Player strength and effort in contests[*]

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Abstract

In competitive settings, disparities in player strength are common. It is intuitively unclear whether a stronger player would opt for larger or smaller effort compared to weaker players. Larger effort could leverage their strength, while lower effort might be justified by their higher probability of winning regardless of effort. We analyze multi-prize contests with three or more heterogeneous players, exploring when stronger players exert larger or lower effort. To rank efforts, it suffices to compare marginal utilities in situations where efforts are equal. Effort ranking depends on differences in hazard rates (which are smaller for stronger players) and reversed hazard rates (which are larger for stronger players). Compared to weaker players, stronger players choose larger effort in winner-takes-all contests and lower effort in loser-gets-nothing contests. Effort rankings can be non-monotonic in contests with several identical prizes, and they depend on the slopes of players' pdfs in contests with linear prize structure.

Keywords: contest theory, heterogeneity, player strength

JEL classification: C72, D74, D81

1 Introduction

In many contest applications, such as research competitions or competitions for promotion in firms, participants naturally differ in strength or ability. The contest theory literature has looked into different forms of player heterogeneity, such as different valuations for the contest prizes or different cost functions. These analyses have in common that they predict that players who are stronger or have some advantage over others will also choose higher effort compared to the other players. While this is certainly convincing in a range of applications, it seems that, intuitively, this prediction is not representative of all contest-like situations. We believe that weaker players compensating lower ability with higher effort in order to be competitive, or, equivalently, stronger players choosing lower effort as they have a higher probability to win at any effort level is, intuitively, at least as convincing a prediction.

In this paper, we model players' strength through their output or contribution to the contest. We show that, with this approach, stronger players choose either higher or lower effort than weaker players in equilibrium, in line with the intuition that both should be possible equilibrium outcomes. We investigate how the contest prize structure affects the ranking of effort between stronger and weaker players. In addition to deriving general results, we study a number of commonly known contest formats.

Many economic interactions can be modeled as a contest, where players exert effort to win one of several predetermined prizes. Research competitions, for example, involve contestants submitting proposals with winners obtaining grants or monetary prizes. In corporate settings, workers exert effort to outperform their peers in promotion competitions to earn higher wages. College applicants invest effort, e.g., crafting their essay, to gain admission to their preferred institutions. Finally, sporting events are obvious examples of contests.

While all of these examples can formally be modeled as a contest, there is one important difference: In some contests, the organizer wishes the high-ability players to exert high effort, whereas in others, it is important that the low-ability players work hard. In research competitions, often only the best project is implemented, in which case motivating the high-ability players benefits the organizer. In firms, however, poor performance can be very costly, for instance, when workers interact with important customers or handle sensitive data. Here, it might be more important to motivate low-ability workers to work hard and avoid making mistakes. In sports, high effort from strong athletes is desirable, but weaker players should not be discouraged from exerting effort in order to

maintain competitive balance and entertainment value. Summing up, for the contest organizer, understanding the conditions under which stronger players exert higher or lower effort is crucial.

As mentioned, in many contest models, player strength is modeled via their valuation for the prizes or, equivalently, via their effort cost function, with stronger players valuing the prizes more or having lower effort cost (e.g., Baye, Kovenock, and de Vries 1996, Fu and Wu 2020, and Fu, Wu, and Zhu 2023). In these models, stronger players always have a larger incentive to exert effort.[1]

To account for the possibility that weaker players choose larger effort than stronger players, we model player strength via players' output or performance in the contest. We do so by considering a Lazear-Rosen tournament with $n \geq 3$ heterogeneous players and a general prize structure, i.e., n prizes, all of which can be different.[2] Prizes are allocated based on the players' outputs, and a player's output is a function of the player's effort and an individual random variable. Strength is conceptualized through players' random variables, using common stochastic orders comparing the "magnitude" of the random variables. We assume that stronger players' random variables are greater both in the hazard rate order and the reversed hazard rate order.[3]

The analysis of the Lazear-Rosen tournament with n heterogeneous players and a general prize structure is challenging and, to the best of our knowledge, has not been done before.[4] We characterize the players' efforts by the first-order conditions of their maximization problems, and we find that these conditions depend on random variables distributed according to the Poisson binomial distribution. Except for some cases with a small number of players, there is no simple formula for the probability mass function of a Poisson binomial distribution. Thus, we encounter two problems. First, it is difficult to use the first-order conditions to rank the players' efforts, especially when the number of players is large. Second, it is unclear how (or if) these conditions can be related to player strength, i.e., commonly used definitions of stochastic order.

The main theoretical contribution of our paper is to overcome these problems. Con-

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[1]There exist some papers on the all-pay auction, where stronger players have a headstart advantage and choose lower expected effort compared to weaker players. See, e.g., Konrad (2002) and Seel (2014). We discuss the relation of these two papers to ours in more detail in Section 2.

[2]See Lazear and Rosen (1981). In the case of two (heterogeneous) players, there is a symmetric equilibrium, meaning that both players choose the same effort regardless of their strength (Bastani, Giebe, and Gürtler 2022).

[3]Note that both these relations are implied by dominance in the likelihood ratio order.

[4]Drugov and Ryvkin (2020) consider a Lazear-Rosen tournament akin to ours, but they assume homogeneous players.

cerning the first problem, we significantly simplify the analysis by showing that to rank the efforts of two players, it suffices to compare their marginal utilities for identical efforts. To address the second problem, we decompose the pairwise comparison of two players into three terms corresponding to whether the first player performs better than the second, they perform equally well, or the second player performs better than the first. When players perform equally well, the corresponding terms are equal and do not affect the effort ranking. In the other two cases, we derive terms that are symmetric, differing only in that either the reversed hazard rates or the hazard rates of the two players' distributions appear, implying a direct relation to player strength. Since the hazard rates are smaller for stronger players, but the reversed hazard rates are larger, the effects are countervailing. This means that, in general, it can be either the stronger or the weaker player who exerts larger effort.

In a next step, we study players' relative efforts in four common contest formats: the winner-takes-all contest (where a single prize is awarded to the best contestant), the loser-gets-nothing contest (where everyone except the worst performer receives the same prize), a contest with a certain number of identical prizes, and a contest with a linear prize schedule (where the difference between any two adjacent prizes is constant). We find that, in a winner-takes-all contest, stronger players always choose larger effort than weaker ones. The opposite is true in the loser-gets-nothing contest. In a contest with a certain number of identical prizes, the effort profile can be non-monotone in strength (meaning that workers of intermediate strength exert the largest effort). Finally, under a linear prize structure, the ranking of efforts depends on the slopes of the players' probability density functions.

The paper is organized as follows: The next section discusses related literature, while Section 3 describes the model. Section 4 characterizes the equilibrium and derives a condition for ranking players' efforts. Section 5 uses this condition to rank players' efforts in different types of contest. We present two model extensions in Section 6. Finally, Section 7 concludes. All proofs are relegated to the appendix.

2   Related literature

We contribute to the literature on multi-prize contests with heterogeneous players by considering a Lazear-Rosen contest with general prize structure and players of different

strengths.[5] In many contest models, player strength is modeled via a player's valuation of the prize (with stronger players valuing the prize relatively more) or, equivalently, the marginal effort cost (with stronger players having lower cost).[6] The result obtained from these models is that stronger players always exert larger effort than weaker ones (unless the stronger player's advantage is counterbalanced by some other forces such as handicaps).[7]

In contrast, our paper models player strength via players' output in the contest. In particular, we assume that output is stochastic and that, for a given level of effort, the distribution of output of stronger players dominates that of weaker ones. Heterogeneity with respect to players' output in the contest has been assumed in other contest models as well (e.g., Bastani, Giebe, and Gürtler 2022, Kirkegaard 2023a, Kirkegaard 2024), where some models use the state-space formulation (as we do in the current paper) and others the Mirrlees formulation (see Conlon 2009). One of the most closely related papers is Bastani, Giebe, and Gürtler (2022), who consider two-player contests with a single prize, in which the equilibrium is symmetric and both players exert the same effort. In contrast, we consider a contest with more than two players and allow for general prize structures. The equilibrium is no longer symmetric and we investigate the circumstances, under which stronger or weaker players exert relatively larger effort.

There exists a large literature studying the "Tullock contest", where a player's winning probability is given by an increasing function of their effort (referred to as their "impact function") divided by the sum of all players' impact functions. Heterogeneity in these contests is sometimes modeled by assuming that players have different impact functions (e.g., Fu and Wu, 2020, Fu, Wu, and Zhu, 2022). As shown by, e.g., Jia (2008) and Fu and Lu (2012), the Tullock contest is strategically equivalent to a contest where players' outputs depend on their efforts and a random variable, and prizes are awarded based on the ranking of outputs. This means that the Tullock contest sometimes represents a special case of our contest model and that our definition of player strength is consistent with the differences in the impact function studied in the literature. For example, in the concluding section of his paper, Jia (2008) allows for asymmetric distributions and shows how this results in a case with asymmetric impact functions that differ by a multiplicative constant. The distributions considered by Jia satisfy the mono-

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[5]Multi-prize all-pay auctions with heterogeneous players have been studied by, e.g., Clark and Riis (1998), Bulow and Levin (2006), Siegel (2010), González-Díaz and Siegel (2013), and Xiao (2016).

[6]See, e.g., Baye, Kovenock, and de Vries (1996), Fu and Wu (2020), and Fu, Wu, and Zhu (2023).

[7]When the contest game has a mixed-strategy equilibrium as in the all-pay auction, stronger players have a larger *expected* effort.

tone likelihood ratio property and thus relate to our definition of player strength. We relate our findings to Jia (2008) in Subsection 6.2.

A main contribution of our paper is to demonstrate that weaker players sometimes choose larger effort than stronger ones and to provide conditions under which this is the case. Kirkegaard (2023b) presents another contest model where weaker players sometimes choose larger effort than stronger ones. He examines a situation where higher effort increases the probability of drawing output from a "good" distribution rather than a "bad" one, though the distributions themselves do not depend on effort. The incentive to exert effort then hinges on how different the two distributions are. It is possible that this difference is larger for weaker players compared to stronger ones, thus providing them with a relatively stronger incentive to exert effort.

Konrad (2002) and Seel (2014) both study the all-pay auction, where stronger players have a headstart advantage and choose lower expected effort as a result. As the all-pay auction has only a mixed-strategy equilibrium, whereas our focus is on pure-strategy equilibria, the results differ significantly. Their result that weaker players may choose relatively larger expected effort is derived in a situation with two players. In contrast, the equilibrium would always be symmetric with two players in the setting that we are studying.

3   Model description

We consider a Lazear-Rosen tournament with $n \geq 3$ risk-neutral players who compete for n (finite) prizes that are ordered as $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$ (with at least one inequality being strict). All players $i \in \{1, \ldots, n\} =: \mathcal{N}$ simultaneously choose effort $e_i \geq 0$, and the cost of effort $c(e_i)$ is described by a twice continuously differentiable, strictly increasing, and strictly convex function satisfying $c(0) = c'(0) = 0$. Consequently, there exists $\bar{e} > 0$ such that $w_1 = c(\bar{e})$.

Apart from effort, player i's output in the contest depends on the realization θ_i of a random variable Θ_i. This random variable captures the strength of player i and we refer to its realization as i's skill.[8] The realization θ_i is unknown to all players, including player i. It is commonly known, however, that Θ_i is independently and absolutely continuously distributed according to the cdf F_{Θ_i} and pdf f_{Θ_i}.[9] We assume that, for all

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[8]In Subsection 6.2, we show that our results can also be applied to situations where player heterogeneity is modeled via different production functions rather than skill distributions.

[9]The random variables are also referred to as "noise" in the contest literature. Our model specification

$i \in \mathcal{N}$, f_{Θ_i} has convex support denoted by $\mathrm{supp}(f_{\Theta_i}) = \{x : f_{\Theta_i}(x) > 0\}$, with lower bound $a_i \in \mathbb{R} \cup \{-\infty\}$ and upper bound $b_i \in \mathbb{R} \cup \{+\infty\}$.

