\eta \in (0, \bar{\eta})$, then the following holds:*

$$R(s) := \left[(1 - \delta + \beta\delta) w_a + \beta\delta\left(-\rho w_a + w_s\right)\right]\left(\left(1 - \rho\right)s, s\right) \leq 0 \tag{16}$$

By the definition of function γ, (8), we have $(1-\delta+\beta\delta)\gamma(s, \hat{\delta}) = R(s)$. We define function R for convenience.

Proof. *If $w_a((1-\rho)s_o, s_o) \leq 0$, then since $w_s < 0$, (16) holds. We now assume $w_a((1-\rho)s_o, s_o) > 0$. Since w is twice continuously differentiable, there is an $\eta' > 0$ such that $w_a(z - \rho(s_o + \eta), s_o + \eta) > 0$ for all $\eta \in (0, \eta')$ and $z \in (s_o, s_o + \eta)$. Since $Z(s_o + \eta) \geq Z(s_o)$ (as Z is weakly increasing) and $Z(s_o) = s_o$, we have $Z(s_o + \eta) \geq s_o$. Assume from now that $\eta \in (0, \min\{\eta', \bar{\eta}\})$. By definitions of η' and $\bar{\eta}$ (defined in this lemma), since $\eta \in (0, \min\{\eta', \bar{\eta}\})$, we have $Z(s_o + \eta) \in [s_o, s_o + \eta]$ and*

$$w\left(Z\left(s_o + \eta\right) - \rho\left(s_o + \eta\right), s_o + \eta\right) < w\left(\left(1 - \rho\right)\left(s_o + \eta\right), s_o + \eta\right). \tag{17}$$

Note that since $s_o < \bar{s}$, if η is small enough, then $s_o + \eta \in (\rho s_o, \rho s_o + y)$; that is, $s_o + \eta$ is feasible at s_o. Since $Z(s_o) = s_o$, by the optimality of s_o at s_o, we have

$$w\left(\left(1 - \rho\right)s_o, s_o\right) + \beta\delta V\left(s_o\right) \geq w\left(\left(1 - \rho\right)s_o + \eta, s_o\right) + \beta\delta V\left(s_o + \eta\right).$$

Rearranging terms,

$$0 \geq \left(w\left(\left(1-\rho\right)s_o+\eta, s_o\right)-w\left(\left(1-\rho\right)s_o, s_o\right)\right)+\beta\delta\left(V\left(s_o+\eta\right)-V\left(s_o\right)\right).$$

Using (6) *to get rid of* V*, we have*

$$\begin{aligned} 0 &\geq \left(w\left(\left(1-\rho\right)s_o+\eta, s_o\right)-w\left(\left(1-\rho\right)s_o, s_o\right)\right)+\delta\left(W\left(s_o+\eta\right)-W\left(s_o\right)\right) \\ &\quad -\left(1-\beta\right)\delta\left(w\left(Z\left(s_o+\eta\right)-\rho\left(s_o+\eta\right), s_o+\eta\right)-w\left(\left(1-\rho\right)s_o, s_o\right)\right). \end{aligned} \tag{18}$$

Using (15) *to get rid of* W *and applying* (17)*, we have*

$$\begin{aligned} 0 &\geq \left(w\left(\left(1-\rho\right)s_o+\eta, s_o\right)-w\left(\left(1-\rho\right)s_o, s_o\right)\right)+\delta\left(w\left(s_o-\rho\left(s_o+\eta\right), s_o+\eta\right)-w\left(s_o-\rho\left(s_o\right), s_o\right)\right) \\ &\quad -\left(1-\beta\right)\delta\left(w\left(\left(1-\rho\right)\left(s_o+\eta\right), s_o+\eta\right)-w\left(\left(1-\rho\right)s_o, s_o\right)\right). \end{aligned}$$

Dividing both sides by η *and taking limit as* $\eta \downarrow 0$*, we obtain* (16). ■

6.3 A useful inequality

Recall that $U(s) \equiv (1+\beta\delta/(1-\delta))w\left(\left(1-\rho\right)s, s\right)$. The following lemma is analogous to Lemma 3 of Cao and Werning (2018). Here, because w_a can be negative, we cannot directly apply their proof.

Lemma 4. *Let* W *and* V *be the value functions of an MPE* Z*. If* $Z(s) < s$*, then* $W(s) \geq U(s)$ *and* $V(s) \geq \frac{1}{1-\delta}w\left(\left(1-\rho\right)s, s\right)$.

Proof. *Let* s *be such that* $Z(s) < s$*. We have two cases.*

Case 1. $w\left(Z\left(s\right)-\rho s, s\right) \leq w\left(\left(1-\rho\right)s, s\right).$

By the optimality of $Z(s)$ *and the feasibility of* s *at state* s*, we have*

$$\begin{aligned} W\left(s\right) &\geq w\left(\left(1-\rho\right)s, s\right)+\beta\delta V\left(s\right) \\ &= w\left(\left(1-\rho\right)s, s\right)+\delta W\left(s\right)-\left(1-\beta\right)\delta w\left(Z\left(s\right)-\rho s, s\right) \\ &\geq \left[1-\left(1-\beta\right)\delta\right]w\left(\left(1-\rho\right)s, s\right)+\delta W\left(s\right) \\ &= \left(1-\delta\right)U\left(s\right)+\delta W\left(s\right), \end{aligned}$$

where the second line follows from (6) and the last from the definition of U*. Rearranging terms, we have* $W(s) \geq U(s)$*. Since Case 1 is true, using (6) again, we obtain* $V(s) \geq \frac{1}{1-\delta}w\left(\left(1-\rho\right)s, s\right)$*. The proof for Case 1 is completed.*

Before turning to Case 2, we use the inequality obtained for Case 1 to show that the following is impossible: $Z(0) = 0$ *and there is a* $\bar{\eta} > 0$ *such that* $Z(\eta) \leq \eta$ *for all* $\eta \in (0, \bar{\eta})$.

Suppose by negation that such an $\bar{\eta}$ exists. Then, for all $\eta \in (0, \bar{\eta})$, Case 1 holds. We thus have $W(\eta) \geq U(\eta)$. Moreover, since $Z(0) = 0$, we have $W(0) = U(0)$, and hence $W(\eta) - W(0) \geq U(\eta) - U(0)$. At state 0, η is feasible if it is small enough. We now show that the net gain from deviating to η at state 0, which is the right-hand side of (18) with $s_o = 0$, is strictly positive. Applying $W(\eta) - W(0) \geq U(\eta) - U(0)$, $w_a(0,0) > 0$, and $Z(\eta) \leq \eta$ to the right-hand side of (18), the net gain from the deviation is at least

$$
\begin{aligned}
& (w(\eta, 0) - w(0,0)) + \delta (U(\eta) - U(0)) - (1-\beta)\delta (w((1-\rho)\eta, \eta) - w(0,0)) \\
= \quad & (w(\eta, 0) - w(0,0)) + \frac{\beta\delta}{1-\delta} (w((1-\rho)\eta, \eta) - w(0,0)).
\end{aligned}
$$

Since $w_a(0,0) > 0$ and $[(1-\rho)w_a + w_s](0,0) > 0$ (because of the assumption that moderation is ideal in Section 2.1), if η is small enough, the second line is strictly positive, and the deviation is thus profitable. This contradicts Z being an equilibrium.

Case 2. $w(Z(s) - \rho s, s) > w((1-\rho)s, s)$. *Since $Z(s) < s$, this case is true only if $w_a((1-\rho)s, s) < 0$. This further implies that $s > 0$.*

Define the sequence of consumption stocks $(Z^k(s))_{k \in \mathbb{N}}$ recursively by defining $Z^0(s) \equiv s$, and for each $k \in \mathbb{N}$, $Z^{k+1}(s) \equiv Z(Z^k(s))$. Since $Z(s) < s$ and Z is weakly increasing, $(Z^k(s))_{k \in \mathbb{N}}$ is a weakly decreasing sequence that converges to a steady state, denoted as z_∞, with $z_\infty \leq Z(s') < s'$ for all s' in some right neighborhood of z_∞. The paragraph before Case 2 shows that $z_\infty > 0$. Applying Lemma 3 by replacing s_o with z_∞, we have, (16) holds at z_∞. This further implies $[(1-\rho)w_a + w_s]((1-\rho)z_\infty, z_\infty) < 0$. And since w is strongly concave, by Section 6.1, we have, for all $s' > z_\infty$, $[(1-\rho)w_a + w_s]((1-\rho)s', s') < 0$ holds. Since $z_\infty \leq Z^k(s) < s$ for all $k \geq 1$, we have

$$
w\left((1-\rho) Z^k(s), Z^k(s)\right) > w((1-\rho)s, s) \tag{19}
$$

Let T be the smallest k that satisfies

$$
w\left(Z^{k+1}(s) - \rho Z^k(s), Z^k(s)\right) \leq w\left((1-\rho) Z^k(s), Z^k(s)\right).
$$

If there is no $k \in \mathbb{N}$ that satisfies this inequality, set $T = \infty$. If $T < \infty$, then Case 1 is satisfied at state $Z^T(s)$, that is, if we replace s and $Z(s)$ in Case 1 with $Z^T(s)$ and $Z^{T+1}(s)$, respectively. Since Lemma 4 holds when Case 1 is true, we have

$$
\delta^T V(Z^T(s)) \geq \frac{\delta^T}{1-\delta} w\left((1-\rho) Z^T(s), Z^T(s)\right). \tag{20}
$$

If $T = \infty$, then $\delta^T = 0$. Hence (20) also holds.

Since $W(s)$ is the current value from the sequence of consumption stocks $(Z^k(s))_{k\in\mathbb{N}}$, and $w\left(Z^{k+1}(s)-\rho Z^k(s),Z^k(s)\right)>w\left((1-\rho)\,Z^k(s),Z^k(s)\right)$ for all $k\in\{1,2,...,T-1\}$, we have

$$\begin{aligned}W(s) &= w(Z(s)-\rho s,s)+\beta[\sum_{k=1}^{T-1}\delta^k w\left(Z^{k+1}(s)-\rho Z^k(s),Z^k(s)\right)+\delta^T V\left(Z^T(s)\right)] \\ &\geq w((1-\rho)\,s,s)+\beta[\sum_{k=1}^{T-1}\delta^k w\left((1-\rho)\,Z^k(s),Z^k(s)\right)+\delta^T V\left(Z^T(s)\right)].\end{aligned}$$

Using (20), we have

$$\begin{aligned}W(s) &\geq w((1-\rho)\,s,s)+\beta[\sum_{k=1}^{T-1}\delta^k w\left((1-\rho)\,Z^k(s),Z^k(s)\right)+\frac{\delta^T}{1-\delta}w\left((1-\rho)\,Z^T(s)\,,Z^T(s)\right)] \\ &\geq w((1-\rho)\,s,s)+\beta[\sum_{k=1}^{T-1}\delta^k w((1-\rho)\,s,s)+\frac{\delta^T}{1-\delta}w((1-\rho)\,s,s)]=U(s).\end{aligned}$$

The second inequality follows from (19) and the last equality uses the definition of U. Thus, $W(s)\geq U(s)$. Replacing W with V and β with 1 in the above inequalities, we obtain $V(s)\geq\frac{1}{1-\delta}w((1-\rho)\,s,s)$, as desired.

■

Lemma 5. *If Z is an MPE and $s_o\in[0,\bar{s})$ is a steady state, then*

1. *either there is an $\eta>0$ such that for all $s'\in(s_o,s_o+\eta)$, $Z(s')>s'$,*
2. *or there is an $\eta>0$ such that for all $s'\in(s_o,s_o+\eta)$, $Z(s')<s'$.*

Note that because Z is weakly increasing, Z can only have upward jumps. This implies that for any s_1 and s_2 in S with $s_1<s_2$, $Z(s_1)>s_1$ and $Z(s_2)<s_2$, there is a $\hat{s}\in(s_1,s_2)$ such that $Z(\hat{s})=\hat{s}$.

Proof. *The negation of Part 1 is: for all $\eta>0$, there is an $s'\in(s_o,s_o+\eta)$, such that $Z(s')\leq s'$; or equivalently, there is a sequence (s_k) with $s_k\downarrow s_o$ such that $Z(s_k)\leq s_k$ for all $k\in\mathbb{N}$. From the remark before this proof, this statement is true if and only if Part 2 is true or the following statement, called* ***Statement (a)****, is true: there is a strictly decreasing sequence (s_k) with $s_k\downarrow s_o$ such that $Z(s_k)=s_k$ for all $k\in\mathbb{N}$. We now show that Statement (a) is false and thus Lemma 5 is true.*

Suppose by negation that Statement (a) is true. Take any $j,k\in\mathbb{N}\cup\{o\}$. At s_j, a current self's payoff (from choosing s_j) is $U(s_j):=(1+\frac{\beta\delta}{1-\delta})w((1-\rho)s_j,s_j)$, and if she deviates to s_k, her payoff would be $w(s_k-\rho s_j,s_j)+\frac{\beta\delta}{1-\delta}w((1-\rho)s_k,s_k)$. Thus, the gain from deviating *to s_k is*

$$
\begin{aligned}
& \int_{s_j}^{s_k} w_a(z-\rho s_j, s_j) + \frac{\beta\delta}{1-\delta}[(1-\rho)w_a + w_s]((1-\rho)z, z)dz \\
= & \int_{s_j}^{s_k} \left(\left[w_a + \frac{\beta\delta}{1-\delta}[(1-\rho)w_a + w_s] \right] ((1-\rho)z, z) - \int_{s_j}^{z} [-\rho w_{aa} + w_{as}](z-\rho s, s)ds \right) dz \\
= & \int_{s_j}^{s_k} \left(\frac{1-\delta+\beta\delta}{1-\delta}\gamma(z, \hat{\delta}) - \int_{s_j}^{z} [-\rho w_{aa} + w_{as}](z-\rho s, s)ds \right) dz. \qquad (21)
\end{aligned}
$$

Since $\delta > \delta_\beta$, by definition of δ_β, for fixed $\hat{\delta}$, $z = s_{\delta,\beta}$ uniquely solves $\gamma(z, \hat{\delta}) = 0$; moreover, $\gamma(s, \hat{\delta}) < 0$ for all $s \in (s_{\delta,\beta}, \bar{s})$.

We consider three exhaustive cases and show that each leads to a contradiction. Then we are done. Case (1): suppose $\gamma(s_o, \hat{\delta}) > 0$. We let $j = o$. Since w_{aa} and w_{as} are bounded, if k is sufficiently large, then for all $z \in (s_j, s_k)$, we have $\frac{1-\delta+\beta\delta}{1-\delta}\gamma(z, \hat{\delta}) > \int_{s_j}^{z}[-\rho w_{aa} + w_{as}](z - \rho s, s)ds$, and thus expression (21) is strictly positive. That is, deviating to s_k is profitable at state s_o, which contradicts Z being an equilibrium. Case (2): suppose $\gamma(s_o, \hat{\delta}) < 0$. Similarly to case (1), deviating to s_o at state s_j is profitable if j is large, and we thus reach a contradiction.

Case (3): suppose $\gamma(s_o, \hat{\delta}) = 0$, and thus $s_o = s_{\delta,\beta} > 0$. Note that since $s_k \downarrow s_o$ and $Z(s_k) \leq s_k$ for all k, there is some $\bar{k} < \infty$ such that if $k > \bar{k}$, then $\rho s_k < Z(s_o) < Z(s_k) < \rho s_o + y$. Thus, $Z(s) \in (\rho s, \rho s + y)$ for all $s \in [s_o, s_k]$; moreover, for any $s, s' \in [s_o, s_k]$, both s and $Z(s)$ are feasible at s'. By Lemma 1, the current value function of equilibrium Z, denoted as W, is continuous on $[s_o, s_k]$. We divide case (3) into two subcases: (3.1), for each $k \geq 1$, there is some $s \in (s_o, s_k)$ such that $Z(s) > s$; and (3.2) $Z(s) \leq s$ on (s_o, s_1).

