Profiling and Hold-up Risk

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October 2025

Abstract

We study how buyer profiling affects market outcome when valuations are endogenous. A buyer can invest to raise his valuation for a seller's good at a privately known cost, and the seller observes a signal about this cost before setting her price. More informative signals improve the seller's ability to predict the buyer's investment decision and extract trade surplus but also undermine the buyer's incentive to invest due to hold-up risk. We analyze this trade-off and characterize the set of feasible investment and welfare outcomes across all signal structures. The buyer and seller are aligned in their preference for maximizing profiling accuracy for high-cost types, while profiling accuracy for low-cost types determines the split of the investment surplus. Shifting welfare from the seller to the buyer necessarily reduces aggregate investment.

Keywords: Hold-up; Profiling; Private investment cost.
JEL Classification: D42, D82, D83.

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1 Introduction

In many economic relationships, one party must decide whether to make a costly, unverifiable investment before transacting with a counterpart who has bargaining power. Such situations give rise to the classic hold-up problem: once the investment cost is sunk, the investor becomes vulnerable to opportunistic pricing and risks losing the surplus generated by the investment. For example, a consumer may spend time learning how to use a brand's product line or integrating his smart devices into a single ecosystem, but he risks facing higher prices after becoming habituated or locked into that brand.[1] A business may reconfigure its workflows to adapt to a particular platform or service provider, but it risks facing higher fees after becoming operationally dependent.[2]

Classic analyses of hold-up focus on complete-information environments, where the central friction arises from incomplete contracts. In many real-world settings, however, the investing party also holds private information that affects his decision to invest, and the investment itself is often unobservable to the counterpart. For example, consumers differ in their privately known costs of learning a brand's product or integrating into its ecosystem, depending on prior experience and technological aptitude. Whether they have exerted this effort is typically unobserved by the seller. Similarly, a business's cost of integration depends on its operational complexity, which is rarely observed and understood by outsiders. Vendors also have limited information about whether their clients are attempting to integrate as opposed to multi-homing or developing in-house substitutes. In this paper, we study how information asymmetry about the investment cost affects hold-up risks and welfare.

Seminal works such as Akerlof (1970), Myerson and Satterthwaite (1983), and Bergemann et al. (2015) have shown how the information environment affects trade outcomes and the division of surplus. What is distinctive in settings with hold-up risk is that information also affects the *size* of the surplus by shaping incentives to invest. When the noninvesting counterpart possesses information about factors influencing investment decisions, she can use it to predict the investing party's action and extract surplus. However, for precisely this reason, the availability of such information to the noninvesting party also increases hold-up risk and discourages the investing party from investing in the first place.

[1] Murray and Häubl (2007) document how consumers develop skill-based habits and cognitive lock-in through repeated use of a product, which amplifies brand dependence.

[2] For instance, a Wall Street Journal article in 2024 reports how companies avoid relying on a single cybersecurity provider due to the cost of "vendor lock-in." As one analyst notes, "As you [...] invest further, it gets harder and harder to get out [...] Vendors can essentially increase their prices without customers having a choice, because the switching cost is very high."

Given this trade-off, how should profiling—which generates information about factors affecting investment incentives—be designed to let the noninvesting party extract trade surplus while still preserving the investor's incentive to invest? If the objective is instead to protect the investor, should profiling be banned altogether, or can the investor benefit from allowing the counterpart to possess information that influences his investment decision? If the objective is to maximize social welfare, should the information policy favor the investing or the noninvesting party? And across these objectives, which types of investors are more or less likely to be profiled accurately? We address these questions in this paper.

We analyze a model in which a buyer (he) initially values a seller's (her) product at v_0 and can privately pay a cost c to invest in his valuation before meeting the seller. The buyer's post-investment valuation is a private draw from a commonly known distribution with support bounded below by v_0. Although the return to investment is stochastic, the buyer's investment is modeled as an "all-or-nothing" decision. This specification fits our motivating examples, where the relevant consideration is not how much to invest, but whether to cross a qualitative threshold. For example, a consumer learning a brand's system can either reach fluency or remain a casual user. A smart-device user can either go "all in" and synchronize all his devices or remain a user of multiple brands. A business can either undertake the costly migration to a single vendor or keep its operations modular.

The buyer privately knows his investment cost c, while the seller profiles the buyer by observing a private signal about c. The signal structure that determines the mapping from c to signals is public knowledge. We examine how profiling affects both investment incentives and the division of trade surplus by deriving all the payoffs that are feasibly attained in equilibrium across all possible signal structures.

This approach is motivated by firms' increasing use of data analytics to profile their clients and how advances in data technology have made it possible for such profiling to be designed flexibly. For example, a seller can control and commit to her profiling mechanism through the design of its login or sign-up process, such as deciding what information to request at registration, whether to track usage histories, and how to combine data across services. Platforms can also control information flows by limiting the granularity of data accessible to third-party sellers, and regulators can also exert influence through regulations governing how firms may collect, link, and use consumer data. Together, these technological and institutional controls create substantial scope for variation in the information that a seller can obtain about its potential buyers.

When the buyer's valuation is *exogenously* determined, giving the seller information about the buyer's private type always (at least weakly) improves the seller's ability to extract surplus—with full information, she can engage in first-degree price discrimination and extract all the gains from trade. In contrast, in our model, information about the buyer's private type can be detrimental to the seller's profit. As we show in Lemma 1 and Lemma 2, full information yields the same profit for the seller as no information. Intuitively, if the seller believes that the buyer has not invested, she charges the low price v_0; if she believes that the buyer has invested, she ratchets up the price. Hence, the buyer retains his investment surplus only when he is mis-profiled as a type not expected to invest in equilibrium. The seller thus faces a trade-off: profiling accuracy improves surplus extraction but weakens investment incentives. If the seller's signal about c is too informative, the buyer protects himself by pooling investment behavior across types to hinder the seller's inference. Consequently, the seller's information about c becomes uninformative about whether the buyer has invested.

Our main analysis is in Section 4. In Lemma 4, we first show that every attainable Pareto-optimal outcome can be attained in an equilibrium described by two cutoffs, ϕ and τ, with $\phi < \tau$. Types with $c \leq \tau$ invest, whereas types with $c > \tau$ do not. Among the investing types, types with $c \leq \phi$ obtain all of the investment surplus—they are always mis-profiled as high-cost noninvestor types. On the other hand, types with $c \in (\phi, \tau]$ have their investment surplus fully captured by the seller—these types are mis-profiled with just enough probability to maintain their investment incentive. In Proposition 1, we derive the bounds on these two cutoffs that are sustainable in equilibrium, thereby characterizing the Pareto frontier. In Proposition 2, we characterize all the signal structures that can attain each Pareto-optimal outcome. In Proposition 3, we provide the trade-off between buyer welfare and seller profit along the Pareto frontier. In Proposition 4, we characterize all other attainable (non-Pareto-optimal) outcomes, showing that any payoff outcome between the Pareto frontier and the payoff outcome under a complete hold-up is attainable.

To highlight some of the qualitative implications of our results, we find that seller-optimality involves *minimizing* profiling error for every type to what is just necessary to sustain the buyer's equilibrium investment strategy. This is intuitive: greater profiling accuracy enables the seller to extract more of the trade surplus. In contrast, buyer-optimality *maximizes* profiling error for low-cost types but still minimizes it for high-cost types. This latter feature shows a partial alignment of interests: both parties prefer high profiling accuracy for high-cost types, whereas their preferences are opposite for low-cost types.

To see why the buyer also benefits from higher profiling accuracy for high-cost types, note first that a noninvesting type's utility is always zero, regardless of the profiling error for him.[3] Since high-cost types do not invest in equilibrium, increasing profiling accuracy for them does not affect ex-ante buyer welfare, but it allows the signal structure to introduce greater profiling error for low-cost (investing) types, while still maintaining the overall profiling accuracy needed for the seller to follow the signal's recommendation. Because an investing type's utility strictly increases with the profiling error for him, reallocating profiling error from noninvesting to investing types raises the buyer's ex-ante expected utility.

These implications shed new light on ongoing concerns about consumer data protection. Policymakers often raise two concerns regarding sellers gaining access to granular consumer data: (i) such information enables personalized pricing, thereby shifting surplus from consumers to sellers, and (ii) data disclosure has externalities, such as spillover learning and privacy loss.[4] Our paper highlights another concern: consumers' value from consumption can often be improved by them taking certain costly actions, and sellers' access to consumer data also affects consumers' incentive to invest in those actions. Our results show that firms may voluntarily commit to limit their data collection—thus partially addressing privacy concerns—but such restraint does not always benefit buyer welfare. Furthermore, both sides can benefit from higher profiling accuracy for consumers with high costs of improving their valuations—in particular, improving a seller's ability to identify such consumers generates positive externalities for low-cost consumers by allowing them to face lower prices more often.[5] More broadly, while privacy remains an important concern for all forms of data collection, our results suggest a distinction across types of information. Data that primarily reveal high-cost consumers are less likely to adversely affect value creation, whereas data that reveal low-cost consumers pose a greater threat to consumer welfare.

1.1 Related Literature

Our paper is related to the literature on how information policies shape market outcomes. Bergemann et al. (2015) characterizes the set of feasible payoffs across all signal structures about the buyer's valuation in a monopoly. Elliot et al. (2025) extend the consideration to oligopoly, and Ichihashi and Smolin (2025) study environments where the seller is privately, but imperfectly, informed. Roesler and Szentes (2017) analyze buyer-optimal learning about

[3]Since the lowest possible valuation is v_0, the seller's price is never below v_0. Thus, a type who does not invest either buys at his valuation or takes his zero outside option, both yielding zero utility.

[4]See, for example, reports by the Council of Economic Advisers (2015), the European Commission (2018), and the Federal Trade Commission (2025).

[5]We discuss issues related to welfare distribution across the consumer population in Section 5.2.

his own valuation in a monopoly, and Armstrong and Zhou (2022) analyze the feasible payoff outcomes under duopoly. Kartik and Zhong (2025) examine information policies for both the seller and the buyer in environments with correlated adverse selection. Compared to these papers, hold-up risk in our setting implies that the information policy affects not only trade probability and surplus division but also the size of the surplus itself.