Player i's output is given by $g(\theta_i, e_i) = \theta_i + e_i$, and prizes are allocated according to the ranking of outputs.[10] That is, the player with the jth largest output receives prize w_j (for $j \in \{1, \ldots, n\}$). Ties are broken randomly.

We adopt a first-order approach and characterize the equilibrium through the players' first-order conditions to their maximization problems. The following set of assumptions ensures the existence of a pure-strategy Nash equilibrium determined by these first-order conditions (see Proposition 1).

Assumption 1. *i)* f_{Θ_i} *is continuously differentiable on* $[a_i, b_i]$ *for all* $i \in \mathcal{N}$.

ii) Both f_{Θ_i} *and* f'_{Θ_i} *are bounded for all* $i \in \mathcal{N}$.

iii) $a_i + \bar{e} < b_k$ *for all* $(i, k) \in \mathcal{N}^2$ *with* $i \neq k$.

iv) c *is sufficiently convex, i.e.,* $\inf_{e \in [0, \bar{e}]} c''(e)$ *is sufficiently large.*

Part i) of the assumption ensures that the players' objective functions are twice continuously differentiable. Parts ii) and iv) ensure that the objective functions are strictly concave. Finally, part iii) ensures that players' marginal return to exerting effort is always positive, thereby ruling out a corner solution at an effort level of zero.

Finally, we introduce some notation. Denote by $\mathbf{e}$ the vector of efforts and by $\mathbf{e}_{-i}$ the vector of efforts excluding e_i, i.e., $\mathbf{e}_{-i} = (e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)$. Player i's expected payoff is denoted by $U_i(\mathbf{e}) = \sum_{j=1}^{n} P_{ij}(\mathbf{e})w_j - c(e_i)$, where $P_{ij}(\mathbf{e})$ denotes the player's probability of receiving prize w_j. For a given effort e_i and skill θ_i of player i and given efforts $\mathbf{e}_{-i}$ (but unknown skill realizations) of the other players, $P_A^{i,s}(e_i, \theta_i, \mathbf{e}_{-i})$ denotes the probability that s of the players from set $A \subseteq \mathcal{N}$ perform better than i. Moreover, for $A \subseteq \mathcal{N}$, denote by A^C the set $\mathcal{N} \setminus A$. We denote equilibrium efforts with an asterisk.

includes the case $\Theta_i = t_i + \Gamma_i$, with t_i being a commonly known constant and $E[\Gamma_i] = 0$. This could be understood as player i's output depending on the commonly known expected skill t_i and the zero-mean random variable Γ_i which can be readily interpreted as noise.

[10]With a general production function $g(\theta_i, e_i)$, our assumptions are not sufficient to ensure the existence of a Nash equilibrium characterized by the players' first-order conditions to their maximization problems. For example, the second derivative of the expected prize may be positive and unbounded, implying that the cost function could never be sufficiently convex for the objective function to be concave. However, when such an equilibrium exists, all main results (i.e., Theorem 1 and Propositions 2 to 4) would be the same as in the case of an additive production function that we focus on in the paper. That is, the results do not depend on whether effort and skill are complements or substitutes. See Subsection 6.1 for details.

4 Equilibrium characterization

We start by considering the existence of a pure-strategy Nash equilibrium.

Proposition 1. *A pure-strategy Nash equilibrium* $\left(e_1^*, \ldots, e_n^*\right)$ *exists in which* e_i^* *($i \in \mathcal{N}$) is characterized by the first-order condition*

$$\sum_{j=1}^{n} \left(\int \left. \frac{\partial P_{\{i\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i})}{\partial e_i} \right|_{\mathbf{e}=\mathbf{e}^*} f_{\Theta_i}(x) \, dx \right) w_j = c'\left(e_i^*\right).$$

The first-order condition equates a player's marginal return to the marginal cost of effort. The marginal return from exerting effort is calculated as the weighted sum of the marginal effects of effort on the probability of achieving a certain rank, with the weights corresponding to the respective prizes. Player i receives prize w_j if he or she is outperformed by $j-1$ of the other players. For a given θ_i, the probability of i receiving w_j can thus be written as $P_{\{i\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i})$. To obtain the unconditional probability of receiving w_j, this probability must be averaged over all possible realizations of Θ_i. Multiplying with w_j and differentiating with respect to e_i then gives rise to the j-th element of the sum in the player's first-order condition.

Notice that, for a given vector of efforts and a given realization of Θ_i, player i's rank follows a Poisson binomial distribution with "success probabilities" $1 - F_{\Theta_l}(e_i + \theta_i - e_l)$ for all $l \neq i$.[11] Consequently, $P_{\{i\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i})$ is also derived from a Poisson binomial distribution. Except for some cases with a small number of draws (i.e., players), there is no simple formula for the probability mass function of a Poisson binomial distribution. Therefore, using the first-order condition from Proposition 1 to compare the efforts of different players is very difficult. Furthermore, it is not clear how (or whether) the conditions can be related to definitions of player strength.

As mentioned in the introduction, the main theoretical contribution of our paper is to overcome these problems. Regarding the first problem, we significantly simplify the analysis by showing that, to rank the efforts of two players, it suffices to compare their marginal utilities in a situation in which their efforts are the same (see Lemma 1). Regarding the second problem, we propose a decomposition of the resulting condition into three terms, corresponding to whether the first player outperforms the second, they perform equally, or the second player outperforms the first (see Theorem 1).

We begin by showing that, to rank the equilibrium efforts of two players i and k, it suffices to compare their marginal utilities in a situation in which their efforts are

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[11]To be precise, the rank equals 1 plus the realization of a random variable that follows this distribution.

the same. When both players exert same effort ($e_i = e_k = e$) and have the same skill realizations ($\theta_i = \theta_k = x$), the probabilities $1 - F_{\Theta_l}(e + x - e_l)$ are the same for the two players (for all $l \neq i, k$). This observation simplifies the comparison of the two players' equilibrium efforts significantly.

Lemma 1. *If* $\left.\frac{\partial U_i(\mathbf{e})}{\partial e_i}\right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} > \left.\frac{\partial U_k(\mathbf{e})}{\partial e_k}\right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*}$ *and c is sufficiently convex, then* $e_i^* > e_k^*$.

In what follows, we assume that the requirement regarding the convexity of c is fulfilled, and we focus on the condition $\left.\frac{\partial U_i(\mathbf{e})}{\partial e_i}\right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} > \left.\frac{\partial U_k(\mathbf{e})}{\partial e_k}\right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*}$. The following theorem decomposes this condition into different terms. We first present the theorem and then explain the decomposition in detail.

Theorem 1. *If*

$$
\sum_{j=1}^{n-1} \left(\int \left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)} \right) F_{\Theta_i}(x) F_{\Theta_k}(x) \left.\frac{\partial P_{\{i,k\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i})}{\partial e_i}\right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} dx \right) w_j >
$$

$$
\sum_{j=1}^{n-1} \left(\int \left(\frac{f_{\Theta_k}(x)}{1 - F_{\Theta_k}(x)} - \frac{f_{\Theta_i}(x)}{1 - F_{\Theta_i}(x)} \right) \left(1 - F_{\Theta_i}(x) \right) \left(1 - F_{\Theta_k}(x) \right) \left.\frac{\partial P_{\{i,k\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i})}{\partial e_i}\right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} dx \right) w_{j+1},
$$

then $e_i^* > e_k^*$.

Theorem 1 provides a condition under which a player i exerts greater effort than another player k. It is evident that the ranking of the reversed hazard rates of the two players' skill distributions, i.e., the sign of $\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)}$, and the ranking of the corresponding hazard rates, i.e., the sign of $\frac{f_{\Theta_k}(x)}{1 - F_{\Theta_k}(x)} - \frac{f_{\Theta_i}(x)}{1 - F_{\Theta_i}(x)}$, play a crucial role in determining the players' relative efforts.[12]

To derive this condition, we rewrite i's probability of getting prize w_j, "extracting" the probabilities of beating (or not beating) player k:

$$
P_{ij}(\mathbf{e}) = \int \left(F_{\Theta_k}(x + e_i - e_k) P_{\{i,k\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i}) + \left(1 - F_{\Theta_k}(x + e_i - e_k) \right) P_{\{i,k\}^C}^{i,j-2}(e_i, x, \mathbf{e}_{-i}) \right) f_{\Theta_i}(x) dx.
$$

From this we obtain a decomposition of player i's marginal winning probability (evalu-

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[12]Note that stating Theorem 1 in terms of (reversed) hazard rates prepares the discussion of player strength in Section 5.

ated at $e_k = e_i^*$ and $\mathbf{e}_{-k} = \mathbf{e}_{-k}^*$):

$$\frac{\partial P_{ij}(\mathbf{e})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} = \int f_{\Theta_i}(x) f_{\Theta_k}(x) \left(P_{\{i,k\}^C}^{i,j-1}\left(e_i^*,x,\mathbf{e}_{-i}^*\right) - P_{\{i,k\}^C}^{i,j-2}\left(e_i,x,\mathbf{e}_{-i}^*\right) \right) dx$$

$$+ \int f_{\Theta_i}(x) F_{\Theta_k}(x) \frac{\partial P_{\{i,k\}^C}^{i,j-1}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)}{\partial e_i} dx$$

$$+ \int f_{\Theta_i}(x)\left(1 - F_{\Theta_k}(x)\right) \frac{\partial P_{\{i,k\}^C}^{i,j-2}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)}{\partial e_i} dx.$$

The three lines above correspond to three events, as we explain in the following. A similar decomposition needs to be done for player k, with the perspectives of i and k interchanged.

In the first event (first line above), player k has the same skill x as player i. Here, the collision density $f_{\Theta_i}(x) f_{\Theta_k}(x)$ represents the "likelihood" that both players i and k have skill x, and the difference $\left(P_{\{i,k\}^C}^{i,j-1}\left(e_i^*,x,\mathbf{e}_{-i}^*\right) - P_{\{i,k\}^C}^{i,j-2}\left(e_i^*,x,\mathbf{e}_{-i}^*\right) \right)$ describes the increase in the probability of receiving prize w_j when player i marginally increases effort, thereby overtaking player k.

Now, notice that, if we determine the corresponding decomposition from player k's point of view, the first line is exactly the same as that just derived for player i. Accordingly, this first event does not play a role when comparing the two players' marginal utilities.

In the second event, player k has a skill strictly smaller than i's, described by the "likelihood" $f_{\Theta_i}(x) F_{\Theta_k}(x)$, and thus i beats k with certainty if they have equal efforts. In this case, i's marginal winning probability for prize w_j depends only on being the jth-best performer among all players except k:

$$f_{\Theta_i}(x) F_{\Theta_k}(x) \frac{\partial P_{\{i,k\}^C}^{i,j-1}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}.$$

Observe that $\frac{\partial P_{\{i,k\}^C}^{i,j-1}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} = \frac{\partial P_{\{i,k\}^C}^{k,j-1}(e_i,x,\mathbf{e}_{-i})}{\partial e_k}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}$, as both i and k have equal efforts, and we only look at players in $\{i,k\}^C$. Therefore, when comparing the two players' marginal utilities, the corresponding difference in terms amounts to

$$\left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)} \right) F_{\Theta_i}(x) F_{\Theta_k}(x) \frac{\partial P_{\{i,k\}^C}^{i,j-1}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*},$$

explaining why the difference in reversed hazard rates is relevant for the comparison of

marginal utilities.

Finally, in the third event, player k has a skill strictly larger than x. Analogous to the preceding case, this event is described by the "likelihood" that k beats i, $f_{\Theta_i}(x)\left(1-F_{\Theta_k}(x)\right)$, and similar to the second event, i's and k's marginal probability of $j-2$ other rivals outperforming them is identical for $e_i = e_k$.