In case (3.1), take a $k > \bar{k}$ and $s' \in (s_o, s_k)$ such that $Z(s') > s'$. Let $\bar{s}' := \sup\{\tilde{s} \in [s', s_k) : Z(s) > s, \forall s \in [s', \tilde{s})\}$ and $\underline{s}' := \inf\{\tilde{s} \in (s_o, s'] : Z(s) > s, \forall s \in (\tilde{s}, s']\}$. By definition, we have $s' \geq s_o$, $Z(\bar{s}') = \bar{s}'$ and $Z(\underline{s}'-) \leq \underline{s}'$. Thus, $W(\bar{s}') = U(\bar{s}')$. Moreover, by Lemma 4, we have $W(\underline{s}'-) \geq U(\underline{s}')$. Since $Z(s) \in (\rho s, \rho s + y)$ for all $s \in [\underline{s}', \bar{s}']$, by Lemma 1, we have

$$
\begin{aligned}
W(\bar{s}') - W(\underline{s}') &= \int_{\underline{s}'}^{\bar{s}'} [-\rho w_a + w_s] (Z(s) - \rho s, s)\, ds \geq \int_{\underline{s}'}^{\bar{s}'} [-\rho w_a + w_s] (s - \rho s, s)\, ds \\
&= \int_{\underline{s}'}^{\bar{s}'} \left(U'(s) - \frac{1-\delta+\beta\delta}{1-\delta}\gamma(s, \hat{\delta}) \right) ds > U(\bar{s}') - U(\underline{s}').
\end{aligned}
$$

where the first inequality is because $Z(s) > s$ on $(\underline{s}', \bar{s}')$ and $-\rho w_{aa} + w_{as} > 0$, the second equality uses the definitions of U and $\gamma(s, \hat{\delta})$, and the last inequality is because $\gamma(s, \hat{\delta}) < 0$ for $s > s_o$. But the inequality contradicts $W(\bar{s}') = U(\bar{s}')$ and $W(\underline{s}'-) \geq U(\underline{s}')$. Thus, case

(3.1) is impossible.

Finally, consider case (3.2). Take a $k > \bar{k}$, where $\bar{k}$ is defined when we begin analyzing case (3). Since $Z(s_k) = s_k$, we have $W(s_k) = U(s_k)$. Moreover, for any $\eta \in (0, s_k - s_o)$, $s_k - \eta$ is feasible at stock s_k and $Z(s_k - \eta) \leq s_k - \eta$. At s_k, the current self's payoff from deviating to $s_k - \eta$ is $w(s_k - \eta - \rho s_k, s_k) + \beta\delta V(s_k - \eta)$. Since $Z(s_k - \eta) \leq s_k - \eta$, by Lemma 4, we have $V(s_k - \eta) \geq \frac{1}{1-\delta} w((1-\rho)(s_k - \eta), s_k - \eta)$. Therefore, following from the same derivation of (21), the net benefit from the deviation is at least

$$\int_{s_k}^{s_k-\eta} \left(\frac{1-\delta+\beta\delta}{1-\delta} \gamma(z, \hat{\delta}) - \int_{s_k}^{z} [-\rho w_{aa} + w_{as}](z - \rho s, s) ds \right) dz.$$

Note that $\gamma(s_k, \hat{\delta}) < 0$, and both w_{aa} and w_{as} are bounded. Since z in the integrand is in $(s_k - \eta, s_k)$, as η goes to 0, the integrand converges to $\frac{1-\delta+\beta\delta}{1-\delta}\gamma(s_k, \hat{\delta}) < 0$. Since $s_k - \eta < s_k$, it follows that if η is small enough, the above net benefit is strictly positive. Again, this contradicts Z being an equilibrium.

■

Lemma 7 of the Online Appendix shows that the continuation value function V of any MPE must be weakly decreasing. Moreover, through the proof, we show that if V is weakly decreasing, then the marginal utility of consumption must be nonnegative in equilibrium: $w_a(Z(s) - \rho s, s) \geq 0$ for all $s \in S$. Combining the assumption that the ordinary good is necessary (defined in Section 2.1), we have the following corollary:

Corollary 1. *In any MPE Z, first, addictive good consumption is less than y: $Z(s) - \rho s < y$ for all $s \in S$; second, $Z(\bar{s}) < \bar{s}$.*

6.4 Proof of Theorem 1

Theorem 1 follows from the following lemma.

Lemma 6. *Assume $\delta > \delta_\beta$. Let Z be an MPE and s_o be an absorbing steady state of Z.*

1. *s_o is the unique absorbing steady state of Z. Moreover, $s_o \geq s_{\delta,\beta} > s_\delta$.*

2. *If Z features a bright-line rule at some s_u, then $s_u < s_{\delta,\beta}$.*

Proof. *Part 1. Fix an MPE Z. We first show that there is a unique $s \in S$ that satisfies the following property, called Property SS: there is an $\epsilon > 0$ such that $Z(s) = s$ and $Z(s') \leq s'$ for all $s' \in (s, s + \epsilon)$.*

We first show existence. Since $Z(\bar{s}) < \bar{s}$ by Corollary 1, we have two cases: (1) $Z(s) < s$ for all $s \in S$; or (2) there is some $s' \in [0, \bar{s})$ such that $Z(s') > s'$. In Case (1), $s = 0$ satisfies

Property SS. In Case (2), because Z is weakly increasing and $Z(\bar{s}) < \bar{s}$, there is some $\tilde{s} < \bar{s}$ that satisfies Property SS.

We now show uniqueness. Suppose by negation that multiple points satisfy the above property. We have two cases: (a) there are (at least) two steady states s_o^1 and s_o^2 with $s_o^1 < s_o^2$ and a point $s' \in [s_o^1, s_o^2]$ such that, for all $s \in (s_o^1, s')$, $Z(s) \leq s$, and for all $s \in (s', s_o^2)$, $Z(s) \geq s$; or (b) there is an s_o and a sequence $s_n \downarrow s_o$ such that $Z(s_o) = s_o$ and $Z(s_k) = s_k$ for all $n \in \mathbb{N}$. (The two cases are exhaustive but not disjoint.) By Lemma 5, Case (b) is impossible. To rule out Case (a), recall that $R(\cdot)$ is defined by (16) and we have

$$U'(s) \quad = \quad [-\rho w_a + w_s]\left((1-\rho)s, s\right) + \frac{R(s)}{1-\delta}. \tag{22}$$

By Lemma 3, we have $R(s_o^1) \leq 0$. And since $\delta > \delta_\beta$, we have, for all $s \in (s_o^1, s_o^2)$, $R(s) < 0$. Since for all $s \in (s', s_o^2)$, $Z(s) \geq s$ holds, we have $Z(s) > \rho s$, and hence Part 1 of Lemma 1 applies. Also, since $Z(s) \geq s$ on (s', s_o^2) and $\rho w_{aa} + w_{ss} < 0$, we have

$$[-\rho w_a + w_s]\left((1-\rho)s, s\right) \leq [-\rho w_a + w_s]\left(Z(s) - \rho s, s\right) = W'(s). \tag{23}$$

By (22) and $R(s) < 0$, we have $W'(s) > U'(s)$. And since $W(s_o^2) = U(s_o^2)$, and that W and U are absolutely continuous over (s', s_o^2), we have $W(s'+) < U(s')$.

Meanwhile, by Lemma 4 and $Z(s') \leq s'$, we have $W(s'-) \geq U(s')$. Hence $W(s'-) > W(s'+)$. But then, if $\eta > 0$ is small enough, at state $s' + \eta$, choosing state $s' - \eta$ (feasible if η is small) is a profitable deviation. This contradicts Z being an equilibrium. So, Case (a) is false, and thus Property SS holds.

Let s_o be the point that satisfies Property SS. Since $R(s_{\delta,\beta}) = 0$ and $R(s) < 0$ for all $s > s_{\delta,\beta}$, the inequality $R(s_o) \leq 0$ implies $s_o \geq s_{\delta,\beta}$. Also, since $R(s_\delta) > 0$, we have $s_{\delta,\beta} > s_\delta$.

Part 2. By the definition of s_o, for all $s \in [s_{\delta,\beta}, s_o]$, $Z(s) \geq s$. We now show that if $s_{\delta,\beta} < s_o$, then for all $s \in [s_{\delta,\beta}, s_o)$, $W(s) < U(s)$, meaning that the current value at s in MPE Z, $W(s)$, is strictly lower than the current value of staying at state s forever. To obtain the inequality, note that for all $s \in [0, s_o)$, since $Z(s) \geq s$ and $w_a(Z(s) - \rho s, s) \geq 0$, (23) holds. By (23), (22), and the fact that $R(s) < 0$ for all $s \in (s_{\delta,\beta}, s_o]$, we have, $U'(s) < W'(s)$ on this interval. Since $U(s_o) = W(s_o)$ and both U and W are continuous on $[s_{\delta,\beta}, s_o]$, this inequality implies that $U(s) > W(s)$ for all $s \in [s_{\delta,\beta}, s_o)$.

Since $W(s_u) = U(s_u)$ at the steady state s_u, we have $s_u \notin [s_{\delta,\beta}, s_o)$. We cannot have $s_u > s_o$ either, since, otherwise, from the fact that $Z(\bar{s}) < \bar{s}$, there is another absorbing steady state above s_u, which contradicts Part 1. Therefore, if $s_o > s_{\delta,\beta}$, then any bright line s_u satisfies $s_u < s_{\delta,\beta}$. Of course, if $s_o = s_{\delta,\beta}$, since s_o is the unique steady state in $[s_{\delta,\beta}, \bar{s}]$

but it is not a bright line, and a bright line s_u is a steady state, it follows that $s_u < s_{\delta,\beta}$. ■

6.5 Bright-line rules

Theorem 2 follows from Proposition 3 and Proposition 4.

Proposition 3. *Assume that $\delta > \delta_\beta$. Let Z be an MPE that does not feature a bright-line rule. Denote its unique absorbing steady state by s_o.*
If $s_o > s_{\delta,\beta}$, then there exists an MPE $\hat{Z}$ that has bright line at s_u with $s_u < s_{\delta,\beta}$, such that $\hat{Z}$ Pareto dominates Z over $[s_u - \Delta, \bar{s}]$, for some $\Delta > 0$.

Proof. *In MPE Z, since there is no other steady state in $[0, s_o)$, we must have $Z(s) > s > \rho s$ for all $s \in [0, s_o)$. Combining Corollary 1, we have, for all $s \in S$, $Z(s) \in (\rho s, \rho s + y)$. Then, by Lemma 1, W is absolutely continuous on $[0, s_o]$. The proof of Theorem 1 establishes that for all $s \in [s_{\delta,\beta}, s_o)$, $U(s) > W(s)$. We thus have two cases: (a) there is an s_u with $0 < s_u < s_{\delta,\beta}$, such that $U(s_u) = W(s_u)$, or (b) $U(s) > W(s)$ for all $s \in (0, s_o)$. Step 1 shows that if (a) holds, then s_u can be sustained as a steady state in some MPE. Step 2 shows that if (b) holds, abstinence (zero stock) can be sustained as a steady state in some MPE. Step 3 shows that these MPE Pareto dominate equilibrium Z.*

Step 1. Consider the same intrapersonal game except that the state space is restricted to $[0, s_u]$ and the choice set is $[\rho s, \rho s + y] \cap [0, s_u]$ for each s. The online appendix shows that this restricted game has an MPE, denoted as $\tilde{Z} : [0, s_u] \to [0, s_u]$.

Since inequality (16) does not hold at any state in $[0, s_u)$, equilibrium $\tilde{Z}$ has no steady state in $[0, s_u)$ that is absorbing on the right; that is, $\tilde{Z}(s) \geq s$ over $[0, s_u]$. Since $\tilde{Z}(s) \leq s_u$ over $[0, s_u]$ by construction, it follows that $\tilde{Z}(s_u) = s_u$. Therefore, s_u is a steady state of $\tilde{Z}$.

We show that the Markov strategy $\hat{Z} : S \to S$ defined by

$$\hat{Z}(s) = \begin{cases} \tilde{Z}(s), & \text{if } s \in [0, s_u] \\ Z(s), & \text{if } s \in (s_u, \bar{s}] \end{cases}$$

is an equilibrium in the original game. Since $\tilde{Z}(s_u) = s_u$, s_u is a steady state of $\hat{Z}$. To this end, we show that each current self has no profitable deviation in $\hat{Z}$. First note that any stock $s \in [0, s_u]$, since $\tilde{Z}$ is an equilibrium of the restricted game, deviations to any $s' \leq s_u$ are unprofitable or infeasible. Similarly, at any $s \in (s_u, \bar{s}]$, since Z is an equilibrium of the original game, deviations to any $s' \geq s_u$ are unprofitable or infeasible.

At state s_u, any upward deviation (i.e. deviation above s_u) is weakly dominated by $Z(s_u)$. Moreover, by the definition of s_u, the agent is indifferent between s_u and $Z(s_u)$. As a result, *there is no profitable deviation at* s_u.

We now show that if there is no profitable upward deviation (above s_u*) at* s_u*, then there is no profitable upward deviation (above* s_u*) at any* $s \in [0, s_u]$. *This implies that there is no profitable deviation at any* $s \in [0, s_u]$. *Take any state* $s \in [0, s_u]$. *For any* $z \in (s_u, \bar{s}]$*, if* z *is feasible at* s*, then, by the optimality of* $\tilde{Z}(s)$ *against* s_u *at* s*, we have*

$$\begin{aligned} & w\left(\tilde{Z}(s) - \rho s, s\right) + \beta\delta V\left(\tilde{Z}(s)\right) - \left[w(z - \rho s, s) + \beta\delta V(z)\right] \\ \geq\ & w(s_u - \rho s, s) + \beta\delta V(s_u) - \left[w(z - \rho s, s) + \beta\delta V(z)\right] \\ \geq\ & w(s_u - \rho s_u, s_u) + \beta\delta V(s_u) - \left[w(z - \rho s_u, s_u) + \beta\delta V(z)\right] \\ \geq\ & 0. \end{aligned}$$

The second inequality follows from $z > s_u$ *and* $s_u > s$*, and the increasing-difference property of* w *in* z *and* s*, that is,* $-\rho w_{aa} + w_{as} > 0$. *(Note also that* $w(z - \rho s_u, s_u)$ *is well-defined because* $0 < z - \rho s_u < z - \rho s \leq y$.*) Therefore, there is no profitable upward deviation (above* s_u*) at* $s \in [0, s_u]$.

Similarly, there is no profitable downward deviation (below s_u*) at any* $s \in (s_u, \bar{s}]$.

We conclude that $\hat{Z}$ *is an equilibrium with* s_u *being a steady state. And since* $\hat{Z}(s) \geq s$ *on* $[0, s_u]$ *and* $\hat{Z}(s) > s$ *on* (s_u, s_o)*,* s_u *is a bright line of* $\hat{Z}$.

Step 2. We show that if (b) holds, then the Markov strategy $\hat{Z} : S \to S$ *defined by*

$$\hat{Z}(s) = \begin{cases} 0, & \text{if } s = 0 \\ Z(s), & \text{if } s \in (0, \bar{s}] \end{cases}$$

is an equilibrium of the original game. Note that 0 *is a bright line of* $\hat{Z}$.

Assume that all the future selves play the strategy $\hat{Z}$. *For a current self, at state* $s = 0$*, any deviation in* $(0, \bar{s}]$ *is dominated by* $Z(0)$. *Since consuming* 0 *gives a current value* $U(0)$*, and* $U(0) > W(0)$*, choosing* $\hat{Z}(0) = 0$ *is optimal. At any state* $s > 0$*, any deviation in* $[\rho s, \rho s + y]$ *is unprofitable because* Z *is an equilibrium of the original game. (Note that although a current self at states sufficiently close to* 0 *wants to choose state* 0*, she can not do so because* 0 *is infeasible.) Therefore,* $\hat{Z}$ *is an equilibrium.*

Step 3. If (b) holds, then $\hat{Z}$ *Pareto dominates* Z *over* S. *If (a) holds, let* $\hat{s} := \inf\{\tilde{s} \in [0, s_u] : \hat{Z}(s) \leq Z(s), \forall s \in [\tilde{s}, s_u]\}$. *We have* $\hat{s} < s_u$ *since* $Z(s_u-) > s_u$*,* $\hat{Z}(s_u) = s_u$*, and both* $\hat{Z}$ *and* Z *are weakly increasing. Moreover, using Lemma 1, we have* $\hat{W}(s) > W(s)$ *on* $[\hat{s}, s_u]$. *Therefore,* $\hat{Z}$ *Pareto dominates* Z *over* $[\hat{s}, s_u]$. ■

6.6 The least excessive-consumption trap

Proposition 4. *Assume $\delta \in (\delta_\beta, 1)$.*

1. *There is an MPE such that $s_{\delta,\beta}$ is the unique absorbing steady state.*

2. *In any MPE Z such that $s_{\delta,\beta}$ is a steady state, there is an $\epsilon > 0$ such that $Z(s) = s_{\delta,\beta}$ for all $s \in [s_{\delta,\beta} - \epsilon, s_{\delta,\beta}]$.*

3. *If complementarity is intermediate at $s_{\delta,\beta}$ (i.e. $w_{as}((1-\rho)s_{\delta,\beta}, s_{\delta,\beta})$ is in $(\underline{K}, \bar{K})$ for some $\underline{K}$, $\bar{K}$ with $0 \le \underline{K} < \bar{K}$), then any MPE without a bright-line rule is Pareto dominated by an MPE with a bright-line rule at states around and above the bright line.*

Proof. *Part 1. First, consider a restricted game in which the state space is $[s_{\delta,\beta}, \bar{s}]$, and for each stock $s \in [s_{\delta,\beta}, \bar{s}]$, the consumer's choice set is $[s_{\delta,\beta}, s] \cap [\rho s, \rho s + y]$; thus, at each s, the consumer can only choose stocks below today's stock. Using an argument analogous to the proof of Proposition 1, it follows that this restricted game has an MPE, denoted as $\tilde{Z}_1 : [s_{\delta,\beta}, \bar{s}] \to [s_{\delta,\beta}, \bar{s}]$. Moreover, we have $\tilde{Z}_1(s) < s$ for all $s \in (s_{\delta,\beta}, \bar{s}]$. To see this, if there is an $\hat{s} \in (s_{\delta,\beta}, \bar{s}]$ such that $\tilde{Z}_1(\hat{s}) = \hat{s}$, then $\underline{D}_{-}\tilde{Z}_1'(\hat{s}) := \liminf_{h\downarrow 0} \frac{\tilde{Z}_1(\hat{s})-\tilde{Z}_1(\hat{s}-h)}{h} \ge 1$. Thus, by (9), at $s = \hat{s}$ the marginal benefit of deviating to a lower stock is at least*

$$-\left(\left[w_a - \beta\delta\left(\rho w_a - w_s\right)\right]\left(\left(1-\rho\right)s, s\right) - \left(1-\beta\right)\delta w_a\left(\left(1-\rho\right)s, s\right)\right), \tag{24}$$

which is strictly positive by the definition of $s_{\delta,\beta}$. But this contradicts Z being an equilibrium.