Several papers study how information policies affect investment incentives in the hold-up problem. Gul (2001), Lau (2008), and Nguyen and Tan (2019) consider investments with deterministic returns, whereas Hermalin and Katz (2009), Halac (2015), and Condorelli and Szentes (2020) consider investments with stochastic returns. In these papers, there is no (ex-ante) asymmetric information about the investment cost; rather, the information policy generates (ex-post) information about the realized value of investment. Such information design exercises implicitly require a party—possibly a mediator—to "monitor" the investor's action and commit to the disclosure rule about it, which is arguably less plausible for individual consumers.[6] We study a different but complementary aspect of how information shapes outcomes in environments with hold-up risks. Our focus is on contexts where sellers profile individual consumers to gather information useful only for predicting whether a consumer will invest in his valuation, but the seller has no means to monitor the consumer's action.

More broadly, our paper is also related to Bayesian persuasion (Kamenica and Gentzkow, 2011). In Bayesian persuasion, the signal structure influences the receiver's action through its signal realization. In our model, the signal structure *itself* also influences a player's action (i.e., the buyer's investment decision). This feature is also present in models where the state about which the signal structure generates information is endogenously determined by a player—for example, see Zapechelnyuk (2020), Boleslavsky and Kim (2021), and Tan (2023). In these models, the signal structure generates ex-post information about an endogenous state. In contrast, in our model, the signal structure generates information about exogenous private information. Ichihashi (2025) also studies a model of this kind, analyzing profiling technologies on criminal types that deter crime under an enforcement resource constraint.

Finally, our paper is also related to the literature on auctions where the bidders can invest to increase their valuations (e.g., Piccione and Tan (1996), Tan (1996), and Gershkov et al. (2021)). This literature examines how the auction/mechanism design affects investment incentives, whereas we study the effects of information design on ex-ante private information.

[6]Such commitment can be motivated by organizational design in a firm or contracting on the inspection frequency of facilities in business-to-business relationships.

2 The Model

A risk-neutral buyer (he) has unit demand for a good produced by a risk-neutral seller (she) at zero marginal cost. Upon meeting the buyer, the seller offers him a take-it-or-leave-it price. Both parties have zero outside options.

The buyer's initial valuation for the good is $v_0 > 0$. He can invest in this valuation by paying a cost c to draw a new valuation v from a commonly known CDF χ (to be described below). The cost c is the buyer's private information—hereafter referred to as the buyer's type—and is distributed according to a commonly known CDF F with positive density f on support $C := [\underline{c}, \bar{c}]$, where $\underline{c} > 0$. As is standard, we model hold-up risk by assuming that the buyer must make his investment decision *before* meeting the seller, and the cost c (if incurred) cannot be recovered, even if the buyer later decides not to buy the good. The idea is to capture the possibility that the seller may alter the price *after* the buyer has made a sunk investment, without modeling the precise timing of such price changes.

If the seller can observe the realization of v after the buyer has invested, the classic hold-up outcome arises regardless of the buyer's type. The seller always charges the buyer's valuation, and since investing is costly, the buyer optimally chooses not to invest.[7] In equilibrium, the buyer's utility is zero, and the seller's profit is v_0. We refer to these payoffs as the *hold-up payoffs*. Henceforth, we assume that the seller *cannot* observe the realization of v after the buyer has invested.

The valuation CDF χ has support on the interval $[v_0, \bar{v}]$, with $v_0 < \bar{v} < \infty$. The problem $\max_p p[1 - \chi(p)]$ has a unique solution, denoted by r, which we assume satisfies $r > v_0$.[8] If the seller knows that the buyer has not invested, she offers the price $p = v_0$, whereas if she knows that the buyer has invested, she $r$atchets up the price to r. However, because the seller does not know the buyer's realized valuation v from investing, the buyer earns an expected information rent $\alpha := \int_{v_0}^{\bar{v}} \max\{v - r, 0\} d\chi(v)$.

Assumption 1. *$\alpha < \underline{c} < \bar{c} < \gamma$, where $\gamma := E[v] - v_0$.*

γ is the expected gain in valuation from investing. The inequality $\bar{c} < \gamma$ implies that investing is always socially efficient. The inequality $\alpha < \underline{c}$ implies that every type still faces hold-up risk even when the seller cannot observe the outcome of the investment, because the expected rent α is always insufficient to cover the buyer's investment cost. This assumption also implies that if the seller can observe whether the buyer has invested (but not the realized

[7] Throughout, we assume (without loss of generality) that the buyer still buys the good when indifferent.

[8] All our results readily extend to the case in which χ is degenerate (i.e., deterministic investment).

valuation), the equilibrium is unique: every type of buyer still chooses not to invest, and both the buyer and the seller obtain their hold-up payoffs.[9] Henceforth, we assume that the seller *cannot* observe the buyer's investment decision.

We model profiling by introducing an initial stage in which the seller learns about the buyer's type through a *privately* observed signal drawn from a commonly known signal structure $\{\mathbb{S}, \pi\}$.[10] Here, $\mathbb{S}$ denotes the set of possible signals, and $\pi_s(c)$ is the probability that a type-c buyer generates signal $s \in \mathbb{S}$.[11]

Since the seller privately observes the signal and the buyer privately chooses whether to invest, it is without loss of generality to assume that the two players choose their strategies simultaneously. Given a signal structure, the seller's strategy specifies a distribution over prices after observing signal s, while the buyer's strategy maps each type c to a probability of investing. The equilibrium concept is Bayes-Nash equilibrium.

Our objective is to study how the seller's information about the buyer's investment cost affects hold-up risk. In particular, we characterize the set of payoff outcomes that are feasible under some signal structure and identify the properties of signal structures that attain each feasible outcome. Since both players are risk-neutral, it suffices to consider each player's expected payoff, and we omit the term "expected" when doing so does not lead to ambiguity.

3 Benchmarks

Let $R := r[1 - \chi(r)]$. Suppose that, after receiving a signal about the buyer's type, the seller's posterior belief is that the buyer has invested with probability β. In this case, the seller's expected profit from charging price p is:

$$Profit(p) = \begin{cases} p & \text{if } p = v_0 \\ \beta p[1 - \chi(p)] & \text{if } p > v_0 \end{cases}.$$

[9]Assumption 1 is made only to simplify exposition. Allowing $c > \gamma$ implies that some types will never invest, whereas allowing $c < \alpha$ implies that some types will always invest. Both cases can be incorporated without affecting the main insights, but doing so requires repeatedly qualifying the results for these types.

[10]Private signals fit well with our motivating examples, where a firm's clients rarely observe the profiles that the firm has constructed for them. In Section 5.3, we discuss what happens when the buyer can also observe the seller's signal.

[11]We assume that $\mathbb{S}$ is finite; as explained later, this is without loss of generality.

The seller's optimal price for belief β is therefore:

$$\arg\max_p \; Profit(p) \;=\; \begin{cases} v_0 & \text{if } v_0 > \beta R \\ r & \text{if } v_0 < \beta R \\ v_0 \text{ and } r & \text{if } v_0 = \beta R \end{cases} . \tag{1}$$

(1) implies that, regardless of the seller's belief, she sets either $p = v_0$ or $p = r$. If the seller sets the low price $p = v_0$ with probability λ, type c's expected payoff from investing is $\lambda\gamma + (1 - \lambda)\alpha - c$, whereas his expected payoff from not investing is always zero. Hence, type c prefers to invest if

$$\lambda\gamma + (1 - \lambda)\alpha \geq c \qquad \Longleftrightarrow \qquad \lambda \geq \frac{c - \alpha}{\gamma - \alpha} . \tag{2}$$

Lemma 1. *(Benchmark I) If the signal structure is fully informative about c, the players' expected payoffs in equilibrium are their hold-up payoffs.*

The reasoning is as follows. If the seller expects the buyer to invest, she sets $p = r$, but since $\underline{c} > \alpha$, every type of buyer prefers not to invest. Conversely, if the seller expects the buyer not to invest, she sets $p = v_0$, but since $\bar{c} < \gamma$, every type of buyer prefers to invest. Hence, in equilibrium, both players must randomize. From (1) and (2), the only strategy profile sustaining mutual indifference is that every type of buyer invests with probability $\frac{v_0}{R}$, and the seller sets $p = v_0$ and $p = r$ with probabilities $\frac{c-\alpha}{\gamma-\alpha}$ and $\frac{\gamma-c}{\gamma-\alpha}$, respectively, to type c. Since every type plays "not invest" with positive probability, his equilibrium utility must equal that from not investing, which is zero because the seller's price is never below v_0. Similarly, because the seller sets $p = v_0$ with positive probability for every type in equilibrium, her equilibrium profit is that from setting $p = v_0$. Since this offer is always accepted by the buyer, the seller's equilibrium profit is v_0.

Lemma 2. *(Benchmark II) If the signal structure is completely uninformative about c (i.e., the seller receives no information about the buyer's type), then in equilibrium, the seller's expected payoff is v_0 (her hold-up payoff), whereas the buyer's ex-ante expected payoff is strictly positive (higher than his hold-up payoff).*

Suppose that the seller receives no information about the buyer's type. If the seller always sets $p = v_0$, every type of buyer invests, and the seller would then profitably deviate to setting $p = r$. Conversely, if the seller always sets $p = r$, no type invests. Since trade then always breaks down, the seller would profitably deviate to $p = v_0$. Hence, the seller must randomize her price in equilibrium. As explained for Lemma 1, if the seller randomizes in equilibrium, her equilibrium profit is v_0. Let λ^o denote the equilibrium probability that the seller sets $p = v_0$, and define the cutoff type $c^o := \lambda^o \gamma + (1 - \lambda^o)\alpha$. All types $c < c^o$ invest, whereas types $c > c^o$ do not. Since the seller must be indifferent between setting $p = v_0$ and $p = r$, (1) implies that c^o must satisfy the condition $F(c^o) = \frac{v_0}{R}$. This implies that c^o is strictly within $(\underline{c}, \bar{c})$. Consequently, the buyer's ex-ante expected utility is $\int_{\underline{c}}^{c^o} [c^o - c] f(c) dc > 0$.

The distinctive feature of our model is that a buyer's investment decision is endogenous to the seller's information. In Lemma 1, full information causes every buyer type to behave identically, so the seller learns nothing about the buyer's investment action, even with perfect knowledge of his type. In Lemma 2, the seller simply has no information from which to infer the buyer's investment behavior. Hence, in both benchmarks, the seller obtains only her hold-up payoff.

We show later that there exist (partially informative) signal structures under which the seller's payoff exceeds her hold-up payoff. Furthermore, although the buyer's payoff in Benchmark II is higher than his hold-up payoff, there also exist (partially informative) signal structures that further raise his payoff.