When comparing the two players' marginal utilities, the corresponding difference in terms amounts to[13]

$$\left(\frac{f_{\Theta_k}(x)}{1-F_{\Theta_k}(x)} - \frac{f_{\Theta_i}(x)}{1-F_{\Theta_i}(x)}\right)\left(1-F_{\Theta_i}(x)\right)\left(1-F_{\Theta_k}(x)\right)\left.\frac{\partial P_{\{i,k\}^C}^{i,j-2}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*},$$

explaining why the difference in hazard rates is relevant for the comparison of marginal utilities.

It is possible to restate the condition in a different form, based on the differences between adjacent prizes rather than absolute prize levels. This alternative condition is presented in the next lemma. It simplifies the analysis in Section 5 because, for the first three prize structures (contest forms) that we consider in this section, all differences between adjacent prizes except one are equal to zero. For the fourth prize structure, the difference between adjacent prizes is constant and positive.

As differences between adjacent prizes are non-negative, the condition shows that the signs of the terms on the left-hand side and right-hand side are entirely determined by the ranking of reversed hazard rates and hazard rates, respectively.

Lemma 2. *The condition from Theorem 1 can be restated as*

$$\int \left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)}\right)F_{\Theta_i}(x)F_{\Theta_k}(x)\cdot$$
$$\left(\sum_{j=1}^{n-2}\left(\sum_{l_i\in\{i,k\}^C} f_{\Theta_{l_i}}\left(e_i^*+x-e_{l_i}^*\right)P_{\{i,k,l_i\}^C}^{i,j-1}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)\right)(w_j-w_{j+1})\right)dx$$
$$> \int \left(\frac{f_{\Theta_k}(x)}{1-F_{\Theta_k}(x)} - \frac{f_{\Theta_i}(x)}{1-F_{\Theta_i}(x)}\right)\left(1-F_{\Theta_i}(x)\right)\left(1-F_{\Theta_k}(x)\right)\cdot$$
$$\left(\sum_{j=1}^{n-2}\left(\sum_{l_i\in\{i,k\}^C} f_{\Theta_{l_i}}\left(e_i^*+x-e_{l_i}^*\right)P_{\{i,k,l_i\}^C}^{i,j-1}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)\right)(w_{j+1}-w_{j+2})\right)dx.$$

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[13]Notice that in Theorem 1, we have reindexed prizes on the right-hand side of the condition, such that only $P^{i,j-1}$ appears. The prize w_j corresponds to the summation index $j-1$ on the right-hand side of the inequality. See the proof of Theorem 1 for details.

5   Do stronger players exert greater effort?

In this section, we investigate in which situations stronger players exert greater effort than weaker ones. We begin by defining players' strength.

As player heterogeneity refers to the distributions of Θ, stronger players should have distributions that are more likely to yield large realizations. Therefore, a definition of player strength should relate to the "magnitude" of Θ. An obvious candidate is the likelihood ratio order, where player i is assumed to be stronger than player k if Θ_i is greater than Θ_k in the likelihood ratio order. However, for our results, the likelihood ratio order is unnecessarily strong. Instead, we adopt a weaker definition based on the hazard rate order and the reversed hazard rate order.[14]

Definition 1. *Player i is said to be stronger than k if Θ_i is greater than Θ_k in both the hazard rate order and the reversed hazard rate order, i.e., $\frac{f_{\Theta_i}(x)}{1-F_{\Theta_i}(x)} \leq \frac{f_{\Theta_k}(x)}{1-F_{\Theta_k}(x)}$ and $\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} \geq \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)}$ for all $x \in \mathbb{R}$.*

Throughout this section, we assume that all players are heterogeneous. By this, we mean that for any pair (i,k) of players, their distributions differ on a subset of $\text{supp}(f_{\Theta_i}) \cup \text{supp}(f_{\Theta_k})$ of positive measure. A direct implication is that whenever one player is stronger than another according to Definition 1, the inequalities between the hazard rates and reversed hazard rates, respectively, are strict on a subset of $\text{supp}(f_{\Theta_i}) \cup \text{supp}(f_{\Theta_k})$ of positive measure.

From Lemma 2, it is now evident that in general, either the stronger or the weaker player may exert relatively larger effort. The reason is that the hazard rates are smaller for stronger players, but the reversed hazard rates are larger, so that both sides of the inequality have the same sign.[15] To delve deeper into the ranking of efforts of players of different strength, we now consider different prize structures that have received considerable attention both in theoretical analyses and in practice, and we study players' relative efforts in these cases.

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[14]As shown in Theorem 1.C.1. in Shaked and Shanthikumar (2007), if Θ_i is greater than Θ_k in the likelihood ratio order, then it is also greater in both the hazard rate order and the reversed hazard rate order (and also in the usual stochastic order). That is, player i would be stronger than k according to Definition 1 if $f_{\Theta_i}/f_{\Theta_k}$ were non-decreasing on $\text{supp}(f_{\Theta_i}) \cup \text{supp}(f_{\Theta_k})$.

[15]Note that the prize w_1 appears only on the left-hand side and w_n only on the right-hand side of the inequality in Lemma 2. This implies that, if player i is stronger than player k, any increase in w_1 or in w_n makes the inequality easier to satisfy.

5.1 Winner takes all

The winner-takes-all contest is the one that is most extensively studied in the contest literature. It awards a single prize to the winner and nothing to the other players.[16] Setting $w_1 = w > 0$ and $w_2 = w_3 = \cdots = w_n = 0$, the condition from Lemma 2 can be stated as

$$\int \left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)} \right) F_{\Theta_i}(x) F_{\Theta_k}(x) \left(\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + x - e_{l_i}^* \right) P_{\{i,k,l_i\}^C}^{i,0} \left(e_i^*, x, \mathbf{e}_{-i}^* \right) \right) w \, dx > 0.$$

When player i performs worse than k, he or she has no chance of winning the prize, and there is no marginal incentive to exert effort. Therefore, in the above condition, all of the terms relating to the hazard rates have disappeared, leaving only the terms relating to the reversed hazard rates. As stronger players have larger reversed hazard rates than weaker ones, we obtain the following proposition.

Proposition 2. *In winner-takes-all contests, if player i is stronger than k, then equilibrium efforts satisfy $e_i^* > e_k^*$.*

We illustrate this result with an example.

Example 1. *Consider a winner-takes-all contest with $n = 3$ players, prize $w = 2$ and cost function $c(e) = 2e^2$. The players' skills are drawn from Exponential distributions with pdfs $f_{\Theta_i} = \lambda_i \exp(-\lambda_i x)$, $x \geq 0$, where $(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{2}, 1, \frac{3}{2} \right)$, i.e., the players are ordered by strength according to Definition 1 with player 1 being the strongest player. The contest has an equilibrium with effort vector $\mathbf{e}^* = (0.1475, 0.1334, 0.1068)$.*

5.2 Loser gets nothing

Next, we consider the loser-gets-nothing contest. In this type of contest, $n - 1$ prizes of equal size are awarded, meaning $w = w_1 = \cdots = w_{n-1} > w_n = 0$.[17] The condition from Lemma 2 can be stated as

$$\int \left(\frac{f_{\Theta_i}(x)}{1 - F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{1 - F_{\Theta_k}(x)} \right) \left(1 - F_{\Theta_i}(x) \right) \left(1 - F_{\Theta_k}(x) \right)$$
$$\cdot \left(\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + x - e_{l_i}^* \right) P_{\{i,k,l_i\}^C}^{i,n-3} \left(e_i^*, x, \mathbf{e}_{-i}^* \right) \right) w \, dx > 0.$$

---

[16] Winner-takes-all contests have been shown to be optimal in some settings. See, e.g., Moldovanu and Sela (2001) and Zhang (2024).

[17] A loser-gets-nothing contest can be optimal when the contest designer can select both the prize structure and the contest-success function, see Letina, Liu, and Netzer (2023).

When player i performs better than k, he or she secures a prize and there is no marginal incentive to exert effort. Therefore, in the above condition all of the terms relating to the reversed hazard rates have disappeared and only the terms relating to the hazard rate play a role. As stronger players have smaller hazard rates than weaker ones, we obtain the following proposition.

Proposition 3. *In loser-gets-nothing contests, if player i is stronger than k, then equilibrium efforts satisfy $e_i^* < e_k^*$.*

We illustrate this result with an example.

Example 2. *Consider a loser-gets-nothing contest with $n = 3$, prize vector $(w_1, w_2, w_3) = (1, 1, 0)$ and cost function $c(e) = e^2$. The players' skills are drawn from Exponential distributions with pdfs $f_{\Theta_i} = \lambda_i \exp(-\lambda_i x)$, $x \geq 0$, where $(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{2}, 1, \frac{3}{2}\right)$, i.e., the players are ordered by strength according to Definition 1 with player 1 being the strongest player. The contest has an equilibrium with effort vector $\mathbf{e}^* = (0.1951, 0.3080, 0.3385)$.*

5.3 $m \in \{2, \ldots, n-2\}$ identical prizes

The winner-takes-all and loser-gets-nothing contests are special cases of a contest in which a certain number of identical prizes are awarded. Now, we study the remaining cases of this type of contest assuming that the players compete for m identical prizes of value w, where $m \in \{2, \ldots, n-2\}$ and $n \geq 4$.[18] We observe that $w_j - w_{j+1} = 0$ for all $j \neq m$ and $w_m - w_{m+1} = w$. The condition from Lemma 2 then simplifies to

$$\int \left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)} \right) F_{\Theta_i}(x) F_{\Theta_k}(x)$$
$$\cdot \left(\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + x - e_{l_i}^* \right) P_{\{i,k,l_i\}^C}^{i,m-1} \left(e_i^*, x, \mathbf{e}_{-i}^* \right) \right) w \, dx$$
$$> \int \left(\frac{f_{\Theta_k}(x)}{1 - F_{\Theta_k}(x)} - \frac{f_{\Theta_i}(x)}{1 - F_{\Theta_i}(x)} \right) \left(1 - F_{\Theta_i}(x) \right) \left(1 - F_{\Theta_k}(x) \right)$$
$$\cdot \left(\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + x - e_{l_i}^* \right) P_{\{i,k,l_i\}^C}^{i,m-2} \left(e_i^*, x, \mathbf{e}_{-i}^* \right) \right) w \, dx.$$

It is difficult to derive a general result regarding the ranking of efforts. The main reason is that effort can now be non-monotonic in strength, meaning that players of intermediate strength may choose the largest effort. We now explain why this is the

---

[18]Such a prize structure is considered by, e.g., Siegel (2010) and Morgan, Tumlinson, and Várdy (2022), where the latter assume a continuum of players.

case. First, notice that $P^{i,m-1}_{\{i,k,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$ appears on the left-hand side of the inequality, whereas $P^{i,m-2}_{\{i,k,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$ appears on the right-hand side. This is because the left-hand side corresponds to the situation in which the other player, being compared to the considered player, performed worse, implying that the player would still win a prize even if $m-1$ of the remaining players performed better. In contrast, the right-hand side of the inequality corresponds to the situation in which the other player performed better, so that the considered player only wins a prize if at most $m-2$ of the remaining players perform better.

Recall that both $P^{i,m-2}_{\{i,k,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$ and $P^{i,m-1}_{\{i,k,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$ are calculated from the probability mass function of the Poisson binomial distribution. From Darroch (1964), we know that this function has either a unique mode or two consecutive ones, where such mode differs from the mean by less than 1.

Now, suppose that players are ordered by decreasing strength, and we are comparing the efforts of players 1 and 2. Let both these players be much stronger than the others. Given sufficiently large differences in strength, we observe $P^{1,m-2}_{\{1,2,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right) > P^{1,m-1}_{\{1,2,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$, as the mode of the corresponding probability mass function is low.[19] Then the right-hand side of the inequality from Lemma 2 tends to be larger than the left-hand side, implying that e_2^* tends to be larger than e_1^*.

Next, suppose we are comparing the efforts of players $n-1$ and n, assuming that these players are much weaker than the others. With a similar line of reasoning, e_n^* would tend to be smaller than e_{n-1}^*. As a result, a non-monotonic effort sequence would arise. The following example illustrates this.