Second, consider a restricted game in which the state space is $[0, s_{\delta,\beta}]$, and for each stock $s \in [0, s_{\delta,\beta}]$, the consumer's choice set is $[0, s_{\delta,\beta}] \cap [\rho s, \rho s + y]$. The same reasoning from the proof of Proposition 3 implies that this restricted game has an MPE, denoted $\tilde{Z}_2 : [0, s_{\delta,\beta}] \to [0, s_{\delta,\beta}]$, and in the original game, the following patched strategy

$$\hat{Z}(s) = \begin{cases} \tilde{Z}_2(s), & \text{if } s \in [0, s_{\delta,\beta}] \\ \tilde{Z}_1(s), & \text{if } s \in (s_{\delta,\beta}, \bar{s}] \end{cases}$$

is an equilibrium, with the unique absorbing steady state $s_{\delta,\beta}$.

Part 2. Take an MPE Z in which $s_{\delta,\beta}$ is an absorbing steady state. Suppose by contradiction that Part 2 is not true. Since Z is weakly increasing and $Z(s_{\delta,\beta}) = s_{\delta,\beta}$, it follows that $Z(s) < s_{\delta,\beta}$ for all $s < s_{\delta,\beta}$. Since $s_{\delta,\beta}$ is an absorbing steady state, there is an $\eta > 0$ such that $Z(s) > s$ for all $s \in (s_{\delta,\beta} - \eta, s_{\delta,\beta})$. Take a stock $s_1 \in (s_{\delta,\beta} - \eta, s_{\delta,\beta})$. For each $k \in \mathbb{N}$, define $s_{k+1} := Z(s_k)$. Let $M_k := \frac{s_{k+3}-s_{k+2}}{s_{k+2}-s_{k+1}}$ for each k, which is well defined since $s < Z(s) < s_{\delta,\beta}$

for all $s \in (s_{\delta,\beta} - \eta, s_{\delta,\beta})$. Note that $Z(s_k) > s_k$ for all k and $Z(s_\infty) = s_\infty = s_{\delta,\beta}$. We will show that there is some $k > 0$ such that at stock s_k, deviating to stock s_{k+2} instead of following $s_{k+1} = Z(s_k)$ is profitable, which contradicts Z being an equilibrium.

We first derive an upper bound for $w(s_{k+3} - \rho s_{k+2}, s_{k+2}) - w(s_{k+2} - \rho s_{k+1}, s_{k+1})$. Subtracting from and adding to it the term $w(s_{k+2} - \rho s_{k+2}, s_{k+2})$, we have

$$\begin{aligned}
& w(s_{k+3} - \rho s_{k+2}, s_{k+2}) - w(s_{k+2} - \rho s_{k+1}, s_{k+1}) \\
= \; & \int_{s_{k+2}}^{s_{k+3}} w_a(z - \rho s_{k+2}, s_{k+2}) dz + \int_{s_{k+1}}^{s_{k+2}} [-\rho w_a + w_s](s_{k+2} - \rho s, s) ds \\
\leq \; & \int_{s_{k+2}}^{s_{k+3}} w_a(s_{k+2} - \rho s_{k+2}, s_{k+2}) dz + \int_{s_{k+1}}^{s_{k+2}} [-\rho w_a + w_s](Z(s) - \rho s, s) ds \\
= \; & \int_{s_{k+1}}^{s_{k+2}} (w_a(s_{k+2} - \rho s_{k+2}, s_{k+2}) M_k + [-\rho w_a + w_s](Z(s) - \rho s, s)) ds. \qquad (25)
\end{aligned}$$

The inequality follows from $w_{aa} < 0$, $z \geq s_{k+2}$, $-\rho w_{aa} + w_{as} > 0$ and $Z(s) \geq s_{k+2}$ for all $s \in [s_{k+1}, s_{k+2}]$ (because $Z(s_{k+1}) = s_{k+2}$ and $Z(s_{k+1}) > s_{k+1}$). By (6), at state s_k, the net benefit from deviating to s_{k+2} instead of following s_{k+1} is

$$\begin{aligned}
& w(s_{k+2} - \rho s_k, s_k) + \delta W(s_{k+2}) - (1-\beta)\delta w(s_{k+3} - \rho s_{k+2}, s_{k+2}) \\
& -(w(s_{k+1} - \rho s_k, s_k) + \delta W(s_{k+1}) - (1-\beta)\delta w(s_{k+2} - \rho s_{k+1}, s_{k+1})) \\
\geq \; & \int_{s_{k+1}}^{s_{k+2}} [w_a(s - \rho s_k, s_k) + \beta\delta[-\rho w_a + w_s](Z(s) - \rho s, s) - (1-\beta)\delta w_a((1-\rho)s_{k+2}, s_{k+2}) M_k] ds \\
\geq \; & \int_{s_{k+1}}^{s_{k+2}} [w_a(s - \rho s_k, s_k) + \beta\delta[-\rho w_a + w_s](s - \rho s, s) - (1-\beta)\delta w_a((1-\rho)s_{k+2}, s_{k+2}) M_k] ds, \; (26)
\end{aligned}$$

where the first inequality uses (25) and Lemma 1; and the second uses $Z(s) > s$ and $-\rho w_{aa} + w_{as} > 0$.

If a subsequence of (M_k) converges to a number $M < 1$, then, along this subsequence, the integrand of (26) converges to

$$[w_a + \beta\delta(-\rho w_a + w_s) - (1-\beta)\delta w_a M](s_{\delta,\beta} - \rho s_{\delta,\beta}, s_{\delta,\beta}) > R(s_{\delta,\beta}) = 0.$$

The inequality holds because $R(s)$ is defined as the left-hand side with M replaced by 1. Therefore, there is some k such that at state s_k, deviating to s_{k+2} is profitable, as desired.

Now suppose (M_k) converges to 1. Since $Z(s_k) > s_k$ for all k and $Z(s_\infty) = s_\infty = s_{\delta,\beta}$, there is a subsequence of (M_k), denoted (M_{k_n}), such that $M_{k_n} < 1$ for all n. Thus, if $k = k_n$, the integrand of (26) is greater than

$$w_a(s - \rho s_k, s_k) + \beta\delta[-\rho w_a + w_s](s - \rho s, s) - (1-\beta)\delta w_a((1-\rho)s_{k+2}, s_{k+2})$$

$$
\begin{aligned}
&= -\int_s^{s_{\delta,\beta}} R'(z)dz + \int_{s_k}^{s}[\rho w_{aa} - w_{as}](s-\rho z, z)dz - (1-\beta)\delta\int_s^{s_{k+2}}[(1-\rho)w_{aa} + w_{as}]((1-\rho)z, z)dz \\
&> -\int_s^{s_{\delta,\beta}} R'(z)dz + \int_{s_k}^{s}[\rho w_{aa} - w_{as}](s-\rho z, z)dz + \int_s^{s_{k+2}}[\rho w_{aa} - w_{as}]((1-\rho)z, z)dz \\
&\geq -(s_{\delta,\beta} - s_{k+2}) \max_{z\in[s_{k+2}, s_{\delta,\beta}]} R'(z) + (s_{k+2} - s_k) \min_{z\in[s_k, s_{k+2}], s\in[s_{k+1}, s_{k+2}]}[\rho w_{aa} - w_{as}](s-\rho z, z) \\
&\geq (s_{k+2} - s_k)\left(-\frac{s_{\delta,\beta} - s_{k+2}}{s_{k+2} - s_k} \max_{z\in[s_1, s_{\delta,\beta}]} R'(z) + \min_{z\in[s_1, s_{\delta,\beta}], s\in[s_1, s_{\delta,\beta}]}[\rho w_{aa} - w_{as}](s-\rho z, z)\right), \qquad (27)
\end{aligned}
$$

where the first equality follows from $R(s_{\delta,\beta}) = 0$*, the first inequality from* $w_{aa} < 0$*,* $w_{as} > 0$*, and* $(1-\beta)\delta < 1$*, the second inequality from* $s \in [s_{k+1}, s_{k+2}]$ *and* $R'(\cdot) < 0$*, and the last inequality from* $s_1 < s_{k+1} < s_{k+2} < s_{\delta,\beta}$*. Let* $N = \frac{\min_{z\in[s_1,s_{\delta,\beta}], s\in[s_1,s_{\delta,\beta}]}[\rho w_{aa} - w_{as}](s-\rho z, z)}{-\max_{z\in[s_1,s_{\delta,\beta}]} R'(z)}$*. We now show that there is some* $k_n > 1$ *such that* $\frac{s_{\delta,\beta} - s_{k_n+2}}{s_{k_n+2} - s_{k_n}} > N$*, implying that (27) is positive and thus at state* s_{k_n}*, deviating to* s_{k_n+2} *is profitable, as desired.*

To show $\frac{s_{\delta,\beta} - s_{k_n+2}}{s_{k_n+2} - s_{k_n}} > N$ *for some* k_n*, note that*

$$
\frac{s_{\delta,\beta} - s_{k+2}}{s_{k+2} - s_k} = \sum_{i=1}^{\infty} \frac{s_{k+2i+2} - s_{k+2i}}{s_{k+2} - s_k} = \sum_{i=1}^{\infty} \frac{s_{k+2i+2} - s_{k+2i}}{s_{k+2i} - s_{k+2i-2}} \cdots \frac{s_{k+4} - s_{k+2}}{s_{k+2} - s_k} = \sum_{i=1}^{\infty} \tilde{M}_{k+2i-2} \cdots \tilde{M}_k,
$$

where $\tilde{M}_k := \frac{s_{k+4} - s_{k+2}}{s_{k+2} - s_k}$*. Since* $M_k := \frac{s_{k+3} - s_{k+2}}{s_{k+2} - s_{k+1}}$ *and* $\lim_{k\to\infty} M_k \to 1$*, we have* $\lim_{k\to\infty} \tilde{M}_k \to 1$*. Thus, for each* $m > 2$*, there is a number* $k'_m < \infty$ *such that if* $k > k'_m$*, then* $\tilde{M}_{k+m-2} \cdots \tilde{M}_k \geq 1 - \frac{1}{m}$*,* $\tilde{M}_{k+m-4} \cdots \tilde{M}_k \geq 1 - \frac{1}{m}$*,..., and* $\tilde{M}_k \geq 1 - \frac{1}{m}$*. Thus, if* $k > k'_m$*, then*

$$
\sum_{i=1}^{\infty} \tilde{M}_{k+2i-2} \cdots \tilde{M}_k > (m-1)(1 - \frac{1}{m}).
$$

For the number N*, choose* m *such that* $(m-1)(1-\frac{1}{m}) > N$ *and* $k_n > k'_m$*. Then* $\frac{s_{\delta,\beta} - s_{k_n+2}}{s_{k_n+2} - s_{k_n}} > N$*, as desired.*

Part 3. Take an MPE Z *that has* $s_{\delta,\beta}$ *as an absorbing steady state, but without a bright-line rule. We have* $W(s_{\delta,\beta}) = U(s_{\delta,\beta})$*, and* $W'(s_{\delta,\beta}) = U'(s_{\delta,\beta})$*. We now show that if* $w_{as}((1-\rho)s_{\delta,\beta}, s_{\delta,\beta})$ *is higher than a threshold, then* $W''(s_{\delta,\beta}-) < U''(s_{\delta,\beta})$*. Then, the same reasoning as in the proof of Proposition 3 implies that* Z *is Pareto dominated by an MPE that features a bright-line rule at states around and above the bright line.*

In the previous step, we have shown that $Z(s) = s_{\delta,\beta}$ *for* $s \in [s_{\delta,\beta} - \epsilon, s_{\delta,\beta}]$*. Therefore,* $Z'(s) = 0$ *at such an* s*. Since* $W'(s) = [-\rho w_a + w_s](Z(s) - \rho s, s)$*, we have*

$$
W''(s_{\delta,\beta}-) \;=\; \frac{d}{ds}[-\rho w_a + w_s](s_{\delta,\beta} - \rho s, s)|_{s=s_{\delta,\beta}}.
$$

We thus have,

$$U''(s_{\delta,\beta}) - W''(s_{\delta,\beta}-) \quad = \quad \frac{R'(s)}{1-\delta} + (-\rho w_{aa} + w_{as})((1-\rho)s, s)|_{s=s_{\delta,\beta}}. \tag{28}$$

Since $w_{aa} < 0$ *and* $w_{as} > 0$*, there are* $\underline{K}$ *and* $\bar{K}$ *with* $0 \leq \underline{K} < \bar{K}$ *such that if* $w_{as}((1-\rho)s, s)|_{s=s_{\delta,\beta}} \in (\underline{K}, \bar{K})$*, then* $R'(s_{\delta,\beta}) < 0$ *and* $U''(s_{\delta,\beta}) - W''(s_{\delta,\beta}-) > 0$*.*

■

6.7 Proof of Proposition 2

Proof. *Let* $s^* \equiv \arg\max_{s\in[0,\bar{s}]} w\left((1-\rho s), s\right)$*, which is the moderation stock level for an infinitely patient consumer* ($\delta = 1$)*. Since* s_δ *is the moderation stock level for a consumer with discount factor* $\delta < 1$ *and* w *is strongly concave, we have* $s^* < s_\delta < s_{\delta,\beta}$*. Moreover, both* s_δ *and* $s_{\delta,\beta}$ *converge to* s^* *as* $\delta \to 1$*.*

Let $s_{\beta\delta}$ *be the moderation stock level of a time-consistent consumer whose discount factor is* $\beta\delta$*. We first show that for each* $s_o \in [s_{\delta,\beta}, s_{\beta\delta})$*, we can construct a constrained Markov equilibrium* $\tilde{Z}(\cdot, s_o)$ *when the state space is restricted to* $[s_o, \bar{s}]$*, and that* s_o *is an (absorbing) steady state of* $\tilde{Z}(\cdot, s_o)$*.*

To see this, consider the following restricted game, in which the state space is restricted to $[s_o, \bar{s}]$*, and for each stock* $s \in [s_o, \bar{s}]$*, the consumer's choice set is restricted to* $[s_o, s] \cap [\rho s, \rho s + y]$*; thus, at each* s*, the consumer is allowed to choose stocks below today's stock. By a similar argument to Proposition 4, this restricted game has an MPE, denoted as* $\tilde{Z}(\cdot, s_o)$*. Moreover, we have* $\tilde{Z}(s, s_o) < s$ *for all* $s \in (s_o, \bar{s}]$*; that is,* s_o *is the unique steady state of* $\tilde{Z}(\cdot, s_o)$*.*

We now use $\tilde{Z}(\cdot, s_o)$ *to define a strategy over a larger interval. Take a small* $\epsilon > 0$*. For each* $s_o \in [s_{\delta,\beta}, s_{\delta,\beta} + \epsilon)$*, define a Markov strategy* $Z(\cdot, s_o) : [s^* - \epsilon, \bar{s}] \to S$ *by*

$$Z(s, s_o) = \begin{cases} s_o, & \text{if } s \in [s^* - \epsilon, s_o] \\ \tilde{Z}(s, s_o), & \text{if } s \in (s_o, \bar{s}] \end{cases}$$

Thus, $Z(\cdot, s_o)$ *extends strategy* $\tilde{Z}(\cdot, s_o)$ *to* $[s^* - \epsilon, \bar{s}]$*, and has a unique (absorbing) steady state at* s_o *We assume that* δ *is high and* ϵ *is small, so the Markov strategy* $Z(\cdot, s_o)$ *is well defined.*