4 Main Analysis

4.1 Preliminaries

We now study the outcomes and payoffs attainable under general signal structures. It suffices to analyze signal structures that make a price recommendation to the seller, subject to the equilibrium incentive-compatibility constraints.[12] As shown in (1), the seller's optimal price under any belief about the buyer's investment decision is either v_0 or r. Hence, we let the signal space be $\mathbb{S} = \{v_0, r\}$.

Let $\mathcal{I}(c)$ denote the probability that type c invests. By (2), the buyer's incentive-compatibility condition is:

$$\pi_{v_0}(c) \geq \frac{c - \alpha}{\gamma - \alpha} \ \text{ if } \ \mathcal{I}(c) = 1 \ ; \quad \pi_{v_0}(c) \leq \frac{c - \alpha}{\gamma - \alpha} \ \text{ if } \ \mathcal{I}(c) = 0 \ ; \quad \pi_{v_0}(c) = \frac{c - \alpha}{\gamma - \alpha} \ \text{ if } \ \mathcal{I}(c) \in (0, 1) \ . \tag{BIC}$$

[12] This follows from an argument similar to the revelation principle, where the recommendation replicates the actual equilibrium prices. See, for example, Bergemann and Morris (2016).

By (1), the seller's incentive-compatibility condition requires that the seller prefers the price recommended by the signal given her posterior belief about investment:

$$\frac{\int_{\underline{c}}^{\bar{c}} \pi_r(c)\mathcal{I}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_r(c)f(c)dc} \underbrace{\geq}_{\text{SIC-}r} \frac{v_0}{R} \underbrace{\geq}_{\text{SIC-}v_0} \frac{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c)\mathcal{I}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c)f(c)dc} \tag{SIC}$$

We say that a pair "$\{\pi, \mathcal{I}\}$ *is an equilibrium*" if it satisfies (BIC) and (SIC). Given $\{\pi, \mathcal{I}\}$, buyer welfare and seller profit are respectively:

$$V_B(\pi, \mathcal{I}) = \int_{\underline{c}}^{\bar{c}} [\pi_{v_0}(c)\gamma + \pi_r(c)\alpha - c]\mathcal{I}(c)f(c)dc, \tag{3}$$

$$V_S(\pi, \mathcal{I}) = \int_{\underline{c}}^{\bar{c}} [\pi_{v_0}(c)v_0 + \pi_r(c)\mathcal{I}(c)R]f(c)dc. \tag{4}$$

We first derive the Pareto frontier of the set of feasible payoff. Throughout this section, Pareto-optimality is defined with respect to the buyer's *ex-ante* expected payoff (i.e., before the realization of his type). We consider the buyer's *interim* expected payoff (i.e., after the realization of his type) in Section 5.2.

Lemma 3. *If $\{\pi, \mathcal{I}\}$ is an equilibrium, then there exists another equilibrium $\{\pi, \hat{\mathcal{I}}\}$ where:*

1. *$\hat{\mathcal{I}}(c)$ is either 0 or 1 for all $c \in C$ (i.e., the buyer does not randomize).*
2. *$V_B(\pi, \hat{\mathcal{I}}) = V_B(\pi, \mathcal{I})$.*
3. *$V_S(\pi, \hat{\mathcal{I}}) \geq V_S(\pi, \mathcal{I})$, with strict inequality if $\mathcal{I}(c) \in (0, 1)$ for a positive measure of c.*

A Pareto-optimal equilibrium cannot involve buyer randomization over his investment decision. Intuitively, the value of information about the buyer's type to the seller depends on how accurately it allows the seller to infer the buyer's investment action. If the buyer plays a pure strategy, his action is inferred deterministically from information about his type. By contrast, if the buyer randomizes, this inference becomes noisier. Hence, removing buyer randomization in equilibrium benefits the seller. At the same time, removing buyer randomization does not lower buyer welfare directly. When a type randomizes, his equilibrium payoff must equal that from not investing, which is already zero. Therefore, it is always possible to construct a Pareto improvement relative to any equilibrium with buyer randomization.

Henceforth, we restrict attention to equilibria in which the buyer plays a pure strategy, represented by the set $I \subset C$ of types who invest.[13] With a slight abuse of the notation, let

[13] I and I' are considered the *same strategy* if the sets I and I' differ only on a zero-measure set of types.

V_B and V_s denote the buyer welfare and seller profit under equilibrium $\{\pi, I\}$:

$$V_B(\pi, I) = \int_I [\alpha + \pi_{v_0}(c)(\gamma - \alpha) - c] f(c)dc. \tag{5}$$

$$V_S(\pi, I) = \int_I [v_0 + \pi_r(c)(R - v_0)] f(c)dc + \int_{C/I} \pi_{v_0}(c) v_0 f(c)dc. \tag{6}$$

The corresponding equilibrium conditions in (BIC) and (SIC) for $\{\pi, I\}$ become:

$$\pi_{v_0}(c) \geq \frac{c - \alpha}{\gamma - \alpha} \quad \forall c \in I, \quad \text{and} \quad \pi_{v_0}(c) \leq \frac{c - \alpha}{\gamma - \alpha} \quad \forall c \notin I; \tag{BIC'}$$

$$\frac{\int_I \pi_r(c) f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_r(c) f(c)dc} \geq \frac{v_0}{R} \tag{SIC-r}$$

$$\frac{\int_I \pi_{v_0}(c) f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c) f(c)dc} \leq \frac{v_0}{R} \tag{SIC-v_0}$$

There is generally a tension between the buyer's and seller's equilibrium constraints. To see this, note that the payoff-relevant information for the seller is whether the buyer has invested. Hence, profiling is considered accurate when the seller is recommended to set $p = r$ when the buyer has invested, and $p = v_0$ when the buyer has not invested. Therefore, we can interpret the "*profiling accuracy*" under $\{\pi, I\}$ as $\pi_r(c)$ for type $c \in I$ and $\pi_{v_0}(c)$ for type $c \notin I$.

Observe that (BIC') requires profiling accuracy to be low, whereas (SIC-r) and (SIC-v_0) require it to be high. Moreover, profiling accuracy affects the players' payoffs in the opposite direction: from (5) and (6), buyer welfare decreases with profiling accuracy, whereas the seller's profit increases with it. However, the nature of these constraints differs. (BIC') places an upper bound on profiling accuracy *type by type*, whereas (SIC-r) and (SIC-v_0) require sufficient profiling accuracy only in *aggregate*. This distinction introduces flexibility in how profiling error and accuracy can be distributed across types, leaving scope for information design to affect the players' payoffs.

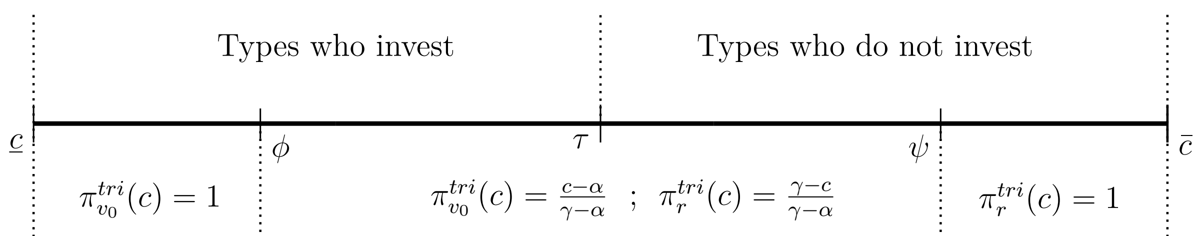
Furthermore, while the seller's profit in (6) increases with higher profiling accuracy for all types, buyer welfare in (5) is affected (negatively) only by the profiling accuracy for the types who invest; it is unaffected by the profiling accuracy for the types who do not invest. This observation suggests that providing the seller with information about the buyer's private information does not always create a direct trade-off between buyer welfare and profit. As we show later, through the equilibrium effects, buyer welfare can in fact strictly increase by raising the profiling accuracy for types who do not invest in equilibrium.

4.2 Dual-cutoff Equilibrium

We first derive a class of equilibria that can attain every Pareto-optimal payoff.

Suppose that in an equilibrium, there exist types $c < c'$ in which the higher-cost type c' invests while the lower-cost type c does not. Then, it remains an equilibrium if we swap the signal distributions $\pi_s(\cdot)$ and the investment actions of these two types—if the lower-cost type c is unwilling to invest, then the higher-cost type c' is even more unwilling to invest under the same signal distribution. Conversely, if higher-cost type c' is willing to invest, then the lower-cost type c is even more willing to invest when under the same signal distribution. In fact, after the swap, we can slightly increase the probability that the lower-cost type c—now an investing type—sends $s = r$ while maintaining his incentive to invest; this increases the seller's profit. Therefore, Pareto efficiency requires *monotone sorting*: investment is made by lower-cost types in equilibrium.

Figure 1: Tri-cutoff Equilibrium



Consider an equilibrium $\{\pi, [\underline{c}, \tau]\}$. We construct a signal structure π^{tri}, illustrated in Figure 1, such that $\{\pi^{tri}, [\underline{c}, \tau]\}$ is an equilibrium that is payoff-equivalent to the original one. Define ϕ by

$$\int_{\underline{c}}^{\phi} 1 f(c)dc + \int_{\phi}^{\tau} \frac{c-\alpha}{\gamma-\alpha} f(c)dc = \int_{\underline{c}}^{\tau} \pi_{v_0}(c) f(c)dc.$$

From (BIC'), the original signal structure satisfies $\pi_{v_0}(c) \geq \frac{c-\alpha}{\gamma-\alpha}$ for all $c < \tau$. The new signal structure π^{tri} preserves the total probabilities of sending signals $s = r$ and $s = v_0$ from the set of investing types $[\underline{c}, \tau]$, but redistributes the probability of $s = v_0$ from higher-cost types near τ to lower-cost types near $\underline{c}$. Each type in $[\underline{c}, \tau]$ still faces at least probability $\frac{c-\alpha}{\gamma-\alpha}$ of sending $s = v_0$, thereby maintaining incentive compatibility. Because the total probability of each signal sent from the investing and noninvesting sets is the same as in the original equilibrium, both the expected buyer welfare and seller profit remain unchanged.