Example 3. *Consider a contest with $n=4$ players, prize vector $(w_1,w_2,w_3,w_4)=(1,1,0,0)$ and cost function $c(e)=e^2$. The players' skills are drawn from Exponential distributions with pdfs $f_{\Theta_i} = \lambda_i \exp(-\lambda_i x)$, $x \geq 0$, where $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\left(\frac{1}{2},1,\frac{3}{2},2\right)$, i.e., players are ordered by strength according to Definition 1 with player 1 being the strongest player. The contest has an equilibrium with effort vector $\mathbf{e}^* = (0.1804, 0.2780, 0.3119, 0.3013)$ which is non-monotonic in strength.*

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[19]Given that the remaining players are much weaker than 1 and 2, their success probabilities are low. Accordingly, the mean number of successes is low as well, which implies a low mode by Darroch's rule for the mode.

5.4 Linear prize structure

Finally, we consider a linear prize structure, where the difference between adjacent prizes is constant. That is, we assume that $w_1 - w_2 = \cdots = w_{n-1} - w_n = w$ and $w_n = 0$.[20]

The condition from Lemma 2 can be restated as

$$
\int f_{\Theta_i}(x) \left(\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + x - e_{l_i}^* \right) \sum_{j=1}^{n-2} P_{\{i,k,l_i\}^C}^{i,j-1} \left(e_i^*, x, \mathbf{e}_{-i}^* \right) \right) dx
$$
$$
> \int f_{\Theta_k}(x) \left(\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + x - e_{l_i}^* \right) \sum_{j=1}^{n-2} P_{\{i,k,l_i\}^C}^{i,j-1} \left(e_i^*, x, \mathbf{e}_{-i}^* \right) \right) dx.
$$

As $\sum_{j=1}^{n-2} P_{\{i,k,l_i\}^C}^{i,j-1} \left(e_i^*, x, \mathbf{e}_{-i}^* \right) = 1$, the condition simplifies to

$$
\int f_{\Theta_i}(x) \left(\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + x - e_{l_i}^* \right) \right) dx > \int f_{\Theta_k}(x) \left(\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + x - e_{l_i}^* \right) \right) dx
$$
$$
\Leftrightarrow E_{\Theta_i} \left[\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + \Theta_i - e_{l_i}^* \right) \right] > E_{\Theta_k} \left[\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}} \left(e_i^* + \Theta_k - e_{l_i}^* \right) \right].
$$

When the prize structure is linear, player i's marginal incentive to exert effort does not depend on whether k performs better or worse, for the following reason. Suppose i performs better than k. Then he or she secures a prize of w and competes against the remaining $n-2$ players for an even larger prize. He or she gains w when beating one of the remaining players, $2w$ when beating two of them, and so on. Now suppose i performs worse than k. Then he or she does not secure any positive prize, but still competes against the remaining players for a positive reward. And, as in the first case, i would gain w by beating one of the remaining players, $2w$ by beating two of them and so on. That is, the structure of rewards in the competition against the players other than k is the same regardless of whether i performs better or worse than k. Consequently, the terms in the condition from Lemma 2 relating to the hazard and reversed hazard rates, respectively, disappear (formally, we add $f_{\Theta_i}(x) F_{\Theta_x}(x)$ and $f_{\Theta_i}(x)(1 - F_{\Theta_x}(x))$, which simplifies to f_{Θ_i}).

Suppose that i is stronger than k according to Definition 1. As Θ_i is greater than Θ_k in the usual stochastic order, the following proposition is immediate.

Proposition 4. *Consider a contest with a linear prize structure and let i be stronger than k.*

---

[20]In the literature, this is also referred to as an arithmetic prize sequence. See, e.g., Xiao (2016).

i) If, for all $(e_i, e_{l_i}) \in [0, \bar{e}]^2$, $\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}}(e_i + x - e_{l_i})$ is increasing in x on $\mathrm{supp}(f_{\Theta_i}) \cup \mathrm{supp}(f_{\Theta_k})$, then the equilibrium efforts satisfy $e_i^* > e_k^*$.

ii) If, for all $(e_i, e_{l_i}) \in [0, \bar{e}]^2$, $\sum_{l_i \in \{i,k\}^C} f_{\Theta_{l_i}}(e_i + x - e_{l_i})$ is decreasing in x on $\mathrm{supp}(f_{\Theta_i}) \cup \mathrm{supp}(f_{\Theta_k})$, then the equilibrium efforts satisfy $e_i^* < e_k^*$.

We illustrate Proposition 4 with an example.

Example 4. *Consider a contest with $n = 3$ players, linear prize structure $(w_1, w_2, w_3) = (2, 1, 0)$ and cost function $c(e) = 8e^2$. We assign the roles of player i and k to 1 and 3, respectively. Then $\{i, k\}^C = \{2\}$. We assume $f_1(x) = 2\left(x - \frac{1}{2}\right)$ on $\left[\frac{1}{2}, \frac{3}{2}\right]$, $f_2(x) = \frac{1}{4} + \frac{x}{4}$ on $[0, 2]$, $f_3(x) = 2 - 2\left(x - \frac{1}{2}\right)$ on $\left[\frac{1}{2}, \frac{3}{2}\right]$. This contest has an equilibrium with effort vector $\mathbf{e}^* = (0.07505, 0.0628, 0.0698)$, illustrating case i) of Proposition 4. If we change player 2's increasing density to the decreasing $f_2(x) = \frac{3}{4} - \frac{x}{4}$ on $[0, 2]$, we get an equilibrium with effort vector $\mathbf{e}^* = (0.0708, 0.0621, 0.0759)$, illustrating case ii) of Proposition 4.*

While Theorem 1 considers two players with different marginal utilities, the results should be transferable to the case where they are the same. That is, in the case of equal marginal utilities, we would expect the two players to choose the same effort, as an effort level fulfilling both players' first-order conditions would exist. The following example confirms this intuition for the case of a linear prize structure and a constant pdf for one of the players.

Example 5. *Consider a contest with $n = 3$ players, linear prize structure $(w_1, w_2, w_3) = (2, 1, 0)$ and cost function $c(e) = 8e^2$. Again, we assign the roles of player i and k to 1 and 3, respectively. We assume $f_1(x) = 2\left(x - \frac{1}{2}\right)$ on $\left[\frac{1}{2}, \frac{3}{2}\right]$, $f_2(x) = \frac{1}{2}$ on $[0, 2]$, $f_3(x) = 2 - 2\left(x - \frac{1}{2}\right)$ on $\left[\frac{1}{2}, \frac{3}{2}\right]$. The contest has an equilibrium with effort vector $\mathbf{e}^* = \left(\frac{7}{96}, \frac{1}{16}, \frac{7}{96}\right) \approx (0.0729, 0.0625, 0.0729)$. We have $e_1^* = e_3^*$ even though player 1 is stronger than player 3.*

5.5 A weaker definition of player strength

Our definition of player strength is relatively strict in that it is not always possible to rank two players in terms of their strength. The reason is that neither skill variable may be greater than the other in both the hazard rate order and the reversed hazard rate order. With a weaker definition of player strength, our results, unsurprisingly, do not necessarily hold.

It seems reasonable to require that any definition of strength should ensure that a stronger player has a larger expected skill than a weaker one, so ranking players

according to their expected skills (in case they are well-defined) amounts to a relatively weak definition of player strength.

In the following, we suppose that strength is defined by the ranking of expected values, and consider some of the implications.

Example 6 shows that equally strong players can have different efforts.

Example 6. *Consider a winner-takes-all contest with $n = 3$ players, prize $w = 1$ and cost function $c(e) = 2e^2$. The players' skills are drawn from Beta distributions with parameters α_i and β_i such that $(\alpha_1, \beta_1) = (1,1)$, $(\alpha_2, \beta_2) = (2,2)$, $(\alpha_3, \beta_3) = (3,3)$. The expected value is given by $\frac{\alpha_i}{\alpha_i + \beta_i}$, i.e., all three players have the same strength of $E[X] = \frac{1}{2}$ if defined by expected value. This contest has an equilibrium with effort vector $\mathbf{e}^* = (0.2476, 0.3001, 0.3045)$.*

In Example 7 we show that weaker players may choose larger effort than stronger ones in a winner-takes-all contest, contrary to Proposition 2. We do this by modifying Example 6, making player 1 (the player with the smallest equilibrium effort) slightly stronger than the other players, resulting in player 1's effort being (still) smaller than the other players' efforts.

Example 7. *Consider a winner-takes-all contest with $n = 3$ players, prize $w = 1$ and cost function $c(e) = 2e^2$. The players' skills are drawn from Beta distributions with parameters α_i and β_i such that $(\alpha_1, \beta_1) = \left(\frac{11}{10}, 1\right)$, $(\alpha_2, \beta_2) = (2,2)$, $(\alpha_3, \beta_3) = (3,3)$. The expected value is given by $\frac{\alpha_i}{\alpha_i + \beta_i}$, i.e., player 1 is stronger than the other two players if defined by expected value. This contest has an equilibrium with effort vector $\mathbf{e}^* = (0.2616, 0.2920, 0.2910)$.*

6 Extensions

In this section, we extend our model in two ways. First, we allow for more general production functions than the additive one considered in our main model. Second, we illustrate that our definition of strength can also encompass differences in the production technology rather than the skill distribution.

6.1 General production technology

With a general production technology, player i wins against k iff

$$g(\theta_i, e_i) > g(\theta_k, e_k),$$

where we assume that g is continuously differentiable, and increasing in θ_i for all $e_i > 0$.[21]

Define $g_e : \mathbb{R}_+ \to \mathbb{R}$ by $g_e(x) = g(x, e)$ and denote the inverse by g_e^{-1}. Player i then performs better than k iff

$$g_{e_k}^{-1}(g(\theta_i, e_i)) > \theta_k.$$

Accordingly, for a given θ_i, i performs better than k with probability $F_{\Theta_k}\left(g_{e_k}^{-1}(g(\theta_i, e_i))\right)$. This means that player i's probability of ending up in position m can be written as

$$
P_{im}(\mathbf{e}) = \int \Big[\Big(F_{\Theta_k}\big(g_{e_k}^{-1}(g(x, e_i))\big) \Big) P_{\{i,k\}^C}^{i,m-1}(e_i, x, \mathbf{e}_{-i})
$$
$$
+ \Big(1 - F_{\Theta_k}\big(g_{e_k}^{-1}(g(x, e_i))\big)\Big) P_{\{i,k\}^C}^{i,m-2}(e_i, x, \mathbf{e}_{-i}) \Big] f_{\Theta_i}(x) \, dx.
$$

With a general production technology, our assumptions are not sufficient to ensure the existence of a Nash equilibrium characterized by the players' first-order conditions to their maximization problems. For example, the second derivative of the expected prize may be positive and unbounded, implying that the cost function could never be sufficiently convex for the objective function to be concave. However, when such an equilibrium exists, all main results (i.e., Theorem 1 and Propositions 2 to 4) would be the same as before. This is shown in the remainder of this section.

First, notice that Lemma 1 does not depend on the production technology, so that it continues to hold for a general technology. The following proposition shows that Theorem 1 continues to hold as well.

Proposition 5. *Consider a general production technology $g(\theta_i, e_i)$. If an equilibrium exists that is characterized by the first-order conditions to the players' maximization problems and if*

$$
\sum_{j=1}^{n-1} \left(\int \left(F_{\Theta_i}(x) F_{\Theta_k}(x) \left. \frac{\partial P_{\{i,k\}^C}^{i,m-1}(e_i, x, \mathbf{e}_{-i})}{\partial e_i} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} \right) \left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)} \right) dx \right) w_j >
$$

$$
\sum_{j=1}^{n-1} \left(\int \left(\left(1 - F_{\Theta_i}(x)\right)\left(1 - F_{\Theta_k}(x)\right) \left. \frac{\partial P_{\{i,k\}^C}^{i,m-2}(e_i, x, \mathbf{e}_{-i})}{\partial e_i} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} \right) \left(\frac{f_{\Theta_k}(x)}{1 - F_{\Theta_k}(x)} - \frac{f_{\Theta_i}(x)}{1 - F_{\Theta_i}(x)} \right) dx \right) w_{j+1},
$$

then $e_i^* > e_k^*$.