We now use the strategy $Z(\cdot, s_o)$ *to construct a new strategy with a bright line. Let* $W(s, s_o)$ *be the current value associated with strategy* $Z(s, s_o)$ *for every* $s \in [s^* - \epsilon, \bar{s}]$*. By* *construction, for every* $s \in [s^* - \epsilon, s_o]$,

$$W(s, s_o) \;=\; w(s_o - \rho s, s) + \frac{\beta\delta}{1-\delta} w((1-\rho)s_o, s_o). \tag{29}$$

The last part of the proof of Proposition 4 implies that there is a $\delta_2 < 1$ *such that if* $\delta \in (\delta_2, 1)$*, then the right-hand side of (28) is negative. Moreover, since* $U''(s_{\delta,\beta}) - W''(s_{\delta,\beta})$ *goes to* $-\infty$ *as* $\delta \to 1$*, we can choose* δ_2 *to be large, so that there is a* $\tilde{\epsilon} \in (0, \epsilon)$ *such that if* $\delta \in (\delta_2, 1)$*, then* $W(s, s_{\delta,\beta}) > U(s)$ *for all* $s \in (s^* - \tilde{\epsilon}, s_{\delta,\beta})$*. Define* $\bar{\delta} := \delta_2$*.*

Note that for every $s \in [s^* - \tilde{\epsilon}, s_{\delta,\beta}]$ *and* $s_o \geq s_{\delta,\beta}$*,* $W(s, s_o)$ *is strictly decreasing in the second argument. Since* $W(s, s_{\delta,\beta}) > U(s)$ *for all* $s \in (s^* - \tilde{\epsilon}, s_{\delta,\beta})$ *and* $W(s_{\delta,\beta}, s_{\delta,\beta}) = U(s_{\delta,\beta})$ *(by definition), there is a* $\Delta > 0$ *such that if* $\delta \in (\bar{\delta}, 1)$*, then for every* $s_u \in [s^* - \Delta, s_{\delta,\beta}]$*, there exists an* $s_o^u > s_{\delta,\beta}$ *such that* $W(s_u, s_o^u) = U(s_u)$ *and* $W(s, s_o^u) < U(s)$ *for all* $s \in (s_u, s_o^u)$*. We now construct an equilibrium that has two steady states, one at* s_u *and the other at* s_o^u*.*

For each $s_u \in [s^* - \Delta, s_{\delta,\beta})$*, let* $\tilde{Z}(\cdot, s_u)$ *be a constrained MPE when the state space is restricted to* $[0, s_u]$*. The proof of Theorem 2 shows its existence and that* s_u *is a steady state of* $\tilde{Z}(\cdot, s_u)$*. Let* $\hat{Z}(\cdot, s_u)$

$$\hat{Z}(s, s_u) = \begin{cases} \tilde{Z}(s, s_u), & \text{if } s \in [s^*, s_u] \\ s_o^u, & \text{if } s \in (s_u, s_o^u] \\ \tilde{Z}(s, s_o^u), & \text{if } s \in (s_o^u, \bar{s}] \end{cases}$$

The same arguments as those in the proof of Theorem 2 imply that $\hat{Z}(s, s_u)$ *is an MPE of the original game. Moreover, by construction, it features a bright-line rule at* s_u*.*

■

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7 Online Appendix

This section proves the monotonicity of current and continuation value functions and the existence of MPE of the original game and the restricted game when the state space is restricted to a subset of S. We then provide formal statements and proofs for Section 4.

7.1 Monotonicity of value functions

Lemma 7. *Let Z be an MPE with value functions W and V. Both W and V are weakly decreasing.*

Proof. *We first show that V is weakly decreasing through 3 steps. Suppose by negation that V is not weakly decreasing.*[19] *Then, there is a point s_0 in $[0, \bar{s}]$ such that, (a) there is a sequence $s_n \uparrow s_0$ such that $V(s_n) < V(s_0)$ for all n, or (b) there is a sequence $s_n \downarrow s_0$, such that $V(s_0) < V(s_n)$ for all n.*

Step 1. We show the following claim:

Claim 1. *If (a) or (b) is true, then*

1. *$w_a (Z(s_0+) - \rho s_0, s_0) \leq 0$; and*
2. *there is a sequence (z_n) such that $z_n \uparrow Z(s_0+)$ and $V(z_n) < V(Z(s_0+))$ for all n, or a sequence (z_n) such that $z_n \downarrow Z(s_0+)$ and $V(z_n) > V(Z(s_0+))$ for all n.*

Suppose (a) is true. *Since $s_n \uparrow s_0$, either (a.1) there is a subsequence of (s_n), denoted (s_{n_k}), such that $W(s_{n_k}) < W(s_0)$ for all k, or (a.2) if n is large enough, then $W(s_n) \geq W(s_0)$. Note that (a.1) is the negation of (a.2).*

We first prove Part 1 of Claim 1 for Case (a.1). If $Z(s_0) - \rho s_0 < y$, then, if k is large enough, we have $Z(s_0) < \rho s_{n_k} + y$; that is $Z(s_0)$ is feasible at s_{n_k}. Since $W(s_{n_k}) < W(s_0)$ (by assumption) and $Z(s_{n_k})$ is optimal at s_{n_k}, we have

$$w(Z(s_0) - \rho s_0, s_0) + \beta\delta V(Z(s_0)) = W(s_0) > W(s_{n_k}) \geq w(Z(s_0) - \rho s_{n_k}, s_{n_k}) + \beta\delta V(Z(s_0)).$$

Subtracting the fourth expression from the first, we have

$$w(Z(s_0) - \rho s_0, s_0) - w(Z(s_0) - \rho s_{n_k}, s_{n_k}) > 0. \tag{30}$$

Applying the fundamental theorem of calculus, this inequality implies

$$\int_{s_{n_k}}^{s_0} [-\rho w_a + w_s] (Z(s_0) - \rho s, s)\, ds > 0. \tag{31}$$

[19] By definition, V is weakly decreasing on $[0, \bar{s}]$ iff for all $s_1, s_o \in [0, \bar{s}]$ with $s_1 < s_o$, $V(s_1) \geq V(s_o)$ holds. We now show that V is weakly decreasing is equivalent to **Statement** α, defined as "for all s_o, for all sequence (s_n) with $s_n \uparrow s_o$, $V(s_n) \geq V(s_o)$ holds, and for all sequence (s_n) with $s_n \downarrow s_o$, $V(s_n) \leq V(s_o)$ holds. First, it is obvious that V is weakly decreasing implies Statement α. To show the converse, suppose Statement α holds and take two points $s'_1 < s_o$ in $[0, \bar{s}]$. Let (s_n) be such that $s_n = s_o + \frac{1}{n}(s'_1 - s_o)$ for all n. Thus $s_1 = s'_1$. Since $V(s_n) \geq V(s_o)$ for all n, we have $V(s'_1) = V(s_1) \geq V(s_o)$, as desired.

We now show $w_a(Z(s_0)-\rho s_0, s_0)<0$. Suppose by contradiction that $w_a(Z(s_0)-\rho s_0, s_0)\geq 0$. Since $w_s<0$, we have $[-\rho w_a+w_s](Z(s_0)-\rho s_0, s_0)<0$. And since w is twice continuously differentiable, there is an $s'<s_0$ such that for all $s\in[s', s_0]$,

$$[-\rho w_a+w_s](Z(s_0)-\rho s, s)<0.$$

But then, (31) does not hold if k is large enough. A contradiction.

Thus, if $Z(s_0)-\rho s_0<y$, then $w_a(Z(s_0)-\rho s_0, s_0)<0$. And since $Z(s_0+)\geq Z(s_0)$ and $w_{aa}<0$, we have $w_a(Z(s_0+)-\rho s_0, s_0)<0$. Part 1 of the claim holds. If $Z(s_0)-\rho s_0=y$, by the assumption that the ordinary good is necessary (defined in Section 2.1), we have $w_a(Z(s_0)-\rho s_0, s_0)<0$. Part 1 thus also holds.

We now prove Part 2 for case (a.1). Since $w_a(Z(s_0+)-\rho s_0, s_0)<0$ (shown above) and Z is weakly increasing, there is an $\epsilon>0$ such that for all $s\in(s_0, s_o+\epsilon)$, $w_a(Z(s)-\rho s, s)<0$. We now show that there is an $\eta>0$ such that for all z in $(Z(s_0+)-\eta, Z(s_0+))$, $V(z)<V(Z(s_0+))$; it then follows that Part 2 holds. Suppose this statement is not true; that is, there is a sequence (z_k) with $z_k\uparrow Z(s_0+)$ such that $V(z_k)\geq V(Z(s_0+))$ for all k. Note that since $w_a(Z(s_0+)-\rho s_0, s_0)<0$, we have $Z(s_0+)-\rho s_0>0$. Now, since $z_k\uparrow Z(s_0+)$ as $k\to\infty$ (by assumption) and $Z(s)\downarrow Z(s_0+)$ as $s\downarrow s_0$, we have, if k is sufficiently large and s is sufficiently close to s_0 with $s>s_0$, we have $z_k-\rho s>0$. Since we also have $z_k<Z(s)$, z_k is thus feasible at s. Since $w_a(Z(s)-\rho s, s)<0$ for $s\in(s_0, s_0+\epsilon)$ and $V(z_k)\geq V(Z(s_0+))$ for all k, if k is sufficiently large and s is sufficiently close to s_0 with $s\in(s_0, s_0+\epsilon)$, then z_k is feasible at s, and we have

$$w(Z(s)-\rho s, s)+\beta\delta V(Z(s))<w(z_k-\rho s, s)+\beta\delta V(z_k),$$

which contradicts the optimality of $Z(s)$ at s.

Thus, for Case (a.1), the claim holds. Note that in all other cases, that is, (a.2) and (b), if $Z(s_0+)-\rho s_0=y$, we have $w_a(Z(s_0+)-\rho s_0, s_0)<0$. The same reasoning as in Case (a.1) implies that the claim holds. Therefore, when proving this claim for (a.2) and (b), we assume $Z(s_0)-\rho s_0<y$.

We now turn to Case (a.2). By equations (3) and (4), we have

$$(1-\beta)\delta V(Z(s)) \;=\; V(s)-W(s). \tag{32}$$

Since Case (a.2) means that if n is large, then $W(s_n)\geq W(s_0)$ and $V(s_n)<V(s_0)$, applying these inequalities to (32), we have $V(Z(s_n))<V(Z(s_0))$ for large n's.

We first prove Part 1. Since $Z(s_0)-\rho s_0<y$ and $s_n\uparrow s_0$, if n is large, then $Z(s_0)$ is feasible at s_n. Since $Z(s_n)$ is optimal at s_n, we have

$$w(Z(s_n)-\rho s_n, s_n)+\beta\delta V(Z(s_n)) \;\geq\; w(Z(s_0)-\rho s_n, s_n)+\beta\delta V(Z(s_0)).$$

Since $V(Z(s_n))<V(Z(s_0))$ for large n's, we have

$$w(Z(s_n)-\rho s_n, s_n)+\beta\delta V(Z(s_n)) \;>\; w(Z(s_0)-\rho s_n, s_n)+\beta\delta V(Z(s_n)),$$

implying that $w(Z(s_n) - \rho s_n, s_n) > w(Z(s_0) - \rho s_n, s_n)$*. We thus have*

$$\int_{Z(s_n)}^{Z(s_0)} w_a(z - \rho s_n, s_n)\,ds < 0. \tag{33}$$

Letting n *go to* ∞*, we have* $w_a(Z(s_0) - \rho s_0, s_0) \leq 0$*. Part 1 of Claim 1 thus holds.*

We now prove Part 2. If Z *is discontinuous at* s_0*, that is,* $Z(s_0+) > Z(s_0)$*, then since* $w_a(Z(s_0) - \rho s_0, s_0) \leq 0$ *and* $w_{aa} < 0$*, we have* $w_a(Z(s_0+) - \rho s_0, s_0) < 0$*. The same proof as in Case (a.1) shows that Part 2 holds. Now suppose* Z *is continuous at* s_0*. Recall that* $V(Z(s_n)) < V(Z(s_0))$ *for large* n*'s. Moreover, since* Z *is continuous and weakly increasing, we have* $Z(s_n) \uparrow Z(s_0)$ *as* $s_n \uparrow s_0$*. Therefore, there is an* N *such that if we define* $z_n = Z(s_{n+N})$*, then* $V(z_n) < V(Z(s_0))$ *for all* $n \in \mathbb{N}$*. And since* $Z(s_0) = Z(s_0+)$*, Part 2 of Claim 1 holds.*

We now turn to Case (b). *Again, since* $s_n \downarrow s_0$*, we have two cases: (b.1) there is a subsequence of* (s_n)*, denoted* s_{n_k}*, such that* $W(s_0) < W(s_{n_k})$ *for all* k*, or (b.2)—the negation of (b.1)—if* n *is large enough, then* $W(s_0) \geq W(s_n)$*. We prove Claim 1 for Case (b). Recall that we are left with the case* $Z(s_0+) < \rho s_0 + y$*.*

If Case (b.1) holds, since $Z(s_0+) < \rho s_0 + y$ *and* $s_{n_k} \downarrow s_0$*, if* k *is large enough, then* $Z(s_{n_k}) < \rho s_0 + y$*; that is* $Z(s_{n_k})$ *is feasible at* s_0*. By the assumption that* $W(s_{n_k}) > W(s_0)$ *(as we are in Case (b.1)) and the optimality of* $Z(s_0)$ *at* s_0*, we have*

$$w(Z(s_{n_k}) - \rho s_{n_k}, s_{n_k}) + \beta\delta V(Z(s_{n_k})) = W(s_{n_k}) > W(s_0) \geq w(Z(s_{n_k}) - \rho s_0, s_0) + \beta\delta V(Z(s_{n_k})),$$

which implies

$$\int_{s_0}^{s_{n_k}} [-\rho w_a + w_s](Z(s_{n_k}) - \rho s, s)\,ds > 0. \tag{34}$$

By similar arguments as in Case (a.1), we obtain $w_a(Z(s_0+) - \rho s_0, s_0) < 0$ *and Claim 1.*

We are left with Case (b.2) when $Z(s_0+) < \rho s_0 + y$*. As in Case (b.1), if* n *is large, then* $Z(s_n)$ *is feasible at* s_0*. By the optimality of* $Z(s_0)$ *at* s_0*, we have*

$$w(Z(s_0) - \rho s_0, s_0) + \beta\delta V(Z(s_0)) \;\geq\; w(Z(s_n) - \rho s_0, s_0) + \beta\delta V(Z(s_n)).$$

Also, as in Case (a.2), $V(Z(s_n)) > V(Z(s_0))$ *for large* n*'s. We thus have*

$$w(Z(s_0) - \rho s_0, s_0) + \beta\delta V(Z(s_0)) \;>\; w(Z(s_n) - \rho s_0, s_0) + \beta\delta V(Z(s_0)).$$

And so

$$\int_{Z(s_0)}^{Z(s_n)} w_a(z - \rho s_0, s_0)\,ds < 0. \tag{35}$$

If Z *is discontinuous at* s_0*, that is,* $Z(s_0) < Z(s_0+)$*, then as* $n \to \infty$*, we have* $\int_{Z(s_0)}^{Z(s_0+)} w_a(z - \rho s_0, s_0)\,ds \leq 0$*. Since* $Z(s_0) < Z(s_0+)$ *and* $w_{aa} < 0$*, we have* $w_a(Z(s_0+) - \rho s_0, s_0) < 0$*. The same proof in Case (a.1) shows the Claim 1 holds.*

If Z *is continuous at* s_0*, then (35) implies that* $w_a(Z(s_0) - \rho s_0, s_0) \leq 0$*; the same proof* *of Case (a.1) implies that Part 1 holds. To prove Part 2, recall that* $V(Z(s_0)) < V(Z(s_n))$ *for large* n*'s. Moreover, since* Z *is weakly increasing and continuous at* s_0*, as* $z_n \downarrow s_0$*, we have* $Z(s_n) \downarrow Z(s_0+)$*. Thus, Part 2 of Claim 1 holds.*

Step 2. Let s_{st} *be the limit of the sequence* $(s_0, Z(s_0+), Z(Z(s_0+)+), ...)$*. The limit exists since the sequence is monotone and bounded. Note that in Claim 1 (and in statements (a) and (b)), if we replace* s_0 *with* $Z(s_0+)$ *and* $Z(s_0+)$ *with* $Z(Z(s_0+)+)$*, then Claim 1 still holds. Continuing in this way, by induction, we have* $w_a((1-\rho)s_{st}, s_{st}) \leq 0$*.*

We show that there is a profitable deviation at some state near s_{st}*. It then contradicts the hypothesis that* Z *is an equilibrium. Consequently, the continuation value function* V *must be weakly decreasing.*

We have two cases: (2.1) there exists some $\hat{\epsilon} > 0$ *such that* $Z(s_{st} - \epsilon) \geq s_{st} - \epsilon$ *for all* $\epsilon \in (0, \hat{\epsilon})$*, and (2.2) for all* $\hat{\epsilon} > 0$*, there exists* $\epsilon \in (0, \hat{\epsilon})$*, such that* $Z(s_{st} - \epsilon) < s_{st} - \epsilon$*.*