Next, define ψ by

$$\int_{\tau}^{\psi} \frac{\gamma - c}{\gamma - \alpha} f(c)dc + \int_{\psi}^{\bar{c}} 1 f(c)dc = \int_{\tau}^{\bar{c}} \pi_r(c) f(c)dc.$$

Analogously, the original signal structure satisfies $\pi_r(c) \geq \frac{\gamma - c}{\gamma - \alpha}$ for the set of noninvesting types $(\tau, \bar{c}]$. The new signal structure π^{tri} redistributes the probability of $s = r$ from lower-cost types near τ to higher-cost types near $\bar{c}$, while keeping the total probability of each signal unchanged and the noninvestment incentives intact. This adjustment leaves the equilibrium buyer welfare and seller profit unchanged. Hence, the "tri-cutoff equilibrium" in Figure 1 is payoff-equivalent to the original equilibrium.

Next, fixing τ, increasing ψ improves profiling accuracy on noninvesting types ($c > \tau$). This increases the seller's profit by reducing the probability that she charges $p = r$ to noninvesting type, which would otherwise cause trade breakdown. At the same time, since noninvesting types always obtain zero utility, improving profiling accuracy on them (by increasing ψ) does not affect buyer welfare. Therefore, $\psi = \bar{c}$ must hold in any Pareto-optimal equilibrium. In contrast, decreasing ϕ improves profiling accuracy on investing types ($c < \tau$), which raises the seller's profit but lowers the utility of investing types. Thus, ϕ determines the split of the investment surplus created by investing types in $[\underline{c}, \tau]$.

We formalize the discussion above in Lemma 4. Define

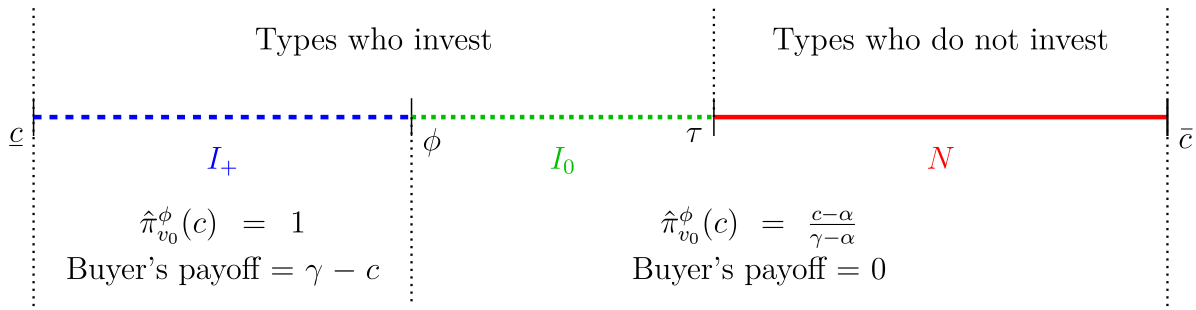
$$\hat{\pi}^{\phi}_{v_0}(c) := \begin{cases} 1 & \text{if } c \leq \phi \\ \frac{c-\alpha}{\gamma-\alpha} & \text{if } c > \phi \end{cases} . \tag{7}$$

A pair $\{\pi, I\}$ is a "*dual-cutoff (DC) equilibrium*" if it is an equilibrium with $\pi = \hat{\pi}^{\phi}$ for some $\phi \in [\underline{c}, \bar{c})$, and $I = [\underline{c}, \tau]$ for some $\tau > \phi$.

Lemma 4. *Every equilibrium is weakly Pareto-dominated by some DC equilibrium. Specifically, if $\{\pi, I\}$ is an equilibrium, there exists a DC equilibrium $\{\hat{\pi}^{\phi}, [\underline{c}, \tau]\}$ with $V_B(\hat{\pi}^{\phi}, [\underline{c}, \tau]) \geq V_B(\pi, I)$ and $V_S(\hat{\pi}^{\phi}, [\underline{c}, \tau]) \geq V_S(\pi, I)$.*

Figure 2 illustrates each buyer type's expected payoff in a DC equilibrium. Types in the rightmost interval, $N = (\tau, \bar{c}]$ (solid red) are indifferent between investing and not investing, and they do not invest in equilibrium. Investing types are separated into two intervals: $I_+ = [\underline{c}, \phi]$ (dashed blue), and $I_0 = (\phi, \tau]$ (dotted green). Each type c in I_0 is also made indifferent between investing and not investing, so their equilibrium payoff is zero, just like the noninvesting types in N. Types in I_+ are always mis-profiled and thus always retain

Figure 2: Dual-cutoff (DC) Equilibrium



their entire investment surplus $\gamma - c$. In a DC equilibrium, these types in I_+ are the sole contributors to buyer welfare.

Constraints (SIC-r) and (SIC-v_0) for a DC profile $\{\hat{\pi}^\phi, [\underline{c}, \tau]\}$ can be written as

$$\beta_{v_0}(\phi,\tau) \underbrace{\leq}_{\text{SIC-}v_0} \frac{v_0}{R} \underbrace{\leq}_{\text{SIC-}r} \beta_r(\phi,\tau) \ , \tag{8}$$

$$\text{where} \qquad \beta_{v_0}(\phi,\tau) = \frac{F(\phi) + \int_\phi^\tau \frac{c-\alpha}{\gamma-\alpha} f(c)dc}{F(\phi) + \int_\phi^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)dc}, \quad \text{and} \quad \beta_r(\phi,\tau) = \frac{\int_\phi^\tau \frac{\gamma-c}{\gamma-\alpha} f(c)dc}{\int_\phi^{\bar{c}} \frac{\gamma-c}{\gamma-\alpha} f(c)dc}.$$

Since (BIC') always holds, the DC profile $\{\hat{\pi}^\phi, [\underline{c}, \tau]\}$ is an equilibrium if and only if (8) holds. The following lemma describes how the cutoffs ϕ and τ affect (8).

Lemma 5. $\frac{\partial \beta_{v_0}(\phi,\tau)}{\partial \phi} > 0$, $\frac{\partial \beta_r(\phi,\tau)}{\partial \phi} < 0$, $\frac{\partial \beta_{v_0}(\phi,\tau)}{\partial \tau} > 0$, *and* $\frac{\partial \beta_r(\phi,\tau)}{\partial \tau} > 0$. *Therefore:*

- *Increasing ϕ tightens both (SIC-r) and (SIC-v_0).*
- *Increasing τ relaxes (SIC-r) but tightens (SIC-v_0).*

Referring to Figure 2, raising ϕ shrinks I_0. Since N stays fixed, the signal $s = r$ then originates relatively more often from N—where types do not invest—thereby tightening (SIC-r). At the same time, raising ϕ expands I_+. This increases the likelihood that $s = v_0$ is sent by a type in I_+—which consists of investing types—thereby tightening (SIC-v_0).

Raising τ expands I_0 and shrinks N. The signal $s = r$ is now more often sent by a type in I_0 (rather than in N). Since I_0 contains investing types, this relaxes (SIC-r). However, a smaller N reduces the likelihood that $s = v_0$ is sent by a type in N—which consists of noninvesting types—thereby tightening (SIC-v_0).

4.3 Feasible Payoffs

Suppose that buyer welfare u_B is attainable by some equilibrium. Let $\bar{U}_S(u_B)$ denote the highest profit attainable together with buyer welfare u_B. By Lemma 4, the payoff pair $(u_B, \bar{U}_S(u_B))$ must be attainable by a DC equilibrium. To simplify notation, we henceforth denote a DC equilibrium $\{\hat{\pi}^{\phi}, [\underline{c}, \tau]\}$ by $\{\phi, \tau\}$.

Under $\{\phi, \tau\}$, buyer welfare is $\int_{\underline{c}}^{\phi}(\gamma - c)f(c)dc$, which increases in ϕ and is independent of τ. Hence, the first cutoff ϕ of the DC equilibrium that attains buyer welfare u_B must be $\Phi(u_B)$, defined by

$$\int_{\underline{c}}^{\Phi(u_B)}(\gamma - c)f(c)dc = u_B. \tag{9}$$

The seller's profit is

$$\hat{V}_S(\phi, \tau) \;:=\; \underbrace{v_0 F(\phi)}_{\text{from } I_+} + \underbrace{R\int_{\phi}^{\tau}\frac{\gamma - c}{\gamma - \alpha}f(c)dc}_{\text{from charging } r \text{ to } I_0} + \underbrace{v_0\int_{\phi}^{\bar{c}}\frac{c - \alpha}{\gamma - \alpha}f(c)dc}_{\text{from charging } v_0 \text{ to } I_0 \text{ and } N} \tag{10}$$

$\hat{V}_S(\phi, \tau)$ decreases in ϕ (the seller keeps less of the investment surplus) and increases in τ (more investment surplus is created).[14] From Lemma 5, raising τ relaxes (SIC-r) but tightens (SIC-v_0). Therefore, to attain the profit $\bar{U}_S(u_B)$, (SIC-v_0) must bind.

Define $T(u_B)$ to be the second cutoff τ for which (SIC-v_0) binds when $\phi = \Phi(u_B)$—i.e.,[15]

$$\beta_{v_0}\Big(\Phi(u_B), T(u_B)\Big) = \frac{v_0}{R}. \tag{11}$$

The DC equilibrium that attains the payoff pair $(u_B, \bar{U}_S(u_B))$ is $\{\Phi(u_B), T(u_B)\}$. From (9), $\Phi(u_B)$ increases in u_B. Since raising ϕ tightens (SIC-v_0) and also β_{v_0} also increases in τ, keeping f (SIC-v_0) binding requires τ to fall. Hence, $T(u_B)$ decreases in u_B.

The seller's profit on the Pareto frontier can be expressed as

$$\bar{U}_S(u_B) = \hat{V}_S\left(\Phi(u_B), T(u_B)\right) = F\Big(T(u_B)\Big)R.$$

This is because when (SIC-v_0) binds, the seller is indifferent between charging either $p = v_0$ or $p = r$ following signal $s = v_0$. Thus, in a DC equilibrium $\{\phi, \tau\}$ where (SIC-v_0) binds, the

[14] By Leibniz rule, $\frac{\partial \hat{V}_S(\phi,\tau)}{\partial \phi} = -f(\phi)\left(\frac{\gamma - \phi}{\gamma - \alpha}\right)(R - v_0) < 0$, and $\frac{\partial \hat{V}_S(\phi,\tau)}{\partial \tau} = f(\tau)\left[\frac{\gamma - \tau}{\gamma - \alpha}\right]R > 0$.

[15] $T(u_B)$ is well defined because for any ϕ, $\beta_{v_0}(\phi, \tau)$ increases in τ and $\beta_{v_0}(\phi, \bar{c}) = 1$.

seller's equilibrium profit is equal to what she would earn by always setting $p = r$, regardless of the signal, while the buyer invests if and only if $c \leq \tau$. Since trade then breaks down with all noninvesting types, the expected profit from doing so is $F(\tau)R$.