---

[21]The assumption that g is only increasing in θ_i for positive e_i is made to include multiplicative production functions of the form $g(\theta_i, e_i) = \theta_i e_i$. Throughout this section, we assume that all efforts are positive (as they would be in an equilibrium characterized by the players' first-order conditions to their maximization problems).

The proposition generalizes Theorem 1 (keeping in mind that the "success probabilities" determining $P^{i,j}_{\{i,k\}^C}(e_i, x, \mathbf{e}_{-i})$ are now given by $1 - F_{\Theta_l}\left(g^{-1}_{e_l}(g(x, e_i))\right)$ rather than $1 - F_{\Theta_l}(e_i + x - e_l)$). As the theorem continues to hold and as the results from Section 5 are derived from the theorem, they continue to hold as well.

6.2   Differences in production technology

In our model, player heterogeneity referred to differences in the skill distributions. Our results, however, can also be applied to situations where players' skills are i.i.d. but the production technologies differ. We illustrate this by means of the Tullock contest.

Suppose that player i's production in the contest is given by the multiplicative function $g_i(\theta_i, e_i) = \delta_i \theta_i e_i$, where $\delta_i > 0$. Let the Θ_i be i.i.d. and the cdf be given by $F_\Theta(x) = \exp(-x^{-r})$ for $x > 0$ and $F_\Theta(x) = 0$ for $x \le 0$, with $r > 0$. Notice that the cdf corresponds to the one in condition (5) in Jia (2008). Accordingly, if δ_i were the same for all players and there were a single prize, the setting would correspond to a symmetric Tullock contest with discriminatory power r.

We allow for different δ_i though. Defining $\Psi_i := \delta_i \Theta_i$, one can show that the cdf of Ψ_i can be stated as $F_{\Psi_i}(x) = F_\Theta\left(\frac{x}{\delta_i}\right) = \exp\left(-\left(\frac{x}{\delta_i}\right)^{-r}\right)$ for $x > 0$ and $F_{\Psi_i}(x) = 0$ for $x \le 0$. Hence, for $x > 0$, we obtain $f_{\Psi_i}(x) = \exp\left(-\delta_i^r x^{-r}\right) r \delta_i^r x^{-r-1}$. This pdf corresponds to the one in condition (10) of Jia (2008). This implies that, in the case of a single prize, the model would now describe an asymmetric Tullock contest, where player i's probability of winning is given by $\frac{\delta_i^r e_i^r}{\sum_{l=1}^n \delta_l^r e_l^r}$.

Let $\delta_i > \delta_k$. It is very easy to show that Ψ_i is greater than Ψ_k in the likelihood ratio order. To see this, notice that the likelihood ratio is given by

$$\frac{f_{\Psi_i}}{f_{\Psi_k}}(x) = \frac{\delta_i^r}{\delta_k^r} \exp\left(-\left(\delta_i^r - \delta_k^r\right) x^{-r}\right).$$

As $\delta_i > \delta_k$, this expression is increasing in x.

Summing up, while we have considered a situation with different production technologies and i.i.d. skills, this model can be transformed into one with identical production technologies, but different skill distributions. The new skill distributions satisfy the monotone likelihood ratio property, meaning that a player with a larger δ_i would be stronger according to Definition 1. It follows that our previous results from Section 5 can be applied to this setting (if an equilibrium characterized by the players' first-order conditions to their maximization problems exists).

7 Conclusion

We have studied multi-prize contests with players of different strengths. We have derived a condition that allows us to rank the efforts of two players in a contest with general prize structure. The condition is symmetric except for the differences in the reversed hazard rates and the hazard rates of the two players' skill distributions, and can thus be related to the players' relative strength. Two players' efforts can be ranked if their strength can be compared according to our definition. This holds even if other players in the contest cannot be ranked by their strength. We have used the condition to compare players' efforts in four widely used contest formats.

While we have not investigated a contest-design problem, we can still derive some implications from our results for the designer's choice of prize structure. Whenever the designer wishes to ensure that the strongest players exert the largest effort, a winner-takes-all contest is an attractive option. In contrast, when the designer wishes to motivate the weaker players, a loser-gets-nothing contest could be preferred.

A Appendix

A.1 Proofs

For a given e_i, θ_i, and $\mathbf{e}_{-i}$ (but unknown skill realizations of players other than i), denote by $R_i(e_i, \theta_i, \mathbf{e}_{-i}) + 1$ a random variable that gives the final rank of player i (where rank 1 corresponds to prize w_1 etc.). Denote probabilities by P. We obtain the following lemma:

Lemma A.1. *Let $\hat{e}_i > \tilde{e}_i$ and $0 < F_{\Theta_j}(e_i + \theta_i - e_j) < 1$ for $e_i \in \{\hat{e}_i, \tilde{e}_i\}$, given θ_i and $\mathbf{e}_{-i}$, and all $j \in \{i\}^C$. Then $R_i(\hat{e}_i, \theta_i, \mathbf{e}_{-i})$ is smaller than $R_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i})$ in the usual stochastic order, that is, $P(R_i(\hat{e}_i, \theta_i, \mathbf{e}_{-i}) \le x) \ge P(R_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le x)$ for all $x \in \{0, \dots, n-1\}$. The inequality is strict for all $x \in \{0, \dots, n-2\}$.*

Proof of Lemma A.1. The random variable $R_i(e_i, \theta_i, \mathbf{e}_{-i})$ is distributed according to the Poisson binomial distribution with "success probabilities" $1 - F_{\Theta_j}(e_i + \theta_i - e_j)$. If $\hat{e}_i > \tilde{e}_i$ and $0 < F_{\Theta_j}(e_i + \theta_i - e_j) < 1$ for $e_i \in \{\hat{e}_i, \tilde{e}_i\}$, then $1 - F_{\Theta_j}(\hat{e}_i + \theta_i - e_j) < 1 - F_{\Theta_j}(\tilde{e}_i + \theta_i - e_j)$ for all $j \in \{i\}^C$. W.l.o.g., let $i = n$.

We prove the lemma by showing that, if we replace $F_{\Theta_j}(\tilde{e}_i + \theta_i - e_j)$ by $F_{\Theta_j}(\hat{e}_i + \theta_i - e_j)$ for *one* player $j \in \{n\}^C$, the resulting distribution of R_i is smaller than the original distribution in the usual stochastic order with the corresponding inequality being strict for

all $x \in \{0, \dots, n-2\}$. As a switch from $\tilde{e}_i$ to $\hat{e}_i$ leads to a change from $F_{\Theta_j}\left(\tilde{e}_i + \theta_i - e_j\right)$ to $F_{\Theta_j}\left(\hat{e}_i + \theta_i - e_j\right)$ for *all* players $j \in \{n\}^C$, the claim of the lemma follows from repeatedly applying the same argument for all $j \in \{n\}^C$.

We write $R_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) = Z_1 + \cdots + Z_{n-1}$, where the Z_j are independent Bernoulli variables with $P\left(Z_j = 1\right) = 1 - F_{\Theta_j}\left(\tilde{e}_i + \theta_i - e_j\right)$. We further define $R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) = Z_1 + \cdots + Z_{n-2}$. We obtain

$$
\begin{aligned}
& P\left(R_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y\right) \\
=\ & P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y \wedge Z_{n-1} = 0\right) + P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y - 1 \wedge Z_{n-1} = 1\right) \\
=\ & P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y\right) F_{\Theta_{n-1}}(\tilde{e}_i + \theta_i - e_{n-1}) \\
& + P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y - 1\right)\left(1 - F_{\Theta_{n-1}}(\tilde{e}_i + \theta_i - e_{n-1})\right) \\
=\ & P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y\right) F_{\Theta_{n-1}}(\tilde{e}_i + \theta_i - e_{n-1}) \\
& + \left(P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y\right) - P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) = y\right)\right)\left(1 - F_{\Theta_{n-1}}(\tilde{e}_i + \theta_i - e_{n-1})\right) \\
=\ & P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y\right) \\
& - P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) = y\right)\left(1 - F_{\Theta_{n-1}}(\tilde{e}_i + \theta_i - e_{n-1})\right).
\end{aligned}
$$

For $y = n-1$, we have $P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) = y\right) = 0$. In this case, we obtain $P\left(R_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y\right) = P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y\right) = 1$. For $y \le n-2$, we have $P\left(R'_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) = y\right) > 0$. Hence, all else equal, $P\left(R_i(\tilde{e}_i, \theta_i, \mathbf{e}_{-i}) \le y\right)$ is strictly increasing in $F_{\Theta_{n-1}}(\tilde{e}_i + \theta_i - e_{n-1})$. This means that, if we would replace $F_{\Theta_{n-1}}(\tilde{e}_i + \theta_i - e_{n-1})$ by $F_{\Theta_{n-1}}(\hat{e}_i + \theta_i - e_{n-1})$, the term would increase. $\qquad\square$

Proof of Proposition 1. We first show that a pure-strategy Nash equilibrium exists. The optimal e_i will always belong to the set $[0, \bar{e}]$, meaning that we can restrict attention to $(e_1, \dots, e_n) \in [0, \bar{e}]^n$ (which is compact and convex). By Rosen's theorem (Rosen, 1965, Vojnović, 2016, p.658), it remains to show that a player i's expected payoff is continuous and concave in e_i. Notice that i's expected payoff can be stated as

$$
U_i(\mathbf{e}) = \sum_{j=1}^{n} \left(\int P^{i,j-1}_{\{i\}^C}(e_i, x, \mathbf{e}_{-i}) f_{\Theta_i}(x)\, dx\right) w_j - c(e_i).
$$

By our assumptions, U_i is twice continuously differentiable and, thus, continuous. Moreover, it is strictly concave if

$$
\sum_{j=1}^{n} \left(\int \frac{\partial^2 P^{i,j-1}_{\{i\}^C}(e_i, x, \mathbf{e}_{-i})}{\partial e_i^2} f_{\Theta_i}(x)\, dx\right) w_j < c''(e_i),
$$

which is fulfilled for sufficiently convex c if $\int \frac{\partial^2 P^{i,j-1}_{\{i\}^C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i^2} f_{\Theta_i}(x)dx$ is bounded from above. We return to this issue at the end of the proof.

Next, we show that the optimal efforts are characterized by the players' first-order conditions to their maximization problems. By the concavity of the expected payoff functions, it suffices to rule out a corner solution at $e_i = 0$. Since $c'(0) = 0$, the optimal effort is positive if $\sum_{j=1}^n \left(\int \left. \frac{\partial P^{i,j-1}_{\{i\}^C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right|_{e_i=0} f_{\Theta_i}(x)dx \right) w_j > 0$.

Recall that $P^{i,j-1}_{\{i\}^C}(e_i,x,\mathbf{e}_{-i})$ is the probability that $j-1$ of player i's opponents perform better than i when i chooses effort e_i and has skill level $\theta_i = x$. The probability that a randomly selected player $l \neq i$ performs better than i can be stated as $1 - F_{\Theta_l}(e_i + x - e_l)$. As such, $P^{i,j-1}_{\{i\}^C}(e_i,x,\mathbf{e}_{-i})$ can be calculated by means of the Poisson binomial distribution with "success probabilities" $1 - F_{\Theta_l}(e_i + x - e_l)$. Notice that, by part iii) of Assumption 1, $F_{\Theta_l}(e_i + x - e_l) \in (0,1)$ for all $e_i, e_l \in [0,\bar{e}]$ and all $l \in \{i\}^C$. Accordingly, we can evoke Lemma A.1, according to which a marginal increase in effort leads to a rank distribution that is smaller in the usual stochastic order than the original distribution. As prizes are decreasing in the rank with at least one inequality strict, it follows that $\sum_{j=1}^n \left(\int \left. \frac{\partial P^{i,j-1}_{\{i\}^C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right|_{e_i=0} f_{\Theta_i}(x)dx \right) w_j > 0$.