In case (2.1), we have $Z(s_{st}) = s_{st}$*. Since* $w_a(s_{st} - \rho s_{st}, s_{st}) \leq 0$ *and* $-\rho w_{aa} + w_{as} > 0$*, we have* $w_a(s_{st} - \rho(s_{st} - \epsilon), s_{st} - \epsilon) < 0$*. We have*

$$\begin{aligned} & w(s_{st} - \rho(s_{st} - \epsilon), s_{st} - \epsilon) - w\left(Z(s_{st} - \epsilon) - \rho(s_{st} - \epsilon), s_{st} - \epsilon\right) \\ = & \int_{Z(s_{st}-\epsilon)}^{s_{st}} w_a(z - \rho(s_{st} - \epsilon), s_{st} - \epsilon) dz \\ \geq & \int_{Z(s_{st}-\epsilon)}^{s_{st}} w_a(s_{st} - \rho(s_{st} - \epsilon), s_{st} - \epsilon) dz \\ \geq & w_a(s_{st} - \rho(s_{st} - \epsilon), s_{st} - \epsilon)\epsilon, \end{aligned} \tag{36}$$

where the first inequality is due to $w_{aa} < 0$ *and* $z < s_{st}$*, and the last inequality follows from* $Z(s_{st} - \epsilon) \geq s_{st} - \epsilon$ *(in Case (2.1)) and that the integrand is negative.*

For Z *to be an equilibrium, a current self at state* s_{st} *must prefer not to undercut her consumption by any* $\epsilon > 0$*. That is,*

$$w((1-\rho)s_{st}, s_{st}) + \beta\delta V(s_{st}) \geq w((1-\rho)s_{st} - \epsilon, s_{st}) + \beta\delta V(s_{st} - \epsilon).$$

Using equation (6) to get rid of V*, the above inequality becomes*

$$\begin{aligned} & w((1-\rho)s_{st}, s_{st}) + \delta(W(s_{st}) - (1-\beta)w((1-\rho)s_{st}, s_{st})) \\ \geq & w((1-\rho)s_{st} - \epsilon, s_{st}) + \delta(W(s_{st} - \epsilon) - (1-\beta)w(Z(s_{st} - \epsilon) - \rho(s_{st} - \epsilon), s_{st} - \epsilon)). \end{aligned}$$

Applying (15) to get rid of the terms with W*, we have*

$$\begin{aligned} & w((1-\rho)s_{st}, s_{st}) - w((1-\rho)s_{st} - \epsilon, s_{st}) - \delta(w(s_{st} - \rho(s_{st} - \epsilon), s_{st} - \epsilon) - w((1-\rho)s_{st}, s_{st})) \\ & \geq (1-\beta)\delta(w((1-\rho)s_{st}, s_{st}) - w(Z(s_{st} - \epsilon) - \rho(s_{st} - \epsilon), s_{st} - \epsilon)), \end{aligned}$$

Applying (36), dividing both sides by ϵ*, and taking limit as* $\epsilon \to 0$*, we have*

$$w_a((1-\rho)s_{st}, s_{st}) \geq \beta\delta[\rho w_a - w_s]((1-\rho)s_{st}, s_{st}) + (1-\beta)\delta w_a((1-\rho)s_{st}, s_{st}), \tag{37}$$

which is impossible since $w_a((1-\rho)s_{st}, s_{st}) \leq 0$ *and* $w_s((1-\rho)s_{st}, s_{st}) < 0$*.*

Therefore, in Case (2.1), a current self at state s_{st} *strictly prefers to undercut her consumption by* ϵ*.*

We now turn to Case (2.2). Let $(\epsilon_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\epsilon_k \downarrow 0$ and that $Z(s_{st}-\epsilon_k) < s_{st}-\epsilon_k$ for all $k \in \mathbb{N}$. The payoff of a current self from deviating to stock $s_{st}-\epsilon_k$ at state s_{st} is

$$w\left((1-\rho)\, s_{st}-\epsilon_k, s_{st}\right) + \beta\delta V(s_{st}-\epsilon_k),$$

which, from Lemma 4, is at least

$$w\left((1-\rho)\, s_{st}-\epsilon_k, s_{st}\right) + \frac{\beta\delta}{1-\delta} w\left((1-\rho)\left(s_{st}-\epsilon_k\right), s_{st}-\epsilon_k\right).$$

Subtracting the current self's equilibrium payoff at s_{st}, which is $U(s_{st})$, dividing by ϵ_k, and then letting k go to infinity, we have,

$$-w_a\left((1-\rho)\, s_{st}, s_{st}\right) - \frac{\beta\delta}{1-\delta}\left[(1-\rho)\, w_a + w_s\right]\left((1-\rho)\, s_{st}, s_{st}\right),$$

which is strictly positive since $w_a\left((1-\rho)\, s_{st}, s_{st}\right) \leq 0$ and $w_s < 0$. Therefore, there is some k such that at state s_{st}, undercutting today's equilibrium consumption by ϵ_k is profitable.

We now show that W is weakly decreasing. From the proof of Claim 1 for Cases (a.1) and (b.1), we have established that if W is not weakly decreasing, then V is not weakly decreasing. And since we have shown that V is weakly decreasing, we are done. ■

7.2 Existence of MPE: Proof of Proposition 1

Proof. *To rule out pathological equilibria, I focus on equilibria whose continuation values are bounded from below.*[20] *For each s, define $a^\star(s) = \arg\max_{a\in[0,y]} w(a,s)$ as the myopic optimal consumption level at stock s. A natural lower bound for V, denoted by $\underline{V}$, is the continuation value of fixing the stock at the worst state forever (the highest stock $\bar{s}$) and letting the consumer behave myopically: $\underline{V} \equiv \frac{1}{1-\delta} w\left(a^\star(\bar{s}), \bar{s}\right)$. A natural upper bound for V, denoted by $\bar{V}$, is the continuation value of letting the consumer behave myopically but still stay at the best state (0 stock): $\bar{V} \equiv \frac{1}{1-\delta} w\left(a^\star(0), 0\right)$.*[21]

Let $\mathcal{F}$ be the set of left-continuous, weakly-decreasing functions from S to $[\underline{V}, \bar{V}]$:

$$\mathcal{F} := \left\{ V : S \to \mathbb{R} \mid \underline{V} \leq V(s) \leq \bar{V} \text{ for all } s \in S,\ V \text{ is weakly decreasing and left continuous} \right\}.$$

Let $\mathcal{P}(S)$ be the power set of S. For each $V \in \mathcal{F}$, define the value function $W_V : S \to \mathbb{R}$ and

[20] Requiring V to be bounded is to rule out the equilibrium $\left(Z(s) = \rho s + y\right)_{s\in S}$ such that the consumer spends all her income on the addictive good at all states. If the ordinary good is indispensable, that is, if $w(y,s) = -\infty$, then this strategy is an equilibrium and it is the worst equilibrium. This is because if all future selves use this strategy, the current self's continuation value is $-\infty$ irrespective of her consumption; and hence indeed she has no incentive not to spend all her income on the addictive good. Then, using this equilibrium as a punishment, any behavior can be sustained as an equilibrium. Such equilibria are pathological, and hence the hyperbolic discounting literature typically rules them out.

[21] $\underline{V}$ corresponds to the hypothetical world in which the consumer's stock is refilled to $\bar{s}$ at the beginning of each period, and $\bar{V}$ to the hypothetical world in which her stock is cleaned to 0 at the beginning of each period.

the solution correspondence $\Phi_V : S \to \mathcal{P}(S)$ *by*

$$W_V(s) = \max_{z\in[\rho s,\rho s+y]} w(z-\rho s, s) + \beta\delta V(z), \tag{38}$$

$$\Phi_V(s) = \arg\max_{z\in[\rho s,\rho s+y]} w(z-\rho s, s) + \beta\delta V(z). \tag{39}$$

Let $Z_V : S \to S$ *be given by*

$$Z_V(s) = \min \Phi_V(s). \tag{40}$$

Let the function $T : \mathcal{F} \to \mathcal{F}$ *be defined by,*

$$TV(s) = w(Z_V(s) - \rho s, s) + \delta V(Z_V(s)), \ \forall s \in S. \tag{41}$$

Endow $\mathcal{F}$ *with the topology of weak convergence.*[22] *We first show that all the functions are well-defined. We then show that* T *is a continuous self-map. Finally, we show that* T *has a fixed point in* $\mathcal{F}$*. The existence of an MPE follows immediately.*

Step 1. We show that for each $V \in \mathcal{F}$*,*

1. W_V*,* Z_V*, and* TV *are well defined;*
2. W_V *is weakly decreasing; if* $Z_V(s) > \rho s$*,* W_V *is continuous.*
3. Z_V *is weakly increasing and left continuous.*

Take a function $V \in \mathcal{F}$*. Since* V *is weakly decreasing and left-continuous and* w *is continuous, for each* s*, the objective function* $w(z-\rho s, s) + \beta\delta V(z)$ *is left-continuous with respect to* z*, with possibly downward jumps, and is thus upper semi-continuous. Together with compactness of the choice set* $[\rho s, \rho s + y]$*, it implies that the maximization problem has a solution,* $W_V(s)$ *is well defined, and* $\Phi_V(s)$ *is nonempty and compact.* $\min \Phi_V(s)$ *thus exists, and* Z_V *and* TV *are well defined.* Z_V *is weakly increasing by Observation 1.*

We now show that W_V *is weakly decreasing. Take two points* s_1 *and* s_2 *in* S *that are sufficiently close, with* $s_1 < s_2$*. Since* $Z(s_1)$ *is optimal at* s_1 *and* $Z(s_2) - \rho s_2 + \rho s_1$ *is feasible, we have*

$$W_V(s_1) \geq w(Z(s_2) - \rho s_2, s_1) + \beta\delta V(Z(s_2) - \rho s_2 + \rho s_1)$$

[22] Since each V in $\mathcal{F}$ is weakly decreasing and left-continuous, we can transform V into a distribution function. For every $V \in \mathcal{F}$, let $F_V : \mathbb{R} \to \mathbb{R}$ be defined by

$$F_V(x) = \begin{cases} 0 & \text{if } x < -\bar{s}; \\ \frac{V(-x)-V(\bar{s})}{V(0)-V(\bar{s})} & \text{if } x \in [-\bar{s}, 0]; \\ 1 & \text{if } x > 0. \end{cases}$$

F_V is then a distribution function with support $[-\bar{s}, 0]$. Note that a distribution function corresponds to a unique probability measure, and weak convergence (or weak* convergence, to be precise) of probability measures is equivalent to "convergence at all continuity points of the limiting distribution function." In the sequel, to prove weak convergence of a sequence of functions in $\mathcal{F}$, we will prove convergence of the sequence at all continuity points of the limiting function.

$$\geq \quad w\left(Z\left(s_{2}\right)-\rho s_{2}, s_{2}\right)+\beta \delta V\left(Z\left(s_{2}\right)-\rho s_{2}+\rho s_{2}\right)=W_{V}\left(s_{2}\right).$$

The second inequality follows from $w_s < 0$ and the fact that V is weakly decreasing.

We now show that W_V is continuous if $Z_V(s) > \rho s$. Under this condition, $Z_V(s)$ is feasible at $s+\eta$, for a small $\eta > 0$. Suppose by negation that W_V is not continuous at s, that is, $W_V(s-) > W_V(s+)$. Then for sufficiently small $\eta > 0$, choosing $Z_V(s)$ at $s+\eta$ gives a value that is sufficiently close to $W_V(s-)$, whereas choosing an element in $\Phi_V(s+\eta)$ gives a value that is at most $W_V(s_+)$ since W_V is weakly decreasing. That is, $Z_V(s)$ is strictly better than any element in $\Phi_V(s+\eta)$, which contradicts the optimality of $\Phi_V(s+\eta)$.

We now show that Φ_V is upper hemi-continuous at s if $Z_V(s) > \rho s$. We have shown that W is continuous at s. Take a sequence $(s_k)_{\in\mathbb{N}}$ that converges to s, a sequence $(z_k)_{k\in\mathbb{N}}$ that converges to some z, with $z_k \in \Phi_V(s_k)$ for all $k \in \mathbb{N}$. If z is not in $\Phi_V(s)$, then W_V cannot be continuous at s.

To show Z_V is left-continuous, take a sequence $s_k \uparrow s$. The sequence $(Z_V(s_k))_{k\in\mathbb{N}}$ is a bounded, weakly increasing sequence, and the limit thus exists and is in $[\rho s, \rho s+y]$. If $Z_V(s) > \rho s$, then since Φ_V is upper hemi-continuous at s, $\lim_{k\to\infty} Z_V(s_k) \in \Phi_V(s)$. And since $Z_V(s_k) \leq z$ for all z in $\Phi_V(s)$, $\lim_{k\to\infty} Z_V(s_k) = \min \Phi_V(s) \equiv Z_V(s)$. That is, Z_V is left-continuous. If $Z_V(s) = \rho s$, since $\lim_{k\to\infty} Z_V(s_k) \geq \rho s$ and $\lim_{k\to\infty} Z_V(s_k) \leq Z_V(s)$ (as Z_V is weakly increasing), we have $\lim_{k\to\infty} Z_V(s_k) = Z_V(s)$, i.e., Z_V is left-continuous.

Next, we prove the following useful lemma.

Lemma 8. *For every $V \in \mathcal{F}$,*

1. *For every $s \in S$ such that $Z_V(s) > \rho s$ and $Z_V(s+) < \rho s + y$, there is an $\eta > 0$, such that W_V is Lipschitz continuous on $(s-\eta, s+\eta)$. Moreover, we have*

$$W_V'(s+) \quad = \quad \left[-\rho w_a + w_s\right]\left(Z_V(s+) - \rho s, s\right), \tag{42}$$

$$W_V'(s-) \quad = \quad \left[-\rho w_a + w_s\right]\left(Z_V(s) - \rho s, s\right). \tag{43}$$

2. *For every $s \in S$ such that $Z_V(s) = \rho s$, we have*

$$\limsup_{s_1 \uparrow s} \frac{W_V(s) - W_V(s_1)}{s - s_1} \leq \left[-\rho w_a + w_s\right]\left(Z_V(s) - \rho s, s\right), \tag{44}$$

   where $W_V'(s-)$ could be $-\infty$.

3. *For every $s \in S$ such that $Z_V(s+) = \rho s + y$, we have*

$$\liminf_{s_1 \downarrow s} \frac{W_V(s_1) - W_V(s)}{s_1 - s} \geq \left[-\rho w_a + w_s\right]\left(Z_V(s+) - \rho s, s\right) \tag{45}$$

Proof. *Part 1. We first derive (42) and (43). The proof is a modification of that of Theorem 3 in Milgrom and Segal (2002).*

For all $s \in [0, \bar{s}]$, let $Z^\star(s) := \arg\max_{z\in[\rho s, \rho s+y]} w(z - \rho s, s)$ be the myopic optimal choice at s. Since $w_a(y, s) < 0$ for all $s \in [0, \bar{s}]$ by assumption, we have $Z^*(s) < \rho s + y$. Since V is *weakly decreasing, we have $Z_V(s) \leq Z^\star(s)$ for all s. Take an s such that $Z_V(s) > \rho s$. And since $Z_V(s) \leq Z^*(s) < \rho s + y$, we have $Z_V(s) \in (\rho s, \rho s + y)$. Since Z_V is left continuous and weakly increasing, there is an $\eta > 0$ such that for all $s' \in (s-\eta, s+\eta)$, $\rho s + \eta < Z_V(s') < \rho(s-\eta) + y$. Therefore, for all s_1 and s_2 in $(s-\eta, s+\eta)$ with $s_2 < s_1$, $Z_V(s_1)$ is feasible at s_2 and $Z_V(s_2)$ is feasible at s_1.*

Let $\underline{z} := Z_V(s-\eta)$, $\bar{z} := Z^\star(s+\eta)$, and $\tilde{g}(z,s) := w(z-\rho s, s)$. We have $\tilde{g}_s(z,s) = [-\rho w_a + w_s](z-\rho s, s)$. Since $w_{aa} < 0$ and $w_{as} > 0$, for all $(z,s') \in [\underline{z}, \bar{z}] \times [s-\eta, s+\eta]$ we have $\tilde{g}_s(\underline{z}, s-\eta) \leq \tilde{g}_s(z,s') \leq \tilde{g}_s(\bar{z}, s+\eta)$. The set of functions $\{\tilde{g}_s(z,\cdot)\}_{z\in(\underline{z},\bar{z})}$ are thus equicontinuous and $\{\tilde{g}(z,\cdot)\}_{z\in(\underline{z},\bar{z})}$ are equidifferentiable on $(s-\eta, s+\eta)$.[23] Let $g(z,s) := w(z-\rho s, s) + \beta\delta V(z)$.