Next, we derive the bounds of attainable buyer welfare u_B. From Lemma 5, lowering the first cutoff ϕ relaxes both constraints (SIC-r) and (SIC-v_0). Hence, ϕ can be lowered all the way to $\underline{c}$, implying that the lowest attainable buyer welfare is $u_B = 0$. The highest buyer welfare must occur in a DC equilibrium where both constraints bind. If (SIC-r) binds but (SIC-v_0) is slack, raising τ relaxes (SIC-r) while keeping (SIC-v_0) satisfied, which then permits raising ϕ. Conversely, if (SIC-v_0) binds but (SIC-r) is slack, lowering τ relaxes (SIC-v_0) while keeping (SIC-r) satisfied, again allowing ϕ to increase.

Define τ_S^* and τ_B^* by

$$\int_{\underline{c}}^{\tau_S^*} (c-\alpha) f(c) dc = \frac{v_0}{R}\Big(E[c]-\alpha\Big), \qquad \text{and} \qquad F(\tau_B^*) = \frac{v_0}{R}\,. \tag{12}$$

Define u_B^* and ϕ_B^* by

$$u_B^* := \frac{v_0/R}{1-v_0/R}\left(E[c] - E[c|c \leq \tau_B^*]\right), \qquad \text{and} \qquad \int_{\underline{c}}^{\phi_B^*} (\gamma - c) f(c) dc = u_B^*. \tag{13}$$

In the proof of Proposition 1, we show that $T(0) = \tau_S^*$, and that both constraints (SIC-r) and (SIC-v_0) bind in a DC equilibrium if and only if $\{\phi, \tau\} = \{\phi_B^*, \tau_B^*\}$.

The following summarizes our discussion and characterizes the Pareto frontier.

Proposition 1. *The set of attainable buyer welfare levels is* $[0, u_B^*]$. *For each* $u_B \in [0, u_B^*]$, *the highest attainable seller profit is* $\bar{U}_S(u_B) = F\Big(T(u_B)\Big)R$, *and the payoff pair* $\Big(u_B, \bar{U}_S(u_B)\Big)$ *is attained by the DC equilibrium* $\{\Phi(u_B), T(u_B)\}$.

- $\Phi(u_B)$ *is strictly increasing in* u_B, *with* $\Phi(0) = \underline{c}$, *and* $\Phi(u_B^*) = \phi_B^*$.
- $T(u_B)$ *is strictly decreasing in* u_B, *with* $T(0) = \tau_S^*$, *and* $T(u_B^*) = \tau_B^*$.
- $\bar{U}_S(u_B)$ *is strictly decreasing in* u_B, *with* $\bar{U}_S(0) = F(\tau_S^*)R$ *and* $\bar{U}_S(u_B^*) = v_0$.

Next, we characterize the other (non-DC) Pareto-optimal equilibria.

Proposition 2. *For any* $u_B \in [0, u_B^*]$, $\{\pi, I\}$ *is an equilibrium that attains the payoff pair* $\Big(u_B, \bar{U}_S(u_B)\Big)$ *if and only if* $\{\pi, I\}$ *satisfies:*

1. $I = [\underline{c}, T(u_B)]$.

2. $\pi_{v_0}(c) = \frac{c-\alpha}{\gamma-\alpha}$ *for all* $c > T(u_B)$.

3. $\pi_{v_0}(c) \geq \frac{c-\alpha}{\gamma-\alpha}$ *for all* $c \leq T(u_B)$, *and* $\int_{\underline{c}}^{T(u_B)}[\alpha + \pi_{v_0}(c)(\gamma - \alpha) - c]f(c)dc = u_B$.

In every such equilibrium, (SIC-v_0) binds, and all types $c > T(u_B)$ receive zero utility.

In general, each Pareto-optimal payoff pair can supported by multiple different equilibria, but these equilibria share common structural features. First, the set of investing types is identical across all equilibria that attain the same Pareto-optimal payoff pair. Multiplicity arises from how buyer welfare u_B is distributed among the equilibrium investing types $c \leq T(u_B)$, and this depends on the signal distribution for these types. In the DC equilibrium that attains this payoff pair, buyer welfare is accrued only to types $c \leq \Phi(u_B)$, while types $c > \Phi(u_B)$ receive zero utility. More generally, welfare can actually be distributed among any investing types—including types in $\left(\Phi(u_B), T(u_B)\right]$—subject to the overal "budget constraint" that the aggregate buyer welfare equals u_B. We return to this issue in Section 5.2.

Next, in every Pareto-optimal equilibrium, any noninvesting type c must face the signal distribution $\pi_{v_0}(c) = \frac{c-\alpha}{\gamma-\alpha}$. To satisfy (BIC'), $\pi_{v_0}(c)$ for a noninvesting type c cannot exceed this value. This reflects a general alignment of preference between buyer and seller: *both* prefer maximal profiling accuracy for noninvesting types. It is clear that the seller always benefits from greater profiling accuracy (for any type). For the buyer, a noninvesting type earns zero utility regardless of the profiling error he faces (provided it is sufficient to deter investment). In contrast, an investing type's utility increases with the profiling error he faces, since he retains his investment surplus only when he is mis-profiled (and offered $p = v_0$). Because the aggregate profiling error must be limited to maintain signal credibility for the seller, wasting profiling error on noninvesting types is *strictly* suboptimal for buyer welfare.

The property that $\pi_{v_0}(c) = \frac{c-\alpha}{\gamma-\alpha}$ for all noninvesting types also implies that (SIC-v_0) must bind in every Pareto-optimal equilibrium. To see why, note that under this property, some of the noninvesting types could be shifted to invest without violating (BIC'), thereby increasing total surplus. Such a shift relaxes (SIC-r) and tightens (SIC-v_0). Hence, if (SIC-v_0) were slack, the shift would be feasible and would improve at least one party's payoff, contradicting Pareto optimality.

Corollary 1. *The seller-optimal equilibrium is unique. The signal structure is* $\pi_{v_0}(c) = \frac{c-\alpha}{\gamma-\alpha}$ *for all* $c \in C$, *and the buyer strategy is* $I = [\underline{c}, \tau_S^*]$. *In the seller-optimal equilibrium, the seller's profit is* $u_S^* := F(\tau_S^*)R$, *and every buyer type's expected payoff is zero.*

The seller-optimal equilibrium is unique across *all* possible equilibria and must also be a DC equilibrium. The logic is as follows: if any buyer type has a slack (BIC'), profiling accuracy could be increased for that type to increase the seller's profit. Hence, the seller-optimal signal structure must be $\pi_{v_0}(c) = \frac{c-\alpha}{\gamma-\alpha}$ for all c, which is the DC signal structure $\hat{\pi}^{\phi}$ in (7), with $\phi = \underline{c}$.

As indicated in Lemma 1 and confirmed by Corollary 1, the seller has an incentive to limit her information about the buyer's type, but such restraint does not benefit the buyer: in the seller-optimal equilibrium, every type's utility is zero. The result underscores that information limitation by the seller—while potentially arising endogenously—is not necessarily a form of buyer protection.

Corollary 2. *An equilibrium $\{\pi, I\}$ is buyer-optimal if and only if $I = [\underline{c}, \tau_B^*]$, $\pi_{v_0}(c) = \frac{c-\alpha}{\gamma-\alpha}$ for all $c > \tau_B^*$, $\pi_{v_0}(c) \geq \frac{c-\alpha}{\gamma-\alpha}$ for all $c \leq \tau_B^*$, and $\int_{\underline{c}}^{\tau_B^*}[\pi_{v_0}(c)\gamma - c]f(c)dc = u_B^*$. In every buyer-optimal equilibrium, both constraints (SIC-r) and (SIC-v_0) bind, buyer welfare is u_B^*, every type $c > \tau_B^*$ earns zero payoff, and the seller's profit is v_0.*

As noted after Lemma 2 and confirmed by Corollary 2, buyer-optimality is not achieved by completely withholding information from the seller. Furthermore, binding of constraint (SIC-r) is a distinctive feature of buyer-optimality among Pareto-optimal equilibria. Intuitively, a buyer-optimal equilibrium minimizes the seller's profit, and the seller's lowest feasible profit is v_0, which occurs if and only if (SIC-r) binds. We provide some heuristics on how this property guides improvements in the equilibrium buyer welfare.

Starting from any equilibrium, increasing profiling accuracy on noninvesting types—without overturning their incentive not to invest—makes the signal $s = v_0$ more credible to the seller, as noninvesting types now send signal $s = v_0$ more often. This added credibility can then be used to introduce more profiling error on investing types, allowing them to send $s = v_0$ more often and retain a larger share of the investment surplus. However, such a shift also decreases the credibility of signal $s = r$ to the seller. If $s = r$ remains overly credible (i.e., (SIC-r) is slack), buyer welfare can be further improved by moving higher-cost types *out* of the investing set. Shrinking the set of investing types reduces the credibility of $s = r$ and enhances that of $s = v_0$, which permits additional profiling error on the remaining investing types. This iterative process reallocates profiling error toward lower-cost investing types—who generate more investment surplus—and allows them to retain a larger portion of that surplus.

Next, we consider the total welfare on the Pareto frontier. We have assumed throughout (without loss of generality) that $c < \gamma$ for all c, so the buyer investing is always socially efficient. We introduce a stronger efficiency concept: say that type c's investment is "*efficient under common knowledge of investment*" (or "*cki-efficient*") if $c \leq \int_r^{\bar{v}} v d\chi(v) - v_0$. If the buyer's investment is observed by the seller (i.e., common knowledge), then since the seller sets price r, trade breaks down when the realized v is greater than r. Cki-efficiency means that investment remains socially efficient even accounting for the deadweight loss from trade breakdown, arising due to asymmetric information about the investment outcome.