It remains to be shown that $\int \frac{\partial^2 P^{i,j-1}_{\{i\}^C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i^2} f_{\Theta_i}(x)dx$ is bounded from above. Notice that $P^{i,j-1}_{\{i\}^C}(e_i,x,\mathbf{e}_{-i})$ can be written as a finite sum of terms of the form

$$\left(1 - F_{\Theta_{l_1}}\left(e_i + x - e_{l_1}\right)\right)\cdot...\cdot\left(1 - F_{\Theta_{l_{j-1}}}\left(e_i + x - e_{l_{j-1}}\right)\right)\cdot F_{\Theta_{l_j}}\left(e_i + x - e_{l_j}\right)\cdot...\cdot F_{\Theta_{l_{n-1}}}\left(e_i + x - e_{l_{n-1}}\right).$$

By the product differentiation rule, $\frac{\partial P^{i,j-1}_{\{i\}^C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}$ can be written as a finite sum of terms of the form

$$-\sum_{r=1}^{j-1} f_{\Theta_{l_r}}\left(e_i + x - e_{l_r}\right)\left(\prod_{s=1,s\neq r}^{j-1}\left(1 - F_{\Theta_{l_s}}\left(e_i + x - e_{l_s}\right)\right)\prod_{s=j}^{n-1} F_{\Theta_{l_s}}\left(e_i + x - e_{l_s}\right)\right)$$
$$+\sum_{r=j}^{n-1} f_{\Theta_{l_r}}\left(e_i + x - e_{l_r}\right)\left(\prod_{s=1}^{j-1}\left(1 - F_{\Theta_{l_s}}\left(e_i + x - e_{l_s}\right)\right)\prod_{s=j,s\neq r}^{n-1} F_{\Theta_{l_s}}\left(e_i + x - e_{l_s}\right)\right).$$

Accordingly, $\dfrac{\partial^2 P^{i,j-1}_{\{i\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i^2}$ can be written as a finite sum of terms of the form

$$-\sum_{r=1}^{j-1} f'_{\Theta_{l_r}}\left(e_i+x-e_{l_r}\right)\left(\prod_{s=1,s\neq r}^{j-1}\left(1-F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)\prod_{s=j}^{n-1}F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)$$

$$+\sum_{r=1}^{j-1}\sum_{t=1,t\neq r}^{j-1} f_{\Theta_{l_r}}\left(e_i+x-e_{l_r}\right)f_{\Theta_{l_t}}\left(e_i+x-e_{l_t}\right)\cdot$$

$$\left(\prod_{s=1,s\neq r,t}^{j-1}\left(1-F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)\prod_{s=j}^{n-1}F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)$$

$$-\sum_{r=1}^{j-1}\sum_{t=j}^{n-1} f_{\Theta_{l_r}}\left(e_i+x-e_{l_r}\right)f_{\Theta_{l_t}}\left(e_i+x-e_{l_t}\right)\cdot$$

$$\left(\prod_{s=1,s\neq r}^{j-1}\left(1-F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)\prod_{s=j,s\neq t}^{n-1}F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)$$

$$+\sum_{r=j}^{n-1} f'_{\Theta_{l_r}}\left(e_i+x-e_{l_r}\right)\left(\prod_{s=1}^{j-1}\left(1-F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)\prod_{s=j,s\neq r}^{n-1}F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)$$

$$-\sum_{r=j}^{n-1}\sum_{t=1}^{j-1} f_{\Theta_{l_r}}\left(e_i+x-e_{l_r}\right)f_{\Theta_{l_t}}\left(e_i+x-e_{l_t}\right)\cdot$$

$$\left(\prod_{s=1,s\neq t}^{j-1}\left(1-F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)\prod_{s=j,s\neq r}^{n-1}F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)$$

$$+\sum_{r=j}^{n-1}\sum_{t=j,t\neq r}^{n-1} f_{\Theta_{l_r}}\left(e_i+x-e_{l_r}\right)f_{\Theta_{l_t}}\left(e_i+x-e_{l_t}\right)\cdot$$

$$\left(\prod_{s=1}^{j-1}\left(1-F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right)\prod_{s=j,s\neq r,t}^{n-1}F_{\Theta_{l_s}}\left(e_i+x-e_{l_s}\right)\right).$$

Since both f_{Θ_i} and f'_{Θ_i} are bounded, it follows that $\dfrac{\partial^2 P^{i,j-1}_{\{i\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i^2}$ can be written as a finite sum of bounded terms. This implies that both $\dfrac{\partial^2 P^{i,j-1}_{\{i\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i^2}$ and $\int \dfrac{\partial^2 P^{i,j-1}_{\{i\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i^2}f_{\Theta_i}(x)\,dx$ are bounded as well, completing the proof. $\qquad\square$

Proof of Lemma 1. The proof is by way of contradiction, so suppose that $e_i^* \leq e_k^*$. In equilibrium, we have $\dfrac{\partial U_i(\mathbf{e})}{\partial e_i}\Big|_{\mathbf{e}=\mathbf{e}^*} = \dfrac{\partial U_k(\mathbf{e})}{\partial e_k}\Big|_{\mathbf{e}=\mathbf{e}^*} = 0$. Together with $\dfrac{\partial U_i(\mathbf{e})}{\partial e_i}\Big|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}^*_{-k}} > \dfrac{\partial U_k(\mathbf{e})}{\partial e_k}\Big|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}^*_{-k}}$ this rules out $e_i^* = e_k^*$, so that we can confine attention to the case $e_i^* < e_k^*$. We rewrite the equilibrium conditions as

$$\frac{\partial U_i(\mathbf{e})}{\partial e_i}\Big|_{\mathbf{e}=\mathbf{e}^*} + \frac{\partial U_i(\mathbf{e})}{\partial e_i}\Big|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}^*_{-k}} - \frac{\partial U_i(\mathbf{e})}{\partial e_i}\Big|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}^*_{-k}} = 0,$$

$$\frac{\partial U_k(\mathbf{e})}{\partial e_k}\Big|_{\mathbf{e}=\mathbf{e}^*} + \frac{\partial U_k(\mathbf{e})}{\partial e_k}\Big|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}^*_{-k}} - \frac{\partial U_k(\mathbf{e})}{\partial e_k}\Big|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}^*_{-k}} = 0,$$

which can be written as

$$\left.\frac{\partial U_i(\mathbf{e})}{\partial e_i}\right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} + \int_{e_i^*}^{e_k^*} \left.\frac{\partial^2 U_i(\mathbf{e})}{\partial e_i \partial e_k}\right|_{\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} de_k \;\; = \;\; 0,$$

$$\left.\frac{\partial U_k(\mathbf{e})}{\partial e_k}\right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} + \int_{e_i^*}^{e_k^*} \left.\frac{\partial^2 U_k(\mathbf{e})}{\partial e_k^2}\right|_{\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} de_k \;\; = \;\; 0.$$

Notice that $\frac{\partial^2 U_i(\mathbf{e})}{\partial e_i \partial e_k}$ does not depend on the cost function c. Thus, if c is sufficiently convex, i.e., if $\inf_{e\in[0,\bar{e}]} c''(e)$ is sufficiently large, then $\frac{\partial^2 U_i(\mathbf{e})}{\partial e_i \partial e_k} > \frac{\partial^2 U_k(\mathbf{e})}{\partial e_k^2}$ for all $\mathbf{e} \in [0,\bar{e}]^n$. Together with $\left.\frac{\partial U_i(\mathbf{e})}{\partial e_i}\right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} > \left.\frac{\partial U_k(\mathbf{e})}{\partial e_k}\right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}$ this implies that the two equilibrium conditions cannot be simultaneously fulfilled, leading to the desired contradiction.

$\square$

Proof of Theorem 1. Player i's probability of ending up in position m can be written as

$$P_{im}(\mathbf{e}) = \int \left(F_{\Theta_k}(x + e_i - e_k) P_{\{i,k\}^C}^{i,m-1}(e_i, x, \mathbf{e}_{-i}) + \left(1 - F_{\Theta_k}(x + e_i - e_k)\right) P_{\{i,k\}^C}^{i,m-2}(e_i, x, \mathbf{e}_{-i}) \right) f_{\Theta_i}(x)\, dx,$$

where $P_{\{i,k\}^C}^{i,j}(e_i, x, \mathbf{e}_{-i}) = 0$ for $j \in \{-1, n-1\}$. Differentiating with respect to e_i leads to

$$\frac{\partial P_{im}(\mathbf{e})}{\partial e_i} = \int \left(f_{\Theta_k}(x + e_i - e_k) P_{\{i,k\}^C}^{i,m-1}(e_i, x, \mathbf{e}_{-i}) + F_{\Theta_k}(x + e_i - e_k) \frac{\partial P_{\{i,k\}^C}^{i,m-1}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right) f_{\Theta_i}(x)\, dx$$

$$+ \int \left(-f_{\Theta_k}(x + e_i - e_k) P_{\{i,k\}^C}^{i,m-2}(e_i, x, \mathbf{e}_{-i}) + \left(1 - F_{\Theta_k}(x + e_i - e_k)\right) \frac{\partial P_{\{i,k\}^C}^{i,m-2}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right) f_{\Theta_i}(x)\, dx$$

$$= \int f_{\Theta_i}(x) f_{\Theta_k}(x + e_i - e_k) \left(P_{\{i,k\}^C}^{i,m-1}(e_i, x, \mathbf{e}_{-i}) - P_{\{i,k\}^C}^{i,m-2}(e_i, x, \mathbf{e}_{-i}) \right) dx$$

$$+ \int \frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} F_{\Theta_i}(x) F_{\Theta_k}(x + e_i - e_k) \frac{\partial P_{\{i,k\}^C}^{i,m-1}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\, dx$$

$$+ \int \frac{f_{\Theta_i}(x)}{1 - F_{\Theta_i}(x)} \left(1 - F_{\Theta_i}(x)\right) \left(1 - F_{\Theta_k}(x + e_i - e_k)\right) \frac{\partial P_{\{i,k\}^C}^{i,m-2}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\, dx.$$

Recall that player i's expected payoff is given by $U_i(\mathbf{e}) = \sum_{j=1}^{n} P_{ij}(\mathbf{e}) w_j - c(e_i)$. The

marginal utility can be written as

$$
\frac{\partial U_i(\mathbf{e})}{\partial e_i}
$$

$$
= \sum_{j=1}^{n} \left(\int f_{\Theta_i}(x) f_{\Theta_k}(x + e_i - e_k) \left(P_{\{i,k\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i}) - P_{\{i,k\}^C}^{i,j-2}(e_i, x, \mathbf{e}_{-i}) \right) dx \right) w_j
$$

$$
+ \sum_{j=1}^{n} \left(\int \frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} F_{\Theta_i}(x) F_{\Theta_k}(x + e_i - e_k) \frac{\partial P_{\{i,k\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i})}{\partial e_i} dx \right) w_j
$$

$$
+ \sum_{j=1}^{n} \left(\int \frac{f_{\Theta_i}(x)}{1 - F_{\Theta_i}(x)} \left(1 - F_{\Theta_i}(x) \right) \left(1 - F_{\Theta_k}(x + e_i - e_k) \right) \frac{\partial P_{\{i,k\}^C}^{i,j-2}(e_i, x, \mathbf{e}_{-i})}{\partial e_i} dx \right) w_j - c'(e_i).
$$