We now derive (42). Take s_1 and s_2 in $(s, s+\eta)$ with $s_1 < s_2$. Let $z_1 := Z(s_1)$, and $z_2 := Z(s_2)$. By optimality arguments, we have

$$\frac{\tilde{g}(z_1,s_2) - \tilde{g}(z_1,s_1)}{s_2 - s_1} = \frac{g(z_1,s_2) - g(z_1,s_1)}{s_2 - s_1} \leq \frac{W_V(s_2) - W_V(s_1)}{s_2 - s_1},$$

and

$$\frac{W_V(s_2) - W_V(s_1)}{s_2 - s_1} \leq \frac{g(z_2,s_2) - g(z_2,s_1)}{s_2 - s_1} = \frac{\tilde{g}(z_2,s_2) - \tilde{g}(z_2,s_1)}{s_2 - s_1}.$$

Taking limsup as $s_1 \downarrow s$, and using the fact that W_V is continuous at s_1 and that $\{\tilde{g}(z,\cdot)\}_{z\in[\underline{z},\bar{z}]}$ is equicontinuous at s, we have

$$\limsup_{s_1\downarrow s} \frac{\tilde{g}(z_1,s_2) - \tilde{g}(z_1,s)}{s_2 - s} \leq \frac{W_V(s_2) - W_V(s)}{s_2 - s} \leq \frac{\tilde{g}(z_2,s_2) - g(z_2,s)}{s_2 - s}.$$

Using equidifferentiability of $\{\tilde{g}(z,\cdot)\}_{z\in[\underline{z},\bar{z}]}$ at s, we have

$$\limsup_{s_1\downarrow s} \tilde{g}_s(z_1,s) + \frac{o(s_2 - s)}{s_2 - s} \leq \frac{W_V(s_2) - W_V(s)}{s_2 - s},$$

and

$$\frac{W_V(s_2) - W_V(s)}{s_2 - s} \leq \tilde{g}_s(z_2,s) + \frac{o(s_2 - s)}{s_2 - s}.$$

Now take liminf as $s_2 \downarrow s$, we have

$$\limsup_{s_1\downarrow s} \tilde{g}_s(z_1,s) \leq \liminf_{s_2\downarrow s} \tilde{g}_s(z_2,s),$$

which implies that $\lim_{s'\downarrow s} \tilde{g}_s(Z(s'), s)$ exists. We have

$$W'(s+) = \lim_{s'\downarrow s} \tilde{g}_s(Z(s'), s) = \tilde{g}_s(Z(s+)\, s, s),$$

[23] Let f be a real-valued functions with domain $X \times Y$. The set of functions $\{f(x,\cdot)\}_{x\in X}$ is *equicontinuous* at $y \in Y$, if $f(x,y')$ converges uniformly as $y' \to y$. It is *equicontinuous on* Y if it is equicontinuous at each point of Y. It is *equidifferentiable* at $y \in Y$, if $\frac{(f(x,y') - (f(x,y))}{y'-y}$ converges uniformly as $y' \to y$. A sufficient condition for equidifferentiability of $\{f(x,\cdot)\}_{x\in X}$ on Y is equicontinuity of $\{f_y(x,\cdot)\}_{x\in X}$ at each point of Y.

where the second equality follows from the continuity of $\tilde{g}_s$ and $\lim_{s'\downarrow s} Z_V(s') = Z_V(s+)$. Using the definition of $\tilde{g}_s$, we obtain (42).

A similar argument implies (43).

We now show that W_V is Lipschitz continuous on $(s-\eta, s+\eta)$. By the definition of η, we have $Z_V(\hat{s}) > \rho\hat{s}$, for all $\hat{s} \in (s-\eta, s+\eta)$, implying that (42) and (43) hold on $(s-\eta, s+\eta)$. Since $[-\rho w_a + w_s]$ is strictly increasing in the first argument and strictly decreasing in the second, if we define $-L = [-\rho w_a + w_s](0, s+\eta)$, we have, for all $\hat{s} \in (s-\eta, s+\eta)$,

$$-L < [-\rho w_a + w_s](Z_V(\hat{s}) - \rho\hat{s}, \hat{s}) < 0.$$

Using (42) and (43), we have, W_V is Lipschitz continuous on $(s-\eta, s+\eta)$ with constant L.

Part 2. Take an s such that $Z_V(s) = \rho s$. For all s_1 with $s_1 < s$ and $\rho s < \rho s_1 + y$, ρs is feasible at s_1. By the optimality of $Z(s_1)$ at s_1 and the feasibility of ρs, we have,

$$\begin{aligned} W_V(s_1) &\geq w(\rho s - \rho s_1, s_1) + \beta\delta V(\rho s) \\ &= w(\rho s - \rho s_1, s_1) - w(0, s) + w(0, s) + \beta\delta V(\rho s) \\ &= w(\rho s - \rho s_1, s_1) - w(0, s) + W_V(s). \end{aligned}$$

Therefore,

$$W_V(s) - W_V(s_1) \leq w(0, s) - w(\rho s - \rho s_1, s_1).$$

Dividing both sides by $s - s_1$ and taking limit superior as $s_1 \uparrow s$, we obtain (44).

Part 3. Take an s such that $Z_V(s+) = \rho s + y$. For all s_1 with $s_1 > s$ and $\rho s_1 < Z(s+)$, $Z(s+)$ is feasible at s_1. By the optimality of $Z(s_1)$ at s_1 and the feasibility of ρs, we have,

$$\begin{aligned} W_V(s_1) &\geq w(Z(s+) - \rho s_1, s_1) + \beta\delta V(Z(s+)) \\ &= w(Z(s+) - \rho s_1, s_1) - w(Z(s+) - \rho s, s) + w(Z(s+) - \rho s, s) + \beta\delta V(Z(s+)) \\ &= w(Z(s+) - \rho s_1, s_1) - w(Z(s+) - \rho s, s) + W_V(s). \end{aligned}$$

Therefore,

$$W_V(s_1) - W_V(s) \geq w(Z(s+) - \rho s_1, s_1) - w(Z(s+) - \rho s, s).$$

Dividing both sides by $s_1 - s$ and taking liminf as $s_1 \downarrow s$, we have (45). ■

Step 2. We show that T is continuous, via 5 substeps.

Step 2.1. We show that for all $s \in S$, TV is continuous at s if and only if W and z are continuous at s.

Since V is decreasing, the marginal utility of consumption at an optimal choice is positive: $w_a(Z_V(s) - \rho s, s) \geq 0$. Moreover, since Z_V can have only upward jumps, both $W_V(s)$ and $-(1-\beta)w(Z_V(s-) - \rho s, s)$ can have only downward jumps. Since

$$\beta TV(s) = W_V(s) - (1-\beta)w(Z_V(s) - \rho s, s), \tag{46}$$

we have, TV is continuous at s if and only if both $W_V(s)$ and $-(1-\beta)w(Z_V(s) - \rho s, s)$

are continuous at s.

Step 2.2 Consider a sequence of functions $(V_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ that weakly converges to some $V \in \mathcal{F}$. For each n, let W_n, Φ_n, Z_n, TV_n (respectively, W, Φ, Z, TV) be defined by equations (38) to (41) when V is replaced with V_n (respectively, V).

We now show that TV_n converges to TV at all continuity points of TV, implying that TV_n weakly converges to TV, and the function T is thus continuous with respect to the weak topology.*

Step 2.3 We show that $W_n \to W$ at all continuity points of W.

Suppose by negation that there is a continuity point of W, s, such that $(W_n(s))_n$ does not converge to $W(s)$. Since the pair of subsequences $(W_n(s), Z_n(s))_n$ is bounded, there is a convergent subsequence, $(W_{n_k}(s), Z_{n_k}(s))_{k\in\mathbb{N}}$, with limit $(\tilde{W}(s), \tilde{Z}(s))$ such that $\tilde{W}(s) \neq W(s)$.

Case 1: $\tilde{W}(s) > W(s)$. By the continuity of W at s, there is an $\eta' > 0$, such that for all $s' \in (s-\eta', s)$, $\tilde{W}(s) > W(s')$.

Since V is monotone, the set of continuity points of V is dense in S. Thus, for each $m \in \mathbb{N}$, there are $s_m \in (s - 1/m, s)$ and $x_m \in \left[\tilde{Z}(s) - 1/m, \tilde{Z}(s)\right]$ such that V is continuous at x_m and x_m is feasible at s_m. Since V_{n_k} weakly converges to V, we have $\lim_{k\to\infty} V_{n_k}(x_m) = V(x_m)$. Also, since $Z_{n_k}(s) \to \tilde{Z}(s)$, for each $m \in \mathbb{N}$, there is an $N_m > 0$ such that, for all $n_k > N_m$, $x_m < Z_{n_k}(s)$, and hence $V_{n_k}(x_m) \geq V_{n_k}(Z_{n_k}(s))$. Taking limit as $n \to \infty$, we have $\lim_{k\to\infty} V_{n_k}(x_m) \geq \lim_{k\to\infty} V_{n_k}(Z_{n_k}(s))$.

By definition of $W(s_m)$ and the feasibility of x_m at state s_m, we have $W(s_m) \geq w(x_m - \rho s_m, s_m) + \beta\delta V(x_m)$. Taking limit as $m \to \infty$, we obtain

$$\begin{aligned}
W(s) = \lim_{m\to\infty} W(s_m) &\geq \lim_{m\to\infty} w(x_m - \rho s_m, s_m) + \beta\delta V(x_m) \\
&= \lim_{m\to\infty}\lim_{k\to\infty} w(x_m - \rho s_m, s_m) + \beta\delta V_{n_k}(x_m) \\
&\geq \lim_{m\to\infty}\lim_{k\to\infty} w\left(\tilde{Z}(s) - \rho s, s\right) + \beta\delta V_{n_k}(Z_{n_k}(s)) \\
&= \tilde{W}(s),
\end{aligned}$$

which contradicts the assumption that $\tilde{W}(s) > W(s)$.

Case 2: $\tilde{W}(s) < W(s)$. Similarly as above, there is an $\eta' > 0$, such that for all $s' \in (s, s+\eta')$, $\tilde{W}(s) < W(s')$.

Since the set of continuity points of V is dense in S, we have, for each $m \in \mathbb{N}$, there exist some $s_m \in (s, s+1/m)$ and $x_m \in [Z(s_m) - 1/m, Z(s_m)]$, such that which V is continuous at x_m and x_m is feasible at s. We thus have $\lim_{k\to\infty} V_{n_k}(x_m) = V(x_m)$, for each $m \in \mathbb{N}$. Since W is continuous at s, we have, as $m \to \infty$, $w(Z(s_m) - \rho s, s) + \beta\delta V(Z(s_m)) \to W(s)$. Since $x_m - Z(s_m) \uparrow 0$ as $m \to \infty$, and V is weakly decreasing, we have

$$\lim_{m\to\infty} w(x_m - \rho s, s) + \beta\delta V(x_m) \geq W(s). \tag{47}$$

By the definition of W_{n_k} and that x_m is feasible at s, we have

$$W_{n_k}(s) \;\; \geq \;\; w(x_m - \rho s, s) + \beta\delta V_{n_k}(x_m).$$

Taking limit as $k \to \infty$ and then as $m \to \infty$, and using (47), we have

$$\begin{aligned}\tilde{W}(s) &\geq \lim_{m\to\infty}\lim_{k\to\infty} w(x_m - \rho s, s) + \beta\delta V_{n_k}(x_m) \\ &= \lim_{m\to\infty} w(x_m - \rho s, s) + \beta\delta V(x_m) \\ &\geq W(s),\end{aligned}$$

which contradicts the assumption that $\tilde{W}(s) < W(s)$.

Therefore, we must have $W_n \to W$ at all continuity points of W.

Step 2.4 We show that $Z_n \to Z$ at all points where both Z and W are continuous.

Suppose by negation that there is an s at which both Z and W are continuous, and $(Z_n(s))_n$ does not converge to $Z(s)$. Since the pair of subsequences $(W_n(s), Z_n(s))_n$ is bounded, there is a convergent subsequence, $(W_{n_k}(s), Z_{n_k}(s))_{k\in\mathbb{N}}$, with limit $(\tilde{W}(s), \tilde{Z}(s))$ such that $\tilde{Z}(s) \neq Z(s)$.

Case 1. $\tilde{Z}(s) > Z(s)$. Let $\epsilon \equiv (\tilde{Z}(s) - Z(s))/2$. If k is large, we have $Z_{n_k}(s) > Z(s)$. And since Z_{n_k} is weakly increasing and Z is continuous at s, there is an $\eta' > 0$, such that $Z_{n_k}(s') > Z(s') + \epsilon$ for all $s' \in [s, s+\eta']$. That is, if k is large, then $Z_{n_k}(s') > \rho s'$, for all $s' \in [s, s+\eta']$. By Parts 2 and 3 of Lemma 8, if s_1 and s_2 are in $[s, s+\eta']$ with $s_1 < s_2$, since $Z_{n_k}(s) > Z(s) + \epsilon$ and $-\rho w_{aa} + w_{as} > 0$, we have, for all $s \in [s_1, s_2]$,

$$W_{n_k}(s_2) - W_{n_k}(s_1) \geq (s_2 - s_1)\left[-\rho w_a + w_s\right](Z(s) + \epsilon - \rho s, s).$$

Take two continuity points of W and Z, s_1 and s_2 in $[s, s+\eta']$ with $s_1 < s_2$. Using again $-\rho w_{aa} + w_{as} > 0$, the above inequality implies

$$\begin{aligned}W_{n_k}(s_2) - W_{n_k}(s_1) &> \int_{s_1}^{s_2}\left[-\rho w_a + w_s\right](Z(s) - \rho s, s)\, ds \\ &\geq W(s_2) - W(s_1).\end{aligned}$$

The last inequality is due to (44) and that $W(s_2) - W(s_1) \leq \int_{s_1}^{s_2} W'(s-)\, ds$ for weakly decreasing functions. Taking limit on the left-hand side as $k \to \infty$, and using the fact that W is continuous at s_1 and s_2, we have $W(s_2) - W(s_1) > W(s_2) - W(s_1)$. A contradiction.

Case 2. $\tilde{Z}(s) < Z(s)$. The same argument as in Case 1 shows that this cannot hold.

Step 2.5 From Step 2.2, to show that T is continuous, we only need to show that TV_n converges to TV at all continuity points of TV. From Step 2.1, it suffices to show that (W_n) and (Z_n) converges to W and Z at points where both W and Z are continuous, which are established by Steps 2.3 and 2.4. Therefore, T is continuous.

Step 3. We show that T is a self map on $\mathcal{F}$. Take any $V \in \mathcal{F}$.

First, TV is left-continuous.

Take a sequence $(s_k)_{k\in\mathbb{N}}$ in S such that $s_k \uparrow s$. We have $Z(s_k) \uparrow Z(s)$, and hence $w(Z(s_k) - \rho s_k, s_k) \to w(Z(s) - \rho s, s)$ by the continuity of w. By the monotonicity of V, we have $V(Z(s_k)) \downarrow V(Z(s))$. Therefore, $TV(s_k) \to TV(s)$: TV is left-continuous.

Second, TV is weakly decreasing.

Take $s_1, s_2 \in S$ with $s_1 < s_2$. Since W, V are weakly decreasing and Z is weakly increasing *by Step 1, we have*

$$\begin{aligned} TV(s_1) &= W(s_1) + (1-\beta)\,\delta V(Z(s_1)) \\ &\geq W(s_2) + (1-\beta)\,\delta V(Z(s_2)) \\ &= TV(s_2) \end{aligned}$$

where the second inequality follows from the monotonicity of W, V, and z.

Finally, $\underline{V} \leq TV(s) \leq \bar{V}$ for all $s \in S$.

Take an $s \in S$. Since $Z^\star(\bar{s}) - \rho\bar{s} \equiv a^\star(\bar{s}) \in [0, y]$, we have $Z^\star(\bar{s}) - \rho\bar{s} + \rho s \in (\rho s, \rho s + y) \subseteq S$. We have

$$\begin{aligned} TV(s) &= W(s) + (1-\beta)\,\delta V(Z(s)) \\ &\geq w(Z^\star(\bar{s}) - \rho\bar{s}, \rho s) + \beta\delta V(Z^\star(\bar{s}) - \rho\bar{s} + \rho s) + (1-\beta)\,\delta V(Z(s)) \\ &\geq w(Z^\star(\bar{s}) - \rho\bar{s}, \rho\bar{s}) + \delta\underline{V} \quad = \underline{V}. \end{aligned}$$

The last inequality uses $V(s) \geq \underline{V}$ for all $s \in S$. As W is weakly decreasing, we have

$$\begin{aligned} TV(s) &= W(s) + (1-\beta)\,\delta V(Z(s)) \\ &\leq W(0) + (1-\beta)\,\delta V(Z(s)) \\ &= w(Z(0), 0) + \beta\delta V(Z(0)) + (1-\beta)\,\delta V(Z(s)) \\ &\leq w(Z^\star(0), 0) + \delta\bar{V} \quad = \bar{V}. \end{aligned}$$

The last inequality follows from $Z^\star(0) \geq Z(0)$, $V \leq \bar{V}$, and $w_a(z, 0) \geq 0$ for all $z \in [0, Z^\star(0)]$.