Proposition 3. *The slope of the Pareto frontier is* $\bar{U}_S'(u_B) = -\frac{R-v_0}{T(u_B)-\alpha}$*. Moreover,* $\bar{U}_S'(u_B) < -1$ *if and only if the investment of type* $T(u_B)$ *is cki-efficient.*

- *If the investment of type* τ_S^* *is cki-efficient, the seller-optimal equilibrium maximizes total welfare.*
- *If the investment of type* τ_B^* *is* <u>*not*</u> *cki-efficient, every buyer-optimal equilibrium maximizes total welfare.*

Recall that $T(u_B)$ decreases in u_B. Proposition 3 therefore implies that the Pareto frontier is concave. If the slope of $\bar{U}_S$ is less (greater) than -1, then total welfare decreases (increases) with buyer welfare u_B along the Pareto frontier. Since the equilibrium investment strategy that attains u_B is $I = [\underline{c}, T(u_B)]$, higher buyer welfare corresponds to lower investment in equilibrium. However, whether an incremental investment increases or decreases total welfare depends on the signal structure $\pi_s(c)$. If $\pi_{v_0}(c) = 1$, the seller always charges $p = v_0$, trade always occurs, and the full expected return $\gamma - c$—positive by assumption—is jointly realized by both players. In contrast, if $\pi_r(c) = 1$, the seller always charges $p = r$, and trade breaks down when the post-investment valuation falls below r. In this case, the incremental investment raises total welfare only if it is cki-efficient.

Proposition 3 states that whether an incremental investment increases or decreases total welfare depends on whether the investment is cki-efficient, even though the marginal investment type $c = T(u_B)$ does not have $\pi_r(c) = 1$. The reason is that a change in u_B necessarily alters the equilibrium signal structure for some investing types. Thus, decreasing buyer welfare—while marginally expanding investment—also affects the trade probability of existing investing types. Using the DC equilibrium as an example, lowering buyer welfare from u_B to u_B' raises the investment cutoff ($T(u_B') > T(u_B)$), so more types invest. At the same time, $\Phi(u_B') < \Phi(u_B)$, shrinking the set of types with $\pi_{v_0}(c) = 1$. This shift increases the likelihood of trade breakdown. The overall net effect on total welfare turns out to always rests on only whether the marginal investment type's investment is cki-efficient.

Cki-efficiency is monotonic in c. Since the seller-optimal equilibrium has the highest investment cutoff τ_S^*, if investment is cki-efficient for type τ_S^*, then investment must be cki-efficient for every possible cutoff investment type in a Pareto-optimal equilibrium. In that case, increasing investment always raises total welfare, so the seller-optimal equilibrium maximizes total welfare. Conversely, the buyer-optimal equilibrium has the lowest investment cutoff type τ_B^*. If investment is not cki-efficient for type τ_B^*, then investment cannot be cki-efficient for every possible cutoff investment type in a Pareto-optimal equilibrium. In that case, the buyer-optimal equilibria maximize total welfare.

As discussed above, the difference between overall efficiency ($\gamma - c$) and cki-efficiency arises from asymmetric information about the investment outcome. If the investment outcome is deterministic, every type's investment is cki-efficient. This implies the following observation:

Corollary 3. *If χ is a degenerate distribution (i.e., investment outcome is deterministic), total welfare along the Pareto frontier is strictly decreasing in the buyer welfare attained. The seller-optimal equilibrium maximizes the total welfare.*

Finally, we characterize the rest of the attainable payoffs.

Proposition 4. *Every payoff pair between the hold-up payoffs and the Pareto-frontier is attainable. Specifically, for any $u_B \in [0, u_B^*]$ and $u_S \in \left[v_0, \bar{U}_S(u_B)\right]$, there exists τ such that $\hat{V}_S\left(\Phi(u_B), \tau\right) = u_S$, and the DC equilibrium attains the payoff pair (u_B, u_S).*

From (10), $\hat{V}_S(\phi, \tau)$ is strictly increasing in τ. Starting from a Pareto-optimal DC equilibrium $\{\Phi(u_B), T(u_B)\}$, lowering the second cutoff τ relaxes (SIC-v_0) and tightens (SIC-r). In the proof of Proposition 4, we show that (SIC-r) binds when $\{\Phi(u_B), \tau\}$ yields the seller profit v_0. Thus, for each $u_S \in \left[v_0, \bar{U}_S(u_B)\right]$, there exists a unique τ such that $\hat{V}\left(\Phi(u_B), \tau\right) = u_S$ and $\{\Phi(u_B), \tau\}$ satisfies both (SIC-v_0) and (SIC-r).

Similar to Proposition 2, non-Pareto-optimal payoff pairs can also be attained by non-DC equilibria. In fact, the range of possible equilibrium structures is much broader: the buyer does not have to play a cutoff investing strategy, the equilibrium noninvesting buyer types need not be indifferent between investing and not investing, and some types may play a mixed strategy. Providing a full characterization of all equilibria that attain a given payoff pair is therefore infeasible and (perhaps) adds little insight.

5 Discussion

5.1 Comparative Statics and Buyer Uncertainty

We examine how the set of attainable payoffs varies with the distribution of buyer types. Let G be another distribution satisfying the same regularity conditions as F. For each

$J \in \{F, G\}$, let u_B^{J*} denote the highest attainable buyer welfare, and let $\bar{U}_S^{J*}(\cdot)$ denote the corresponding Pareto frontier characterized in Proposition 1.

Proposition 5. *If F is a mean-preserving spread of G, then the highest attainable buyer welfare satisfies $u_B^{F*} \geq u_B^{G*}$, and the Pareto frontier satisfies $\bar{U}_S^{F*}(u_B) \geq \bar{U}_S^{G*}(u_B)$ for all $u_B \in [0, u_B^{G*}]$.*

The lowest attainable buyer welfare and seller profit are the hold-up payoffs, which are independent of the type distribution. Hence, Proposition 5 implies that a mean-preserving spread of the type distribution expands the set of feasible payoff outcomes.

Proposition 5 can be used to consider how the buyer's uncertainty and learning about his investment cost affect the set of feasible payoffs. This is relevant when a buyer learns the true cost only after committing to invest. For instance, a firm adapting its workflow to a new service provider may discover the actual disruption cost only during implementation.

In such cases, the buyer decides whether to invest under imperfect information about his investment cost. Suppose the buyer—instead of perfectly knowing c—draws a private signal about c before deciding whether to invest, with the signal structure representing the buyer's learning technology. Because c enters the buyer's utility linearly, only the posterior mean of c, based on the signal realization, influences his investment decision. By Blackwell (1951), there exists a signal-generating process that yields a posterior mean distribution G if and only if F is a mean-preserving spread of G. Moreover, one signal process is Blackwell-more informative than another if and only if the induced posterior mean distribution is a mean-preserving spread of the other.

Given this, Proposition 5 implies that greater buyer uncertainty shrinks the set of feasible payoffs. Conversely, a more informed buyer expands the set of feasible payoffs, including attaining outcomes that simultaneously increase both buyer welfare and seller profit. In this sense, in our model, more ex-ante information asymmetry between the buyer and the seller can enhance efficiency, in contrast to the result in the classic lemon market (Akerlof, 1970).

5.2 Distribution of Buyer Welfare

Our model can also be interpreted as one with many buyers of different types distributed according to F. The preceding analysis focused on aggregate buyer welfare without considering how utility is distributed across individual types. This subsection examines how aggregate welfare can be allocated among buyers.

A corollary of Proposition 2 is that in any Pareto-optimal equilibrium achieving aggregate buyer welfare u_B, all types $c > T(u_B)$ receive zero utility because they do not invest. Assigning positive utility to some type $c > T(u_B)$ therefore requires departing from Pareto optimality. This reduces the seller's profit below $\bar{U}_S(u_B)$ and may also affect whether the aggregate welfare level u_B remains attainable.

The key message of this subsection is that which types receive positive utility determines the maximum feasible aggregate buyer welfare. Beyond this aggregate "budget constraint," however, buyer utility can be flexibly distributed across investing types.

To formalize this "flexibility," let $W(c)$ denote the indirect utility of type c. An equilibrium is said to attain W if type c's equilibrium utility equals $W(c)$ for all $c \in C$. Let C_+^W denote the set of types in which $W(c) > 0$, and let $u_B^W := \int_{\underline{c}}^{\bar{c}} W(c)f(c)dc$ denote the aggregate buyer welfare under W. Since type c's utility must lie between 0 and $\gamma - c$, we restrict attention to W satisfying $0 \leq W(c) \leq \gamma - c$ for all $c \in C$.

Proposition 6. *An indirect utility profile W is attainable if and only if there exists a subset $C_0 \subset C \setminus C_+^W$—representing types who invest but receive zero utility—such that:*

$$\frac{\int_{C_+^W} \frac{W(c)+c-\alpha}{\gamma-\alpha} f(c)dc + \int_{C_0} \frac{c-\alpha}{\gamma-\alpha} f(c)dc}{\int_{C_+^W} \frac{W(c)+c-\alpha}{\gamma-\alpha} f(c)dc + \int_{C/C_+^W} \frac{c-\alpha}{\gamma-\alpha} f(c)dc} \leq \frac{v_0}{R} \leq \frac{\int_{C_+^W} \frac{\gamma-c-W(c)}{\gamma-\alpha} f(c)dc + \int_{C_0} \frac{\gamma-c}{\gamma-\alpha} f(c)dc}{\int_{C_+^W} \frac{\gamma-c-W(c)}{\gamma-\alpha} f(c)dc + \int_{C/C_+^W} \frac{\gamma-c}{\gamma-\alpha} f(c)dc}. \tag{14}$$

If W is attainable, then it is attainable by the signal structure

$$\pi_L(c) = \begin{cases} \frac{W(c)+c-\alpha}{\gamma-\alpha} & \text{if } c \in C_+^W \\ \frac{c-\alpha}{\gamma-\alpha} & \text{if } c \notin C_+^W \end{cases}, \tag{15}$$

in an equilibrium with $I = C_+^W \cup C_0$, where C_0 satisfies (14). Moreover, for any W in which $u_B^W \in [0, u_B^*]$, the indirect utility function W is attainable with profit $\bar{U}_S(u_B^W)$ if and only if C_+^W is a subset of $\left[\underline{c}, T(u_B^W)\right]$.

To interpret condition (14), note that an investing type's expected utility is $\alpha + \pi_{v_0}(\gamma-\alpha) - c$. Hence, for any type with $W(c) > 0$, we must have $\pi_{v_0}(c) = \frac{W(c)+c-\alpha}{\gamma-\alpha}$. The signal structure in (15) satisfies this requirement. For types with $W(c) = 0$—i.e., those in $C \setminus C_+^W$— the signal structure in (15) makes them exactly indifferent between investing and not investing. Consequently, constraint (BIC') holds for any choice of subset $C_0 \subset C \setminus C_+^W$ as the set of equilibrium investing types who receive zero utility. Condition (14) then captures constraints

(SIC-r) and (SIC-v_0) under that choice of C_0. When it is satisfied, the pair $\{\pi, C_+^W \cup C_0\}$ is an equilibrium that attains the indirect utility profile W.