Hence,

$$
\left. \frac{\partial U_i(\mathbf{e})}{\partial e_i} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*}
$$

$$
= \sum_{j=1}^{n} \left(\int f_{\Theta_i}(x) f_{\Theta_k}(x) \left(P_{\{i,k\}^C}^{i,j-1}(e_i^*, x, \mathbf{e}_{-i}^*) - P_{\{i,k\}^C}^{i,j-2}(e_i^*, x, \mathbf{e}_{-i}^*) \right) dx \right) w_j
$$

$$
+ \sum_{j=1}^{n} \left(\int \frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} F_{\Theta_i}(x) F_{\Theta_k}(x) \left. \frac{\partial P_{\{i,k\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i})}{\partial e_i} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} dx \right) w_j
$$

$$
+ \sum_{j=1}^{n} \left(\int \frac{f_{\Theta_i}(x)}{1 - F_{\Theta_i}(x)} \left(1 - F_{\Theta_i}(x) \right) \left(1 - F_{\Theta_k}(x) \right) \left. \frac{\partial P_{\{i,k\}^C}^{i,j-2}(e_i, x, \mathbf{e}_{-i})}{\partial e_i} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} dx \right) w_j - c'(e_i^*),
$$

implying that

$$
\left. \frac{\partial U_i(\mathbf{e})}{\partial e_i} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} - \left. \frac{\partial U_k(\mathbf{e})}{\partial e_k} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*}
$$

$$
= \sum_{j=1}^{n} \left(\int f_{\Theta_i}(x) f_{\Theta_k}(x) \left(P_{\{i,k\}^C}^{i,j-1}(e_i^*, x, \mathbf{e}_{-i}^*) - P_{\{i,k\}^C}^{i,j-2}(e_i^*, x, \mathbf{e}_{-i}^*) \right) dx \right) w_j
$$

$$
- \sum_{j=1}^{n} \left(\int f_{\Theta_i}(x) f_{\Theta_k}(x) \left(P_{\{i,k\}^C}^{k,j-1}(e_i^*, x, \mathbf{e}_{-k}^*) - P_{\{i,k\}^C}^{k,j-2}(e_i^*, x, \mathbf{e}_{-k}^*) \right) dx \right) w_j
$$

$$
+ \sum_{j=1}^{n} \left(\int \frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} F_{\Theta_i}(x) F_{\Theta_k}(x) \left. \frac{\partial P_{\{i,k\}^C}^{i,j-1}(e_i, x, \mathbf{e}_{-i})}{\partial e_i} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} dx \right) w_j
$$

$$
- \sum_{j=1}^{n} \left(\int \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)} F_{\Theta_k}(x) F_{\Theta_i}(x) \left. \frac{\partial P_{\{i,k\}^C}^{k,j-1}(e_k, x, \mathbf{e}_{-k})}{\partial e_k} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} dx \right) w_j
$$

$$
+ \sum_{j=1}^{n} \left(\int \frac{f_{\Theta_i}(x)}{1 - F_{\Theta_i}(x)} \left(1 - F_{\Theta_i}(x) \right) \left(1 - F_{\Theta_k}(x) \right) \left. \frac{\partial P_{\{i,k\}^C}^{i,j-2}(e_i, x, \mathbf{e}_{-i})}{\partial e_i} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} dx \right) w_j
$$

$$
- \sum_{j=1}^{n} \left(\int \frac{f_{\Theta_k}(x)}{1 - F_{\Theta_k}(x)} \left(1 - F_{\Theta_k}(x) \right) \left(1 - F_{\Theta_i}(x) \right) \left. \frac{\partial P_{\{i,k\}^C}^{k,j-2}(e_k, x, \mathbf{e}_{-k})}{\partial e_k} \right|_{e_k = e_i^*, \mathbf{e}_{-k} = \mathbf{e}_{-k}^*} dx \right) w_j.
$$

Notice that, for $z \in \{j-2, j-1\}$, we have $P^{i,z}_{\{i,k\}^C}\left(e^*_i, x, \mathbf{e}^*_{-i}\right) = P^{k,z}_{\{i,k\}^C}\left(e^*_i, x, \mathbf{e}^*_{-k}\right)$ and

$$\left.\frac{\partial P^{i,z}_{\{i,k\}^C}(e_i, x, \mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} = \left.\frac{\partial P^{k,z}_{\{i,k\}^C}(e_k, x, \mathbf{e}_{-k})}{\partial e_k}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} \quad . \text{ Thus, the difference in marginal}$$

utilities becomes

$$\left.\frac{\partial U_i(\mathbf{e})}{\partial e_i}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} - \left.\frac{\partial U_k(\mathbf{e})}{\partial e_k}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}}$$

$$= \sum_{j=1}^{n}\left(\int \left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)}\right) F_{\Theta_i}(x) F_{\Theta_k}(x) \left.\frac{\partial P^{i,j-1}_{\{i,k\}^C}(e_i, x, \mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} dx\right) w_j$$

$$+ \sum_{j=1}^{n}\left(\int \left(\frac{f_{\Theta_i}(x)}{1-F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{1-F_{\Theta_k}(x)}\right) \left(1-F_{\Theta_i}(x)\right) \left(1-F_{\Theta_k}(x)\right) \left.\frac{\partial P^{i,j-2}_{\{i,k\}^C}(e_i, x, \mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} dx\right) w_j.$$

As $P^{i,j}_{\{i,k\}^C}(e_i, x, \mathbf{e}_{-i}) = 0$ for $j \in \{-1, n-1\}$, we can rewrite this as

$$\left.\frac{\partial U_i(\mathbf{e})}{\partial e_i}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} - \left.\frac{\partial U_k(\mathbf{e})}{\partial e_k}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}}$$

$$= \sum_{j=1}^{n-1}\left(\int \left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)}\right) F_{\Theta_i}(x) F_{\Theta_k}(x) \left.\frac{\partial P^{i,j-1}_{\{i,k\}^C}(e_i, x, \mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} dx\right) w_j$$

$$+ \sum_{j=1}^{n-1}\left(\int \left(\frac{f_{\Theta_i}(x)}{1-F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{1-F_{\Theta_k}(x)}\right) \left(1-F_{\Theta_i}(x)\right) \left(1-F_{\Theta_k}(x)\right) \left.\frac{\partial P^{i,j-1}_{\{i,k\}^C}(e_i, x, \mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} dx\right) w_{j+1}.$$

The claim from the theorem then follows from an application of Lemma 1. $\qquad \square$

Proof of Lemma 2. The condition from Theorem 1 can be restated as

$$\int \left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)}\right) F_{\Theta_i}(x) F_{\Theta_k}(x) \left(\sum_{j=1}^{n-1} \left.\frac{\partial P^{i,j-1}_{\{i,k\}^C}(e_i, x, \mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} w_j\right) dx$$

$$> \int \left(\frac{f_{\Theta_k}(x)}{1-F_{\Theta_k}(x)} - \frac{f_{\Theta_i}(x)}{1-F_{\Theta_i}(x)}\right) \left(1-F_{\Theta_i}(x)\right) \left(1-F_{\Theta_k}(x)\right) \cdot$$

$$\left(\sum_{j=1}^{n-1} \left.\frac{\partial P^{i,j-1}_{\{i,k\}^C}(e_i, x, \mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e^*_i, \mathbf{e}_{-k}=\mathbf{e}^*_{-k}} w_{j+1}\right) dx.$$

Thus, it suffices to show that

$$
\left.\frac{\partial P^{i,j-1}_{\{i,k\}^C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}
$$

$$
= \; -\sum_{l_i\in\{i,k\}^C} f_{\Theta_{l_i}}\left(e_i^*+x-e_{l_i}^*\right) P^{i,j-2}_{\{i,k,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)
$$

$$
+ \sum_{l_i\in\{i,k\}^C} f_{\Theta_{l_i}}\left(e_i^*+x-e_{l_i}^*\right) P^{i,j-1}_{\{i,k,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)
$$

for $j\in\{1,\ldots,n-1\}$, where $P^{i,-1}_{\{i,k,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)=0$. For $j=1$, we have $P^{i,j-1}_{\{i,k\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)=$ $P^{i,0}_{\{i,k\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)=F_{\Theta_{l_1}}\left(e_i^*+x-e_{l_1}^*\right)\cdot\ldots\cdot F_{\Theta_{l_{n-2}}}\left(e_i^*+x-e_{l_{n-2}}^*\right)$. By the product differentiation rule, we obtain

$$
\left.\frac{\partial P^{i,0}_{\{i,k\}^C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}
$$

$$
= \sum_{l_i\in\{i,k\}^C} f_{\Theta_{l_i}}\left(e_i^*+x-e_{l_i}^*\right)\prod_{l_j\in\{i,k,l_i\}^C} F_{\Theta_{l_j}}\left(e_i^*+x-e_{l_j}^*\right)
$$

$$
= \sum_{l_i\in\{i,k\}^C} f_{\Theta_{l_i}}\left(e_i^*+x-e_{l_i}^*\right) P^{i,0}_{\{i,k,l_i\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right).
$$

Next, let $j\in\{2,\ldots,n-1\}$. As there are $n-2$ players besides i and k, $P^{i,j-1}_{\{i,k\}^C}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$ contains $\binom{n-2}{j-1}$ summands, each of which contains $n-2$ factors. Each summand is of the form $\left(1-F_{\Theta_{l_1}}\left(e_i^*+x-e_{l_1}^*\right)\right)\cdot\ldots\cdot\left(1-F_{\Theta_{l_{j-1}}}\left(e_i^*+x-e_{l_{j-1}}^*\right)\right)\cdot F_{\Theta_{l_j}}\left(e_i^*+x-e_{l_j}^*\right)\cdot\ldots\cdot F_{\Theta_{l_{n-2}}}\left(e_i^*+x-e_{l_{n-2}}^*\right)$, and the $\binom{n-2}{j-1}$ summands capture all permutations of the $n-2$ players in $\{i,k\}^C$.

Accordingly, $\left.\frac{\partial P^{i,j-1}_{\{i,k\}^C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}$ contains $\binom{n-2}{j-1}\cdot(n-2)$ summands, each of which contains $n-2$ factors. Each summand is either of the form

$$
-f_{\Theta_{l_1}}\left(e_i^*+x-e_{l_1}^*\right)\cdot\left(1-F_{\Theta_{l_2}}\left(e_i^*+x-e_{l_2}^*\right)\right)\cdot\ldots\cdot\left(1-F_{\Theta_{l_{j-1}}}\left(e_i^*+x-e_{l_{j-1}}^*\right)\right)\cdot
$$
$$
F_{\Theta_{l_j}}\left(e_i^*+x-e_{l_j}^*\right)\cdot\ldots\cdot F_{\Theta_{l_{n-2}}}\left(e_i^*+x-e_{l_{n-2}}^*\right)
$$

or of the form

$$
\left(1-F_{\Theta_{l_1}}\left(e_i^*+x-e_{l_1}^*\right)\right)\cdot\ldots\cdot\left(1-F_{\Theta_{l_{j-1}}}\left(e_i^*+x-e_{l_{j-1}}^*\right)\right)\cdot
$$
$$
f_{\Theta_{l_j}}\left(e_i^*+x-e_{l_j}^*\right)\cdot F_{\Theta_{l_{j+1}}}\left(e_i^*+x-e_{l_{j+1}}^*\right)\cdot\ldots\cdot F_{\Theta_{l_{n-2}}}\left(e_i^*+x-e_{l_{n-2}}^*\right),
$$

and the $\binom{n-2}{j-1}\cdot(n-2)$ summands again capture all permutations of the $n-2$ players. In particular, $\frac{j-1}{n-2}\cdot\binom{n-2}{j-1}\cdot(n-2)=(j-1)\cdot\binom{n-2}{j-1}$ summands are of the first form, and $\frac{n-2-(j-1)}{n-2}\cdot\binom{n-2}{j-1}\cdot(n-2)=(n-1-j)\cdot\binom{n-2}{j-1}$ are of the second form. As all of these summands are

symmetric, the share of summands in which the pdf of a particular player in $\{i,k\}^C$ (say l_1) occurs, is $\frac{1}{n-2}$. This means that the pdf of this particular player occurs in $\frac{j-1}{n-2}\cdot\binom{n-2}{j-1}$ summands of the first form and in $\frac{n-1-j}{n-2}\cdot\binom{n-2}{j-1}$ summands of the second form.