Step 4. Since $\mathcal{F}$ is endowed with the topology of weak convergence, it is compact. To see this, by footnote 22, it suffices to show that the set of distributions is compact. Since the set is tight, by the Prokhorov theorem, it is relatively compact. Since it is itself closed, it is compact. Since $\mathcal{F}$ is compact and convex, and T is a continuous self-map of $\mathcal{F}$, by Tychonoff's fixed point theorem, T has a fixed point in $\mathcal{F}$. ■

7.3 Existence of restricted MPE for the proof of Proposition 3

Consider the same intrapersonal game except that the state space is restricted to $[0, s_u]$ and the choice set is $[\rho s, \rho s + y] \cap [0, s_u]$ for each s. We show that this restricted game has an MPE. Let $\underline{V}, \bar{V}, a^\star$ and $Z^\star$ be defined as in Section 7. Define $\tilde{\mathcal{F}}$ by

$$\tilde{\mathcal{F}} \equiv \left\{ V : [0, s_u] \to \mathbb{R} \mid \underline{V} \leq V \leq \bar{V}, \text{ is left-continuous and weakly decreasing.} \right\}.$$

Let $\mathcal{P}([0, s_u])$ be the power set of $[0, s_u]$. For each $V \in \tilde{\mathcal{F}}$, define the value function $\tilde{W}_V : [0, s_u] \to \mathbb{R}$ and the solution correspondence $\tilde{\Phi}_V : [0, s_u] \to \mathcal{P}([0, s_u])$ by

$$\tilde{W}_V(s) = \max_{z \in [\rho s, \rho s + y] \cap [0, s_u]} w(z - \rho s, s) + \beta\delta V(z), \tag{48}$$

$$\tilde{\Phi}_V(s) = \arg \max_{z \in [\rho s, \rho s + y] \cap [0, s_u]} w(z - \rho s, s) + \beta \delta V(z). \tag{49}$$

Let $\tilde{Z}_V : [0, s_u] \to [0, s_u]$ be given by

$$\tilde{Z}_V(s) = \min \tilde{\Phi}_V(s). \tag{50}$$

Let the function $T : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$ be defined by,

$$TV(s) = w(Z_V(s) - \rho s, s) + \delta V(Z_V(s)), \ \forall s \in S. \tag{51}$$

Endow $\tilde{\mathcal{F}}$ with the topology of weak convergence. We show that the mapping T has a fixed point in $\tilde{\mathcal{F}}$. We first show that $TV \geq \underline{V}$. Recall the definition of s_u, we have

$$\begin{aligned}
\frac{1-(1-\beta)\delta}{1-\delta} w(s_u - \rho s_u, s_u) &= W(s_u) \\
&\geq w(Z^\star(\bar{s}) - \rho \bar{s}, \bar{s}) + \beta \delta \underline{V} \\
&= \frac{1-(1-\beta)\delta}{1-\delta} w(Z^\star(\bar{s}) - \rho \bar{s}, \bar{s}).
\end{aligned}$$

Therefore, $w(s_u - \rho s_u, s_u) \geq w(Z^\star(\bar{s}) - \rho \bar{s}, \bar{s})$. Take an $s \in [0, s_u]$. If s_u is not feasible at s, the same argument as in Step 3 of the proof of Proposition 1 implies that $TV(s) \geq \underline{V}$. If s_u is feasible at s, we have

$$\begin{aligned}
TV(s) &= w\left(\tilde{Z}_V(s) - \rho s, s\right) + \beta \delta V\left(\tilde{Z}_V(s)\right) + (1-\beta)\delta V\left(\tilde{Z}_V(s)\right) \\
&\geq w(s_u - \rho s, s) + \beta \delta V(s_u) + (1-\beta)\delta \underline{V} \\
&\geq w(s_u - \rho s_u, s_u) + \delta \underline{V} \\
&\geq w(Z^\star(\bar{s}) - \rho \bar{s}, \bar{s}) + \delta \underline{V} \quad \underline{V},
\end{aligned}$$

where the third line uses $-\rho w_a + w_s < 0$ and $s < s_u$. That $\tilde{\mathcal{F}}$ is convex and weak* compact, T is continuous, and TV is left-continuous, weakly decreasing, $TV \leq \bar{V}$, follow from the same reasoning as in the proof of Proposition 1. Therefore, T is a continuous self map on $\tilde{\mathcal{F}}$. Applying Tychonoff's fixed point theorem, T has a fixed point in $\tilde{\mathcal{F}}$.

7.4 Formal statements and proofs for results in Section 4

7.4.1 Abstinence bright lines and unexpected shocks

We prove that non-abstinence bright lines do not survive the unexpected shocks considered in Section 4.1, but abstinence bright line rules can. Consider an MPE Z with a bright-line rule at s_u in the game without shocks. Let W and V denote the DM's corresponding current and continuation value functions. Suppose that DM believes her future selves will play equilibrium Z from period $t+1$ onward.

We first show that if $s_u > 0$, then once the shock hits, the DM will cross the line, i.e., by choosing a stock above s_u. Consider the game without the shock. Assume Z is discontinuous at s_u: $Z(s_u+) > s_u$. In MPE Z, the DM is indifferent between choosing stocks s_u and $Z(s_u+)$, i.e.,

$$w(s_u - \rho s_u, s_u) + \beta\delta V(s_u) = w(Z(s_u+) - \rho s_u, s_u) + \beta\delta V(Z(s_u+)). \tag{52}$$

Meanwhile, by the fundamental theorem of calculus and the fact that $w_{as} > 0$, we have

$$\begin{aligned}&(w(Z(s_u+) - \rho s_u, s_u + \epsilon) - w(s_u - \rho s_u, s_u + \epsilon)) - (w(Z(s_u+) - \rho s_u, s_u) - w(s_u - \rho s_u, s_u))\\ = &\int_{s_u}^{Z(s_u+)} \int_0^{\epsilon} w_{as}(z - \rho s_u, s_u + e) de dz > 0. \end{aligned} \tag{53}$$

Applying this inequality to (52) and rearranging terms, we get

$$w(s_u - \rho s_u, s_u + \epsilon) + \beta\delta V(s_u) < w(Z(s_u+) - \rho s_u, s_u + \epsilon) + \beta\delta V(Z(s_u+)).$$

The left-hand side is the DM's payoff from staying at the bright line after the shock hits, and the right-hand side is her payoff from choosing a higher stock $Z(s_u+)$ after the shock hits. This inequality thus implies that after the shock hits, the DM strictly prefers $Z(s_u+)$ to staying at the bright line s_u. Therefore, the DM will cross the bright line.

If Z is continuous at s_u, then choose a small $\epsilon_1 > 0$ so that $Z(s_u + \epsilon_1)$ is feasible at s_u. We have rule By the optimality of $Z(s_u + \epsilon_1)$ at $s_u + \epsilon_1$, we have

$$w(Z(s_u + \epsilon_1) - \rho(s_u + \epsilon_1), s_u + \epsilon_1) + \beta\delta V(Z(s_u + \epsilon_1)) \geq w(s_u - \rho(s_u + \epsilon_1), s_u + \epsilon_1) + \beta\delta V(s_u). \tag{54}$$

Let ϵ be such that $\epsilon > \epsilon_1$. Let $f(s_u, \epsilon_1, \epsilon) \equiv w(s_u - \rho(s_u + \epsilon_1), s_u + \epsilon_1) - w(s_u - \rho(s_u), s_u + \epsilon)$. (54) implies that

$$\begin{aligned}&w(Z(s_u + \epsilon_1) - \rho(s_u), s_u + \epsilon) + \beta\delta V(Z(s_u + \epsilon_1)) + f(Z(s_u + \epsilon_1), \epsilon_1, \epsilon)\\ \geq\ &w(s_u - \rho(s_u), s_u + \epsilon) + \beta\delta V(s_u) + f(s_u, \epsilon_1, \epsilon).\end{aligned} \tag{55}$$

We now show that for each ϵ, if $\epsilon_1 > 0$ is sufficiently small, then $f(Z(s_u + \epsilon_1), \epsilon_1, \epsilon) - f(s_u, \epsilon_1, \epsilon) < 0$. To see this, note that by the fundamental theorem of calculus,

$$\begin{aligned}&f(Z(s_u + \epsilon_1), \epsilon_1, \epsilon) - f(s_u, \epsilon_1, \epsilon)\\ = &\int_{s_u}^{Z(s_u+\epsilon_1)} [-\int_0^{\epsilon_1} \rho w_{aa}(z - \rho(s_u + e_1), s_u + \epsilon_1) de_1 - \int_{\epsilon_1}^{\epsilon} w_{as}(z - \rho s_u, s_u + e) de] dz.\end{aligned}$$

As $\epsilon_1 \to 0$, the terms in the square brackets converge to $-\int_0^{\epsilon} w_{as}(z - \rho s_u, s_u + e) de < 0$ since $w_{as} > 0$. Thus, if $\epsilon_1 > 0$ is sufficiently small, then $f(Z(s_u + \epsilon_1), \epsilon_1, \epsilon) - f(s_u, \epsilon_1, \epsilon) < 0$. This inequality and (55) imply that

$$w(Z(s_u + \epsilon_1) - \rho(s_u), s_u + \epsilon) + \beta\delta V(Z(s_u + \epsilon_1)) > w(s_u - \rho(s_u), s_u + \epsilon) + \beta\delta V(s_u) \tag{56}$$

The right-hand side is the DM's payoff from staying at the bright line s_u if the shock hits, while the left-hand side is her payoff from choosing a higher stock $Z(s_u + \epsilon_1)$ if the shock hits. Thus, the inequality means that the DM prefers to cross the line (i.e., by choosing $Z(s_u + \epsilon_1)$) if the shock hits.

Next, assume $s_u = 0$ and $W(0) > W(0+)$. Then, there is a $\bar{\epsilon} > 0$ such that for all

$\epsilon \in (0, \bar{\epsilon})$,

$$w(0, \epsilon) + \beta\delta V(0) > W(0+). \tag{57}$$

The left-hand side is the DM's payoff (at stock 0) from maintaining abstinence after the shock hits while the right-hand side is the DM's maximal payoff (at stock 0) from choosing a positive stock if the shock did not hit. The latter is thus an upper bound of the DM's payoff from crossing the bright line after the shock hits. The inequality implies that the DM prefers to maintain abstinence after the shock hits.

7.4.2 Formal results for Section 4.5

We provide a formal statement for the result in Section 4.5 and prove it.

Proposition 5. *Assume $\delta \in (\delta_1, \delta_\beta)$. Let Z be an MPE without a bright-line rule. Let s_o denote the smallest steady state of Z that is not absorbing on the right. Then,*

1. *$s_o \geq s_{\delta,\beta}$. That is, the consumer will fall into an excessive consumption trap if the initial stock is above $s_{\delta,\beta}$.*

2. *If $s_o \notin S_\beta$ or complementarity (i.e., $w_{as}((1-\rho)s, s)$) is not too weak at every $s \in S_\beta$, then there exists an MPE $\hat{Z}$ with a bright-line rule, such that $\hat{Z}$ Pareto dominates Z for all states around and above the bright line.*

Proof. *Part 1 follows from the same reasoning used in the proof of Theorem 1. We now proceed to prove Part 2. By definition of s_o, we have $Z(s) > s$ on $[0, s_o)$. Let W be the current value function of equilibrium Z.*

Step 1. The reasoning used in the proof of Theorem 2 similarly implies that if $s_o \notin S_\beta$ or the complementarity is not too weak at every state in S_β, then $W(s) < U(s)$ at all $s \in (s_o - \epsilon, s_o)$ for some $\epsilon > 0$. We have two cases: (1) $W(0) < U(0)$; and (2) $W(0) \geq U(0)$. In case (1), we define a new strategy $\hat{Z}$ by $\hat{Z}(0) = 0$ and $\hat{Z}(s) = Z(s)$ for all $s > 0$. The same reasoning as in case (b) of the proof of Proposition 3 implies that Part 2 holds.

In case (2), let $s_u := \sup\{\tilde{s} \in [0, s_o] : W(s) > U(s), \forall s \in (0, \tilde{s})\}$. We have $s_u < s_o$ because the same proof as that of Theorem 2 implies that $W(s) < U(s)$ holds on a left neighborhood of s_o. Consider the intrapersonal game when the state space is restricted to $[0, s_u]$. As in Step 1 of the proof of Proposition 3, this restricted game has an MPE, $\tilde{Z}$. If $\tilde{Z}(s_u) = s_u$, then the construction as in the proof of Proposition 3 implies Part 2.

We now prove $\tilde{Z}(s_u) = s_u$. First, if $s_u = 0$, then this equality holds since the restricted game over $[0, s_u]$ has only 0 as the available stock. Second, if $s_u > 0$, let $\tilde{W}$ denote the current value function of equilibrium $\tilde{Z}$. We have $W'(s) = [-\rho w_a + w_s](Z(s) - \rho s, s)$ and $\tilde{W}'(s) = [-\rho w_a + w_s](\tilde{Z}(s) - \rho s, s)$. Suppose by negation that $\tilde{Z}(s_u) < s_u$, then there is some $s_1 < s_u$ such that $\tilde{Z}(s) < s$ for all $s \in (s_1, s_u]$ and $\tilde{Z}(s_1) = s_1$. Since $Z(s) \geq s$ for all $s \in (s_1, s_u]$ and $-\rho w_{aa} + w_{as} > 0$, we have $W'(s) > \tilde{W}'(s)$ for all $s \in (s_1, s_u]$. As $\tilde{Z}(s_u) < s_u$, by Lemma 4, we have $\tilde{W}(s_u) \geq U(s_u)$. Since $W(s_u) = U(s_u)$, we have $\tilde{W}(s_u) \geq W(s_u)$. Combining $W'(s) > \tilde{W}'(s)$ for all $s \in (s_1, s_u]$, we have $W(s_1) < \tilde{W}(s_1)$. Since $Z(s_1) = s_1$, we have $U(s_1) = \tilde{W}(s_1)$. These inequalities imply $U(s_1) > W(s_1)$, which contradicts the definition of s_u. Thus, $\tilde{Z}(s_u) = s_u$, as desired.

■

7.5 Time consistency precludes bright-line rules

We provide a formal statement for the result in Section 4.3 and prove it.

Proposition 6. *Generically, the time-consistent consumer's optimal policy Z^{tc} does not feature a bright-line rule.*

Therefore, if $\hat{s}$ is an unabsorbing steady state of a time-consistent consumer's optimal policy, then it must be repelling: there exists some $\epsilon > 0$ such that $Z^{tc}(s) > s$ for all $s \in (\hat{s}, \hat{s}+\epsilon)$, and $Z^{tc}(s) < s$ for all $s \in (\hat{s}-\epsilon, \hat{s})$.

Proof. *Suppose by contradiction that Z^{tc} has a steady state $\hat{s}$ that is sustained by a bright-line rule. We have two cases: (1) Z^{tc} is discontinuous at $\hat{s}$; (2) Z^{tc} is continuous at $\hat{s}$. We rule them out one by one.*

Step 1. Suppose Case (1) holds. Then there are two cases: (1.1) $Z^{tc}(\hat{s}+) > \hat{s}$ and $Z^{tc}(\hat{s}-) = \hat{s}$; and (1.2) $Z^{tc}(\hat{s}-) < \hat{s}$ and $Z^{tc}(\hat{s}+) = \hat{s}$. We focus on Case (1.1) because Case (1.2) can be analyzed similarly.

In Case (1.1), since $Z^{tc}(\hat{s}-) > \rho\hat{s}$, by Lemma 1, the left derivative of the consumer's continuation value function V exists. Since $\hat{s}$ is optimal, the consumer has no incentive to cut consumption marginally at $s = \hat{s}$; thus, it holds that

$$w_a((1-\rho)s, s) + \delta V'(s-) \geq 0.$$

Moreover, because $V'(s+) = [-\rho w_a + w_s](Z^{tc}(s+) - \rho s, s)$, $V'(s-) = [-\rho w_a + w_s](Z^{tc}(s-) - \rho s, s)$, and $Z^{tc}(s+) > Z^{tc}(s-)$, we have $V'(s+) > V'(s-)$. The above inequalities imply that

$$w_a((1-\rho)s, s) + \delta V'(s+) > 0.$$

That is, at $\hat{s}$, the consumer strictly prefers to increase consumption marginally, rather than staying at $\hat{s}$. This contradicts the optimality of Z^{tc}.