The converse is more subtle. It states that if W cannot be attained by the signal structure in (15), then no signal structure can attain it. For an equilibrium to attain W and satisfy (BIC'), the value of $\pi_{v_0}(c)$ for each investing type c must coincide with that in (15). Any deviation from the signal structure in (15) can only occur for noninvesting types, for whom $\pi_{v_0}(c)$ may be below $\frac{c-\alpha}{\gamma-\alpha}$. If $\pi_{v_0}(c) < \frac{c-\alpha}{\gamma-\alpha}$, type c's noninvestment incentive constraint is slack. As discussed earlier, if this happens, it is possible to increase the aggregate surplus. Thus, the signal structure in (15) provides utility $W(c)$ for types in C_+^W in the most efficient manner. Therefore, if W is not attainable by this signal structure, then it is not feasible to generate enough surplus in equilibrium to attain W at all.

The last statement of Proposition 6 concerns whether W is attainable on the Pareto frontier. The "only if" direction—that W is not attainable on the Pareto frontier if some $c > T\left(u_B^W\right)$ receive positive utility—follows directly from Proposition 2. The "if" direction establishes that if the investing types include only $c < T\left(u_B^W\right)$—so that investment is efficient and the outcome remains Pareto efficient—then the resulting aggregate buyer welfare u_B^W can be allocated across these investing types in any way consistent with incentive compatibility. This confirms the "flexibility" notion stated above.

5.3 Public Signals

This subsection examines the case in which the buyer also observes the seller's signal before making his investment decision.

Fix a signal structure. The seller's strategy remains a mapping from each signal to a price distribution between v_0 and R. A buyer strategy is now a mapping from each type c *and* the observed signal s to a probability of investing. Suppose that the seller sets $p = v_0$ with probability λ_s following signal s. In equilibrium, λ_s cannot be 1: if $\lambda_s = 1$, every buyer type would invest after signal s, and the seller would prefer switching to $\lambda_s = 0$. By symmetry, λ_s also cannot equal 0. Using equation (2), given $\lambda_s \in (0,1)$, every type $c < \alpha + \lambda_s(\gamma - \alpha)$ invests, and every type $c > \alpha + \lambda_s(\gamma - \alpha)$ does not.

To sustain $\lambda_s \in (0,1)$ in equilibrium, the seller must be indifferent between setting $p = v_0$ and $p = R$. Given the buyer's investment rule above, this requires $F(\alpha + \lambda_s(\gamma - \alpha)) = \frac{v_0}{R}$ for *every* s, which implies that the seller's strategy is independent of s. The seller thus ignores the signals, and the equilibrium outcome is unique, corresponding to Benchmark II in Lemma 2. Hence, profiling becomes impossible when signals are public.

An implication that follows is that if the profiling technology is chosen by the seller, mandating that the seller discloses her information—such as through transparency requirements in data collection and usage—can raise consumer welfare. Public signals eliminate the seller-optimal outcome (which yields zero buyer welfare) and ensure instead the strictly positive buyer welfare from Benchmark II. Conversely, if the profiling technology is chosen by a party aligned with consumers' interests, such transparency can backfire because it prevents the attainment of the buyer-optimal equilibrium, which yields higher buyer welfare. In this sense, the buyer may benefit from strategic ignorance.

6 Conclusion

In both business-to-consumer and business-to-business settings, data analytics is increasingly used to profile and infer potential buyers' private information. When the value of trade depends on buyers undertaking costly pre-trade investments, such profiling also affects their incentives to invest. We show that this feature fundamentally changes the welfare implications of information disclosure relative to environments with exogenous valuations. Achieving any welfare objective requires balancing two forces: incentivizing investment and allocating the resulting trade surplus. While the seller has an incentive to limit her own information about buyers, such restraint does not necessarily benefit them. Buyers can benefit ex-ante from more accurate profiling on types who will not invest, if this additional precision is used to offset greater profiling error for investing types, allowing them to retain a larger share of the investment surplus.

Beyond standard hold-up environments, our framework applies more broadly by interpreting "investment" as any costly action that the "investor" benefits from concealing. One example is consumer search: a buyer may exert search effort to improve his outside option before meeting a seller, who wants to infer that outside option to capture more trade surplus. Search is costly, but the buyer risks facing a high price if the seller knows he has a low outside option, which in turn incentivizes the buyer to search.[16] In this context, the seller's information about the buyer's search cost affects both the buyer's ex-ante incentive to search and the seller's ability to extract ex-post surplus from trade.

Beyond IO contexts, another relevant application is law enforcement. Individuals differ in their intrinsic returns from committing crime, and these returns may be inferred from

[16]This abstracts from other benefits of consumer search, such as improving the match value. Tan (2025) considers such an environment and discusses how it parallels hold-up dynamics.

observable traits. Law enforcers want to target their investigative and prosecuting resources toward individuals who have in fact violated the law, and may use profiling to guide this allocation. These profiling mechanisms, in turn, affect individuals' incentives to commit crime.[17] Our analysis thus highlights the law enforcer's trade-off between deterrence and the direct cost of policing.

A Proof Appendix

A.1 Proof of Lemma 3

Proof. Let $\{\pi, \mathcal{I}\}$ be an equilibrium, and let M denote the set of types for which $\mathcal{I}(c) \in (0,1)$, with $\bar{c}_M = \sup M$ and $\underline{c}_M = \inf M$. Suppose that M has positive measure. By the intermediate value theorem, there exists $\hat{c} \in (\underline{c}, \bar{c})$ such that $\int_{M\cap(\underline{c},\hat{c})} f(c)dc = \int_M \mathcal{I}(c)f(c)dc$.

Define a modifed investment strategy $\hat{\mathcal{I}}$ by $\hat{\mathcal{I}}(c) = \mathcal{I}(c)$ if $c \notin M$, $\hat{\mathcal{I}}(c) = 1$ if $c \in M \cap (\underline{c}, \hat{c})$, and $\hat{\mathcal{I}}(c) = 0$ if $c \in M \cap (\hat{c}, \bar{c})$. This implies that $\int_M \hat{\mathcal{I}}(c)f(c)dc = \int_M \mathcal{I}(c)f(c)dc$, so the total investment probability is unchanged.

For $c \in M$, define $\Delta(c) := \int_{[\underline{c},c]\cap M}[\mathcal{I}(z) - \hat{\mathcal{I}}(z)]f(z)dz$. We have $\Delta(\underline{c}_M) = \Delta(\bar{c}_M) = 0$. Because $\mathcal{I}(c) < 1 = \hat{\mathcal{I}}(c)$ for $c < \hat{c}$ and $\mathcal{I}(c) > 0 = \hat{\mathcal{I}}(c)$ for $c > \hat{c}$, $\Delta(c) < 0$ throughout the interior of M. By integration by parts, we obtain

$$\int_M c[\mathcal{I}(c) - \hat{\mathcal{I}}(c)]f(c)\,dc = \Big[c\Delta(c)\Big]_{\underline{c}_M}^{\bar{c}_M} - \int_M \Delta(c)\,dc = -\int_M \Delta(c)\,dc > 0, \tag{16}$$

where the inequality follows from $\Delta(c) < 0$ on the interior of M.

[17] Ichihashi (2025) studies crime-minimizing profiling under a resource constraint. Persico (2002) and Persico and Todd (2005) also study the implications of profiling on crime deterrence.

From (BIC), $\pi_{v_0}(c) = \frac{c-\alpha}{\gamma-\alpha}$ $\forall c \in M$. Hence,

$$\frac{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c)\mathcal{I}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c)f(c)dc} = \frac{\int_{c\notin M} \pi_{v_0}(c)\mathcal{I}(c)f(c)dc + \int_M \frac{c-\alpha}{\gamma-\alpha}\mathcal{I}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c)f(c)dc}$$
$$> \frac{\int_{c\notin M} \pi_{v_0}(c)\hat{\mathcal{I}}(c)f(c)dc + \int_M \frac{c-\alpha}{\gamma-\alpha}\hat{\mathcal{I}}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c)f(c)dc} = \frac{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c)\hat{\mathcal{I}}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c)f(c)dc}$$
$$\frac{\int_{\underline{c}}^{\bar{c}} \pi_r(c)\mathcal{I}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_r(c)f(c)dc} = \frac{\int_{c\notin M} \pi_r(c)\mathcal{I}(c)f(c)dc + \int_M \frac{\gamma-c}{\gamma-\alpha}\mathcal{I}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_r(c)f(c)dc}$$
$$< \frac{\int_{c\notin M} \pi_r(c)\hat{\mathcal{I}}(c)f(c)dc + \int_M \frac{\gamma-c}{\gamma-\alpha}\hat{\mathcal{I}}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_r(c)f(c)dc} = \frac{\int_{\underline{c}}^{\bar{c}} \pi_r(c)\hat{\mathcal{I}}(c)f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_r(c)f(c)dc}$$

The two inequalities above follow from the properties that (i) (i) $\mathcal{I}$ and $\hat{\mathcal{I}}$ differ only on M, (ii) $\int_M c(\mathcal{I}-\hat{\mathcal{I}})f > 0$ from (16), and (iii) $\int_M \hat{\mathcal{I}}f = \int_M \mathcal{I}f$. Those two inequalities imply that $\{\pi, \hat{\mathcal{I}}\}$ also satisfies (SIC). Since (BIC) also holds for $\{\pi, \hat{\mathcal{I}}\}$, $\{\pi, \hat{\mathcal{I}}\}$ is also an equilibrium. For $c \in M$, (BIC) implies that $\pi_{v_0}(c)\gamma + \pi_r(c)\alpha - c = 0$. Since $\mathcal{I}$ and $\hat{\mathcal{I}}$ differ only on M, it follows that $V_B(\pi, \hat{\mathcal{I}}) - V_B(\pi, \mathcal{I}) = 0$, and $V_S(\pi, \hat{\mathcal{I}}) - V_S(\pi, \mathcal{I}) = \int_M \left[\frac{\gamma-c}{\gamma-\alpha}\right] R\left(\hat{\mathcal{I}}(c) - \mathcal{I}(c)\right) f(c)dc > 0$, from properties (ii) and (iii).