The first form of summand contains $j-2$ "successes", and if we focus on player l_1, the $\frac{j-1}{n-2}\cdot\binom{n-2}{j-1}$ summands corresponding to this player capture all permutations of the $n-3$ players in $\{i,k,l_1\}^C$. Notice that $\frac{j-1}{n-2}\cdot\binom{n-2}{j-1}=\frac{(j-1)(n-2)!}{(n-2)(j-1)!(n-1-j)!}=\frac{(n-3)!}{(j-2)!(n-1-j)!}=\binom{n-3}{j-2}$. As all permutations of players in $\{i,k,l_1\}^C$ are considered and as $P_{\{i,k,l_1\}^C}^{i,j-2}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$ contains $\binom{n-3}{j-2}$ summands, the sum of the $\frac{j-1}{n-2}\cdot\binom{n-2}{j-1}$ terms of the first form can be written as $-f_{\Theta_{l_1}}\left(e_i^*+x-e_{l_1}^*\right)P_{\{i,k,l_1\}^C}^{i,j-2}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$.

Likewise, each player in $\{i,k\}^C$ corresponds to $\frac{n-1-j}{n-2}\cdot\binom{n-2}{j-1}$ summands of the second form and they each contain $j-1$ "successes". The $\frac{n-1-j}{n-2}\cdot\binom{n-2}{j-1}$ summands again capture all permutations of the $n-3$ players in $\{i,k,l_1\}^C$. Notice that $\frac{n-1-j}{n-2}\cdot\binom{n-2}{j-1}=\frac{(n-1-j)(n-2)!}{(n-2)(j-1)!(n-1-j)!}=\frac{(n-3)!}{(j-1)!(n-2-j)!}=\binom{n-3}{j-1}$. As all permutations of players in $\{i,k,l_1\}^C$ are considered and as $P_{\{i,k,l_1\}^C}^{i,j-1}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$ contains $\binom{n-3}{j-1}$ summands, the sum of the $\frac{n-1-j}{n-2}\cdot\binom{n-2}{j-1}$ terms of the second form can be written as $f_{\Theta_{l_1}}\left(e_i^*+x-e_{l_1}^*\right)P_{\{i,k,l_1\}^C}^{i,j-1}\left(e_i^*,x,\mathbf{e}_{-i}^*\right)$.

The proof of the lemma then follows by applying the same argument to all other players. $\qquad\square$

Proof of Proposition 5. Recall that player i's probability of ending up in position m can be written as

$$P_{im}(\mathbf{e})=\int\left(F_{\Theta_k}\left(g_{e_k}^{-1}(g(x,e_i))\right)P_{\{i,k\}^C}^{i,m-1}(e_i,x,\mathbf{e}_{-i})\right.$$
$$\left.+\left(1-F_{\Theta_k}\left(g_{e_k}^{-1}(g(x,e_i))\right)\right)P_{\{i,k\}^C}^{i,m-2}(e_i,x,\mathbf{e}_{-i})\right)f_{\Theta_i}(x)dx.$$

Differentiating with respect to e_i yields

$$\frac{\partial P_{im}(\mathbf{e})}{\partial e_i}$$
$$=\int\left(\frac{f_{\Theta_k}\left(g_{e_k}^{-1}(g(x,e_i))\right)}{g_{e_k}'\left(g_{e_k}^{-1}(g(x,e_i))\right)}\frac{\partial g(x,e_i)}{\partial e_i}P_{\{i,k\}^C}^{i,m-1}(e_i,x,\mathbf{e}_{-i})+F_{\Theta_k}\left(g_{e_k}^{-1}(g(x,e_i))\right)\frac{\partial P_{\{i,k\}^C}^{i,m-1}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\right)f_{\Theta_i}(x)dx$$
$$+\int\left(-\frac{f_{\Theta_k}\left(g_{e_k}^{-1}(g(x,e_i))\right)}{g_{e_k}'\left(g_{e_k}^{-1}(g(x,e_i))\right)}\frac{\partial g(x,e_i)}{\partial e_i}P_{\{i,k\}^C}^{i,m-2}(e_i,x,\mathbf{e}_{-i})+\left(1-F_{\Theta_k}\left(g_{e_k}^{-1}(g(x,e_i))\right)\right)\frac{\partial P_{\{i,k\}^C}^{i,m-2}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\right)f_{\Theta_i}(x)dx.$$

It follows that

$$\frac{\partial P_{im}(\mathbf{e})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} - \frac{\partial P_{km}(\mathbf{e})}{\partial e_k}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}$$

$$= \int \left(\frac{f_{\Theta_k}(x)}{g'_{e_i^*}(x)\frac{\partial g(x,e_i)}{\partial e_i}\big|_{e_i=e_i^*}} P_{\{i,k\}^C}^{i,m-1}\left(e_i^*,x,\mathbf{e}_{-i}^*\right) + F_{\Theta_k}(x)\frac{\partial P_{\{i,k\}^C}^{i,m-1}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) f_{\Theta_i}(x)\,dx$$

$$+ \int \left(-\frac{f_{\Theta_k}(x)}{g'_{e_i^*}(x)\frac{\partial g(x,e_i)}{\partial e_i}\big|_{e_i=e_i^*}} P_{\{i,k\}^C}^{i,m-2}\left(e_i^*,x,\mathbf{e}_{-i}^*\right) + \left(1-F_{\Theta_k}(x)\right)\frac{\partial P_{\{i,k\}^C}^{i,m-2}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) f_{\Theta_i}(x)\,dx$$

$$- \int \left(\frac{f_{\Theta_i}(x)}{g'_{e_i^*}(x)\frac{\partial g(x,e_k)}{\partial e_k}\big|_{e_k=e_i^*}} P_{\{i,k\}^C}^{k,m-1}\left(e_i^*,x,\mathbf{e}_{-k}^*\right) + F_{\Theta_i}(x)\frac{\partial P_{\{i,k\}^C}^{k,m-1}(e_k,x,\mathbf{e}_{-k})}{\partial e_k}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) f_{\Theta_k}(x)\,dx$$

$$- \int \left(-\frac{f_{\Theta_i}(x)}{g'_{e_i^*}(x)\frac{\partial g(x,e_k)}{\partial e_k}\big|_{e_k=e_i^*}} P_{\{i,k\}^C}^{k,m-2}\left(e_i^*,x,\mathbf{e}_{-k}^*\right) + \left(1-F_{\Theta_i}(x)\right)\frac{\partial P_{\{i,k\}^C}^{k,m-2}(e_k,x,\mathbf{e}_{-k})}{\partial e_k}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) f_{\Theta_k}(x)\,dx.$$

Now notice that

$$\frac{f_{\Theta_k}(x)}{g'_{e_i^*}(x)\frac{\partial g(x,e_i)}{\partial e_i}\big|_{e_i=e_i^*}} P_{\{i,k\}^C}^{i,m-1}\left(e_i^*,x,\mathbf{e}_{-i}^*\right) f_{\Theta_i}(x) = \frac{f_{\Theta_i}(x)}{g'_{e_i^*}(x)\frac{\partial g(x,e_k)}{\partial e_k}\big|_{e_k=e_i^*}} P_{\{i,k\}^C}^{k,m-1}\left(e_i^*,x,\mathbf{e}_{-k}^*\right) f_{\Theta_k}(x)$$

and

$$\frac{f_{\Theta_k}(x)}{g'_{e_i^*}(x)\frac{\partial g(x,e_i)}{\partial e_i}\big|_{e_i=e_i^*}} P_{\{i,k\}^C}^{i,m-2}\left(e_i^*,x,\mathbf{e}_{-i}^*\right) f_{\Theta_i}(x) = \frac{f_{\Theta_i}(x)}{g'_{e_i^*}(x)\frac{\partial g(x,e_k)}{\partial e_k}\big|_{e_k=e_i^*}} P_{\{i,k\}^C}^{k,m-2}\left(e_i^*,x,\mathbf{e}_{-k}^*\right) f_{\Theta_k}(x).$$

The above difference in marginal probabilities thus simplifies to

$$\frac{\partial P_{im}(\mathbf{e})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} - \frac{\partial P_{km}(\mathbf{e})}{\partial e_k}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}$$

$$= \int \left(F_{\Theta_k}(x)\frac{\partial P_{\{i,k\}^C}^{i,m-1}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) f_{\Theta_i}(x)\,dx$$

$$+ \int \left(\left(1-F_{\Theta_k}(x)\right)\frac{\partial P_{\{i,k\}^C}^{i,m-2}(e_i,x,\mathbf{e}_{-i})}{\partial e_i}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) f_{\Theta_i}(x)\,dx$$

$$- \int \left(F_{\Theta_i}(x)\frac{\partial P_{\{i,k\}^C}^{k,m-1}(e_k,x,\mathbf{e}_{-k})}{\partial e_k}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) f_{\Theta_k}(x)\,dx$$

$$- \int \left(\left(1-F_{\Theta_i}(x)\right)\frac{\partial P_{\{i,k\}^C}^{k,m-2}(e_k,x,\mathbf{e}_{-k})}{\partial e_k}\bigg|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) f_{\Theta_k}(x)\,dx,$$

which can be rewritten as

$$\int \left(F_{\Theta_i}(x) F_{\Theta_k}(x) \left. \frac{\partial P^{i,m-1}_{\{i,k\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) \frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} dx$$

$$- \int \left(F_{\Theta_i}(x) F_{\Theta_k}(x) \left. \frac{\partial P^{k,m-1}_{\{i,k\}C}(e_k,x,\mathbf{e}_{-k})}{\partial e_k} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)} dx$$

$$+ \int \left(\left(1-F_{\Theta_i}(x)\right)\left(1-F_{\Theta_k}(x)\right) \left. \frac{\partial P^{i,m-2}_{\{i,k\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) \frac{f_{\Theta_i}(x)}{1-F_{\Theta_i}(x)} dx$$

$$- \int \left(\left(1-F_{\Theta_i}(x)\right)\left(1-F_{\Theta_k}(x)\right) \left. \frac{\partial P^{k,m-2}_{\{i,k\}C}(e_k,x,\mathbf{e}_{-k})}{\partial e_k} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) \frac{f_{\Theta_k}(x)}{1-F_{\Theta_k}(x)} dx.$$

Notice that $\left. \dfrac{\partial P^{i,m-1}_{\{i,k\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} = \left. \dfrac{\partial P^{k,m-1}_{\{i,k\}C}(e_k,x,\mathbf{e}_{-k})}{\partial e_k} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}$ and

$\left. \dfrac{\partial P^{i,m-2}_{\{i,k\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} = \left. \dfrac{\partial P^{k,m-2}_{\{i,k\}C}(e_k,x,\mathbf{e}_{-k})}{\partial e_k} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}$. We can thus write the preceding

expression as

$$\left. \frac{\partial P_{im}(\mathbf{e})}{\partial e_i} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} - \left. \frac{\partial P_{km}(\mathbf{e})}{\partial e_k} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*}$$

$$= \int \left(F_{\Theta_i}(x) F_{\Theta_k}(x) \left. \frac{\partial P^{i,m-1}_{\{i,k\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) \left(\frac{f_{\Theta_i}(x)}{F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{F_{\Theta_k}(x)} \right) dx$$

$$+ \int \left(\left(1-F_{\Theta_i}(x)\right)\left(1-F_{\Theta_k}(x)\right) \left. \frac{\partial P^{i,m-2}_{\{i,k\}C}(e_i,x,\mathbf{e}_{-i})}{\partial e_i} \right|_{e_k=e_i^*,\mathbf{e}_{-k}=\mathbf{e}_{-k}^*} \right) \left(\frac{f_{\Theta_i}(x)}{1-F_{\Theta_i}(x)} - \frac{f_{\Theta_k}(x)}{1-F_{\Theta_k}(x)} \right) dx.$$

$\square$

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