Step 2. Suppose Case (2) holds. As $\hat{s}$ is sustained by a bright line rule, there is an $\hat{\epsilon} > 0$ such that $Z^{tc}(s) > s$ for $s \in (\hat{s}, \hat{s}+\hat{\epsilon}) \cup (\hat{s}-\hat{\epsilon}, \hat{s})$. There are three subcases: (2.1) $(Z^{tc})'(\hat{s}-) < (Z^{tc})'(\hat{s}+)$; (2.2) $\liminf_{\epsilon\downarrow 0} \frac{Z^{tc}(\hat{s}) - Z^{tc}(\hat{s}-\epsilon)}{\epsilon} < \limsup_{\epsilon\downarrow 0} \frac{Z^{tc}(\hat{s}+\epsilon) - Z^{tc}(\hat{s})}{\epsilon}$; and (2.3) $\liminf_{\epsilon\downarrow 0} \frac{Z^{tc}(\hat{s}) - Z^{tc}(\hat{s}-\epsilon)}{\epsilon} \geq \limsup_{\epsilon\downarrow 0} \frac{Z^{tc}(\hat{s}+\epsilon) - Z^{tc}(\hat{s})}{\epsilon}$, where (2.1) is a special case of (2.2).[24] *Note that since $\limsup_{\epsilon\downarrow 0} \frac{Z^{tc}(\hat{s}) - Z^{tc}(\hat{s}-\epsilon)}{\epsilon} \leq 1 \leq \limsup_{\epsilon\downarrow 0} \frac{Z^{tc}(\hat{s}+\epsilon) - Z^{tc}(\hat{s})}{\epsilon}$, case (2.3) implies that $(Z^{tc})'(\hat{s}-) = (Z^{tc})'(\hat{s}+) = 1$.*

In case (2.1), since Z^{tc} is continuous and $\hat{s}$ is a steady state, there is some $\epsilon > 0$ such that Z^{tc} is interior on $(\hat{s}-\epsilon, \hat{s}+\epsilon)$; moreover, for any $s, s' \in (\hat{s}-\epsilon, \hat{s}+\epsilon)$, both s and $Z^{tc}(s)$ are feasible at s'. This implies that Part 1 of Lemma 1 holds. Thus, for any $s_1, s_2 \in (\hat{s}-\epsilon, \hat{s}+\epsilon)$ with $s_1 < s_2$, we have $V(s_2) - V(s_1) = \int_{t_1}^{t_2} V'(s)ds$, where $V'(s) = [-\rho w_a + w_s](Z^{tc}(s) - \rho s, s)$. Since $(Z^{tc})'(\hat{s}-) < (Z^{tc})'(\hat{s}+)$, we have $V''(\hat{s}-) - V''(\hat{s}+) = [-\rho w_{aa} + w_{as}](Z^{tc}(\hat{s}) - \rho\hat{s}, \hat{s})\left((Z^{tc})'(\hat{s}-) - (Z^{tc})'(\hat{s}+)\right) < 0$.

We first derive a first-order condition at $\hat{s}$. For $s \in (\hat{s}-\epsilon, \hat{s}+\epsilon)$, since $Z^{tc}(\hat{s}) = \hat{s}$ is optimal at $\hat{s}$ and s is feasible, the following inequality holds:

$$w(\hat{s} - \rho\hat{s}, \hat{s}) - w(s - \rho\hat{s}, \hat{s}) + \delta(V(\hat{s}) - V(s)) \geq 0. \tag{58}$$

[24] We prove case (2.1) in detail because it is easier to follow than if we prove case (2.2) only.

Similarly, since $Z^{tc}(s)$ is optimal at s and $\hat{s}$ is feasible, we have

$$w(\hat{s} - \rho s, s) - w(Z^{tc}(s) - \rho s, s) + \delta(V(\hat{s}) - V(Z^{tc}(s))) \leq 0. \tag{59}$$

Applying Taylor expansion to $w(s-\rho\hat{s}, \hat{s})$ and $V(s)$ in (58) for $s = s'$ and $s = s''$ respectively, with $s' > \hat{s}$ and $s'' < \hat{s}$, we get

$$-\left(w_a(\hat{s} - \rho\hat{s}, \hat{s}) + \delta V'(\hat{s}+)\right)(s' - \hat{s}) \geq 0 \text{ and } -\left(w_a(\hat{s} - \rho\hat{s}, \hat{s}) + \delta V'(\hat{s}-)\right)(s'' - \hat{s}) \geq 0.$$

Since $V'(\hat{s}-) = V'(\hat{s}+)$ as Z^{tc} is continuous at $\hat{s}$, this inequality implies

$$w_a(\hat{s} - \rho\hat{s}, \hat{s}) + \delta V'(\hat{s}) = 0. \tag{60}$$

Next, we show that $Z^{tc}(s) < \hat{s}$ for all $s < \hat{s}$. Suppose by negation that this is false. Since $Z^{tc}(\hat{s}) = \hat{s}$ and Z^{tc} is weakly increasing, it follows that there is an $\tilde{\epsilon} > 0$ such that $Z^{tc}(s) = \hat{s}$ on $(\hat{s} - \tilde{\epsilon}, \hat{s}]$. Thus, $V(s) = w(\hat{s} - \rho s, s) + \delta V(\hat{s})$, and $V'(s) = [-\rho w_a + w_s](\hat{s} - \rho s, s)$ on $(\hat{s} - \tilde{\epsilon}, \hat{s}]$. At stock $s \in (\hat{s} - \tilde{\epsilon}, \hat{s})$, if the consumer deviates to $z \in (\hat{s} - \tilde{\epsilon}, \hat{s})$, the net gain, denoted as $\Delta V(z, s)$, satisfies

$$\Delta V(z, s) = w(z - \rho s, s) + \delta V(z) - V(s) = -\int_z^{\hat{s}} \left(w_a(x - \rho s, s) + \delta V'(x)\right) dx.$$

By (60), the integrand is 0 if $s = x = \hat{s}$. Since $-\rho w_{aa} + w_{as} > 0$, for each $s \in (\hat{s} - \tilde{\epsilon}, \hat{s}]$, there is a $\tilde{z} < \hat{s}$ such that $w_a(x - \rho s, s) + \delta V'(x) < 0$ for all $x \in (\tilde{z}, \hat{s})$, and thus $\Delta V(\tilde{z}, s) > 0$. Thus, deviating to $\tilde{z}$ at s is profitable, contradicting the optimality of Z^{tc}.

Finally, we derive a contradiction. Applying Taylor expansion to $w(s - \rho\hat{s}, \hat{s})$ and $V(s)$ in (58) for $s > \hat{s}$ and using (60), we get

$$-\left(w_{aa}(\hat{s} - \rho\hat{s}, \hat{s}) + \delta V''(\hat{s}+)\right)(s - \hat{s})^2/2 + o((s - \hat{s})^2) \geq 0, \tag{61}$$

where $o((s - \hat{s})^2)$ denotes terms of higher order than $(s - \hat{s})^2$. Applying Taylor expansion to $w(Z^{tc}(s) - \rho s, s)$ and $V(Z^{tc}(s))$ in (58) for $s < \hat{s}$ and using (60), we have

$$\begin{aligned} &-\left(w_a(\hat{s} - \rho s, s) + \delta V'(\hat{s}-)\right)(Z^{tc}(s) - \hat{s}) \\ &-\left(w_{aa}(\hat{s} - \rho s, s) + \delta V''(\hat{s}-)\right)(Z^{tc}(s) - \hat{s})^2/2 + o((Z^{tc}(s) - \hat{s})^2) \leq 0. \end{aligned} \tag{62}$$

Since $w_a(\hat{s} - \rho s, s) < w_a(\hat{s} - \rho\hat{s}, \hat{s})$ for $s < \hat{s}$ due to $-\rho w_{aa} + w_{as} > 0$, (62) implies

$$\begin{aligned} &-\left(w_a(\hat{s} - \rho\hat{s}, \hat{s}) + \delta V'(\hat{s})\right)(Z^{tc}(s) - \hat{s}) \\ &-\left(w_{aa}(\hat{s} - \rho\hat{s}, \hat{s}) + \delta V''(\hat{s}-)\right)(Z^{tc}(s) - \hat{s})^2/2 + o((Z^{tc}(s) - \hat{s})^2) \leq 0. \end{aligned} \tag{63}$$

For (60), (61) and (63) to hold, we must have $V''(\hat{s}+) \leq V''(\hat{s}-)$. But this contradicts our assumption $V''(\hat{s}+) > V''(\hat{s}-)$. Thus, Case (2.1) does not hold.

In case (2.2), let $\alpha = \liminf_{\epsilon\downarrow 0} \frac{Z^{tc}(\hat{s}) - Z^{tc}(\hat{s}-\epsilon)}{\epsilon}$ and $\beta = \limsup_{\epsilon\downarrow 0} \frac{Z^{tc}(\hat{s}+\epsilon) - Z^{tc}(\hat{s})}{\epsilon}$. By definition, there are sequences (s_n) and $(\tilde{s}_n)$, with $s_n \uparrow \hat{s}$ and $\tilde{s}_n \downarrow \hat{s}$ such that $\alpha = \lim_{n\to\infty} \frac{Z^{tc}(\hat{s}) - Z^{tc}(s_n)}{\hat{s} - s_n}$ and $\beta = \lim_{n\to\infty} \frac{Z^{tc}(\tilde{s}_n) - Z^{tc}(\hat{s})}{\tilde{s}_n - \hat{s}}$. Since $V'(s) = [-\rho w_a + w_s](Z^{tc}(s) - \rho s, s)$, *we have* $\lim_{n\to\infty} \frac{V'(s_n)-V'(\hat{s})}{s_n-\hat{s}} - \lim_{n\to\infty} \frac{V'(\tilde{s}_n)-V'(\hat{s})}{\tilde{s}_n-\hat{s}} = [-\rho w_{aa} + w_{as}](Z^{tc}(\hat{s}) - \rho\hat{s}, \hat{s})\,(\alpha - \beta) < 0.$

We can rule out case (2.2) by the same steps as in case (2.1), except that we take the Taylor expansion of (58) *and* (59) *for* $s = \tilde{s}_n$ *and for* $s = s_n$*, for all* n*, respectively; and we replace* $V''(\hat{s}-)$ *and* $V''(\hat{s}+)$ *with* $\lim_{n\to\infty} \frac{V'(s_n)-V'(\hat{s})}{s_n-\hat{s}}$ *and* $\lim_{n\to\infty} \frac{V'(\tilde{s}_n)-V'(\hat{s})}{\tilde{s}_n-\hat{s}}$*, respectively.*

In case (2.3), define

$$M(s) = w_a(s - \rho s, s) + \delta[-\rho w_a + w_s](s - \rho s, s). \tag{64}$$

We first show that $M(\hat{s}) = 0$ *and* $M(s) > 0$ *on a left neighborhood of* $\hat{s}$*. Whenever* Z^{tc} *is interior, we have*

$$V'(s+) = [-\rho w_a + w_s](Z^{tc}(s+) - \rho s, s); \tag{65}$$

$$U'(s) = [-\rho w_a + w_s](s - \rho s, s) + \frac{1}{1-\delta}M(s). \tag{66}$$

Since $\hat{s}$ *is a steady state,* $V(\hat{s}) = U(\hat{s})$*. Since* V *is the value function corresponding to the optimal policy, we have* $V(s) \geq U(s)$ *for all* s*. This holds only if* V *and* U *are tangent at* $s = \hat{s}$*. Using* (65)*,* (66)*, and* $Z^{tc}(\hat{s}) = \hat{s}$*, we have* $M(\hat{s}) = 0$*. Moreover, since* $Z^{tc}(s) > s$ *on a left neighborhood of* $\hat{s}$ *and* $-\rho w_{aa} + w_{as} > 0$*, if* $M(s) \leq 0$ *on a left neighborhood of* $\hat{s}$*, by* (65) *and* (66)*, we would have* $V'(s) > U'(s)$ *for* s *in some left neighborhood of* $\hat{s}$*. However, this implies that* $V(s) < U(s)$ *for some* $s < \hat{s}$ *since* $V(\hat{s}) = U(\hat{s})$*, contradicting the optimality of* Z^{tc}*.*

Take an $s_1 < \hat{s}$*. For each* $n \geq 1$*, define* $s_{n+1} = Z^{tc}(s_n)$*. Choose* s_1 *sufficiently close to* $\hat{s}$ *such that* s_n *converges to* $\hat{s}$ *as* $n \to \infty$*. Since* $(Z^{tc})'(\hat{s}-) = 1$ *and* $Z^{tc}(\hat{s}) = \hat{s}$*, we have* $\lim_{n\to\infty} \frac{Z^{tc}(s_n)-Z^{tc}(\hat{s})}{s_n-\hat{s}} = \lim_{n\to\infty} \frac{\hat{s}-s_{n+1}}{\hat{s}-s_n} = 1$*. Moreover, since* (s_n) *is an increasing sequence that converges to* $\hat{s}$*, it follows that* $\lim_{n\to\infty} \frac{s_{n+1}-s_n}{\hat{s}-s_n} \to 0$*. Thus, for each* $\eta > 0$*, there is an* $\bar{n}_\eta < \infty$ *such that if* $n > \bar{n}_\eta$*, then* $s_{n+1} - s_n \in (0, \eta(\hat{s} - s_n))$*. Thus, if* $n > \bar{n}_\eta$*, then*

$$\int_{s_n}^{s_{n+1}} [-\rho w_{aa} + w_{as}](s_{n+1} - \rho x, x)dx \leq \eta(\hat{s} - s_n) \max_{x\in[s_n, s_{n+1}]} [-\rho w_{aa} + w_{as}](s_{n+1} - \rho x, x). \tag{67}$$

Meanwhile, for Z^{tc} *to be optimal, the following first-order condition must hold:*

$$\Delta(s_n) \equiv w_a(s_{n+1} - \rho s_n, s_n) + \delta[-\rho w_a + w_s](s_{n+2} - \rho s_{n+1}, s_{n+1}) = 0.$$

Using the fundamental theorem of calculus, $\Delta(s_n)$ *equals*

$$M(s_{n+1}) - \int_{s_n}^{s_{n+1}} [-\rho w_{aa} + w_{as}](s_{n+1} - \rho x, x)dx + \int_{s_n+1}^{s_{n+2}} [-\rho w_{aa} + w_{as}](x - \rho s_{n+1}, s_{n+1})dx. \tag{68}$$

Since we have shown that $M(\hat{s}) = 0$ *and* $M(s) > 0$ *on a left neighborhood of* $\hat{s}$*, we have* $M'(\hat{s}) < 0$ *generically. And since* $-\rho w_{aa} + w_{as} > 0$*, by expression* (68)*, we have*

$$\Delta(s_n) \geq -M'(s_n)(\hat{s} - s_n) - \eta(\hat{s} - s_n) \max_{x\in[s_n, s_{n+1}]} [-\rho w_{aa} + w_{as}](s_{n+1} - \rho x, x)$$

Let η *be sufficiently small so that* $-M'(\hat{s}) > \eta[-\rho w_{aa} + w_{as}](\hat{s} - \rho\hat{s}, \hat{s})$*. Thus, if* n *is sufficiently large, then* $\Delta(s_n) > 0$*, which contradicts* Z^{tc} *being optimal. The proof is complete.* ■

7.5.1 Naivety and bright-line rules

For simplicity, we focus on the case where the steady state of a time-consistent DM is interior. The case where her steady state is zero or $\bar{s}$ can be analyzed similarly, but involves a more detailed analysis for the corner solution. However, the proposition below still holds.

Proposition 7. *Assume that utility functions are quadratic and that the time-consistent DM's steady state is interior. Then, a naif never uses a bright-line rule.*

Proof. *Given a quadratic utility function, standard arguments show that the time-consistent DM's policy, Z_{tc}, is linear. Let Z_{nf} denote the optimal policy of a naive DM. Since she believes that from tomorrow on she will be time consistent, she believes that her continuation value function is that of a time-consistent consumer's, denoted V_{tc}. Thus, at stock s, we have*

$$Z_{nf}(s) \in \arg\max_{z \in [\rho s, y+\rho s]} w(z - \rho s, s) + \beta\delta V_{tc}(z).$$

Since Z_{tc} is continuous, we have

$$V'_{tc}(s) = [-\rho w_a + w_s](Z_{tc}(s) - \rho s, s).$$

Thus, if z is optimal and interior, then

$$w_a(z - \rho s, s) + \beta\delta[-\rho w_a + w_s](Z_{tc}(z) - \rho z, z) = 0. \tag{69}$$

Since Z_{tc}, w_a, and w_s are linear functions, there is a unique $z \in \mathbb{R}$, denoted as $\tilde{Z}_{nf}(s)$ that solves the above equation, and $\tilde{Z}_{nf}$ is linear. The naif's optimal policy, denoted as Z_{nf}, is then given by $Z_{nf}(s) = \min\{\rho s + y, \max\{\rho s, \tilde{Z}_{nf}(s)\}\}$. Thus, Z_{nf} does not feature a bright-line rule.

■