□

A.2 Proof of Lemma 4

For any $X \subset C$, let $P(X)$ denote its probability measure, and define κ_X and $\bar{\kappa}_X$ by $F(\kappa_X) = P(X)$ and $1 - F(\bar{\kappa}_X) = P(X)$. We first show that the following two properties hold:

$$\int_{\underline{c}}^{\kappa_X} cf(c)dc \leq \int_{c\in X} cf(c)dc, \qquad \text{with strict inequality if } X \neq [\underline{c}, \kappa_X]; \tag{17}$$
$$\int_{\bar{\kappa}_X}^{\bar{c}} cf(c)dc \geq \int_{c\in X} cf(c)dc, \qquad \text{with strict inequality if } X \neq [\bar{\kappa}_X, \bar{c}]. \tag{18}$$

Proof. Define $f_X(c) = f(c)$ for $c \in X$ and $f_X(c) = 0$ otherwise. Let $F_X(c) = \int_{\underline{c}}^{c} f_X(z)\, dz$ and

$$K_X(c) = \begin{cases} F(c), & c \leq \kappa_X, \\ P(X), & c > \kappa_X. \end{cases}$$

Then $F_X(\underline{c}) = 0$, $F_X(\bar{c}) = P(X)$, and $F_X(c) \leq K_X(c)$ for all $c \in (\underline{c}, \bar{c})$, with strict inequality for some $c < \kappa_X$ if $X \neq [\underline{c}, \kappa_X]$. By integration by parts,

$$\int_{c \in X} cf(c)\,dc = \int_{\underline{c}}^{\bar{c}} cf_X(c)\,dc = \bar{c}P(X) - \int_{\underline{c}}^{\bar{c}} F_X(c)\,dc.$$

Using $F_X \leq K_X$,

$$\int_{c \in X} cf(c)\,dc \geq \bar{c}P(X) - \int_{\underline{c}}^{\bar{c}} K_X(c)\,dc \qquad = \kappa_X P(X) - \int_{\underline{c}}^{\kappa_X} F(c)\,dc = \int_{\underline{c}}^{\kappa_X} cf(c)\,dc,$$

where the inequality is strict if $X \neq [\underline{c}, \kappa_X]$. This proves (17). Next, define

$$\bar{K}_X(c) = \begin{cases} 0, & c < \bar{\kappa}_X, \\ F(c) - F(\bar{\kappa}_X), & c \geq \bar{\kappa}_X. \end{cases}$$

Then $\bar{K}_X(\bar{\kappa}_X) = 0$, $\bar{K}_X(\bar{c}) = P(X)$, and $\bar{K}_X(c) \leq F_X(c)$ for all c, with strict inequality for some $c < \bar{\kappa}_X$ if $X \neq [\bar{\kappa}_X, \bar{c}]$. By the same integration-by-parts identity,

$$\int_{\bar{\kappa}_X}^{\bar{c}} cf(c)\,dc = \bar{c}[1 - F(\bar{\kappa}_X)] - \int_{\bar{\kappa}_X}^{\bar{c}} F(c)\,dc = \bar{c}P(X) - \int_{\underline{c}}^{\bar{c}} \bar{K}_X(c)\,dc.$$

Since $\bar{K}_X \leq F_X$, $\int_{\bar{\kappa}_X}^{\bar{c}} cf(c)\,dc \geq \bar{c}P(X) - \int_{\underline{c}}^{\bar{c}} F_X(c)\,dc = \int_{c \in X} cf(c)\,dc$, with strict inequality if $X \neq [\bar{\kappa}_X, \bar{c}]$. This proves (18). □

Proof of Lemma 4:

Proof. Let $\{\pi, I\}$ be an equilibrium.

Step 1: Construct the corresponding DC equilibrium profile.
Let $\tau \in C$ satisfy $F(\tau) = P(I)$ and let $N = C \setminus I$. By (BIC'), $\pi_{v_0}(c) \leq \frac{c-\alpha}{\gamma-\alpha}$ for all $c \in N$. Hence

$$\int_N \pi_{v_0}(c)f(c)\,dc \;\leq\; \int_N \frac{c-\alpha}{\gamma-\alpha} f(c)\,dc \;\leq\; \int_\tau^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)\,dc, \tag{19}$$

where the second inequality follows from (18) and $P(N) = 1 - F(\tau)$.

Define, for $x \in [\underline{c}, \tau]$,

$$\Sigma(x) := F(x) + \int_x^\tau \frac{c-\alpha}{\gamma-\alpha} f(c)\,dc.$$

Then Σ is strictly increasing with $\Sigma'(x) = \frac{\gamma-x}{\gamma-\alpha} f(x) > 0$. Moreover,

$$\Sigma(\underline{c}) = \int_{\underline{c}}^\tau \frac{c-\alpha}{\gamma-\alpha} f(c)\,dc \;\leq\; \int_I \pi_{v_0}(c)f(c)\,dc \;\leq\; \Sigma(\tau) = F(\tau),$$

where the first inequality uses (17) and (BIC'), and the second is trivial. By the intermediate value theorem, there exists $\phi \in [\underline{c}, \tau]$ such that

$$\Sigma(\phi) = \int_I \pi_{v_0}(c) f(c) dc. \tag{20}$$

Define $\hat{\pi}^{\phi}_{v_0}(c) := 1$ for $c \le \phi$ and $\hat{\pi}^{\phi}_{v_0}(c) := \frac{c-\alpha}{\gamma-\alpha}$ for $c > \phi$; set $\hat{I} := [\underline{c}, \tau]$.

Step 2: Check equilibrium conditions for $\{\hat{\pi}^{\phi}, \hat{I}\}$.

Using (19) and (20),

$$\frac{\int_{\hat{I}} \hat{\pi}^{\phi}_{v_0} f}{\int_{\hat{I}} \hat{\pi}^{\phi}_{v_0} f + \int_N \hat{\pi}^{\phi}_{v_0} f} = \frac{\Sigma(\phi)}{\Sigma(\phi) + \int_\tau^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f} \;\le\; \frac{\int_I \pi_{v_0} f}{\int_I \pi_{v_0} f + \int_N \pi_{v_0} f}, \tag{21}$$

so (SIC-v_0) holds for $\{\hat{\pi}^{\phi}, \hat{I}\}$ whenever it holds for $\{\pi, I\}$. Since $P(\hat{I}) - \Sigma(\phi) = \int_{\hat{I}} \hat{\pi}^{\phi}_r f$ and $P(N) - \int_N \hat{\pi}^{\phi}_{v_0} f = \int_N \hat{\pi}^{\phi}_r f$, we obtain similarly

$$\frac{\int_{\hat{I}} \hat{\pi}^{\phi}_r f}{\int_{\hat{I}} \hat{\pi}^{\phi}_r f + \int_N \hat{\pi}^{\phi}_r f} \;\ge\; \frac{\int_I \pi_r f}{\int_I \pi_r f + \int_N \pi_r f}, \tag{22}$$

so (SIC-r) also carries over. (BIC') holds by construction. Thus $\{\hat{\pi}^{\phi}, \hat{I}\}$ is an equilibrium.

Step 3: Payoff comparisons.

Buyer welfare under $\{\hat{\pi}^{\phi}, \hat{I}\}$ is

$$V_B(\hat{\pi}^{\phi}, \hat{I}) = (\gamma - \alpha)\left[\int_{\underline{c}}^{\phi} f + \int_{\phi}^{\tau} \frac{c-\alpha}{\gamma-\alpha} f\right] + \int_{\underline{c}}^{\tau} (\alpha - c) f = (\gamma-\alpha)\,\Sigma(\phi) + \int_{\underline{c}}^{\tau} (\alpha - c) f.$$

Using (20) and (17),

$$V_B(\hat{\pi}^{\phi}, \hat{I}) \;\ge\; (\gamma - \alpha)\int_I \pi_{v_0}(c) f(c)\, dc + \int_I (\alpha - c) f(c)\, dc = V_B(\pi, I). \tag{23}$$

For the seller,

$$\begin{aligned} V_S(\pi, I) &= P(I) v_0 + (R - v_0)\int_I \big[1 - \pi_{v_0}(c)\big] f(c)\, dc + v_0 \int_N \pi_{v_0}(c) f(c)\, dc \\ &\le P(\hat{I}) v_0 + (R - v_0)\big[P(\hat{I}) - \Sigma(\phi)\big] + v_0 \int_\tau^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)\, dc = V_S(\hat{\pi}^{\phi}, \hat{I}), \end{aligned} \tag{24}$$

using (19) and (20). Hence both payoffs weakly improve. Therefore, the original equilibrium $\{\pi, I\}$ is Pareto-dominated by the DC equilibrium $\{\hat{\pi}^{\phi}, [\underline{c}, \tau]\}$. □

We prove an additional result in relation to Lemma 4 that will be used later.

Lemma 6. *In Lemma 4, both inequalities are strict if I is not a cutoff strategy,*[18] *or if $\pi_{v_0}(c) \neq \frac{c-\alpha}{\gamma-\alpha}$ for a subset of types $c \notin I$ with positive measure.*

Proof. Suppose first that $I \neq [\underline{c}, \tau]$, i.e., $N = C \setminus I \neq [\tau, \bar{c}]$. Then (18) implies that the second inequality in (19) is strict. Alternatively, if $I = [\underline{c}, \tau]$ but $\pi_{v_0}(c) \neq \frac{c-\alpha}{\gamma-\alpha}$ for some positive measure of $c > \tau$, then the first inequality in (19) is strict. In either case,

$$\int_N \pi_{v_0}(c) f(c) dc < \int_\tau^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c) dc. \tag{25}$$

Given (25), choose ϕ to satisfy

$$\Sigma(\phi) = \int_I \pi_{v_0}(c) f(c)\, dc + \varepsilon, \tag{26}$$

for some small $\varepsilon > 0$. Since the gap in (25) is positive, there exists such an ε for which both inequalities (21) and (22) continue to hold, ensuring that $\{\hat{\pi}^\phi, [\underline{c}, \tau]\}$ remains an equilibrium.

With $\varepsilon > 0$, the inequalities in (23) and (24) become strict, so that both $V_B(\hat{\pi}^\phi, [\underline{c}, \tau]) > V_B(\pi, I)$ and $V_S(\hat{\pi}^\phi, [\underline{c}, \tau]) > V_S(\pi, I)$. This proves the result. □

A.3 Proof of Lemma 5

Proof.

$$\frac{\partial \beta_{v_0}(\phi,\tau)}{\partial \phi} = \frac{f(\phi)\frac{\gamma-\phi}{\gamma-\alpha}\int_\tau^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)dc}{\left[F(\phi) + \int_\phi^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)dc\right]^2} > 0 \quad ; \quad \frac{\partial \beta_{v_0}(\phi,\tau)}{\partial \tau} = \frac{\frac{\tau-\alpha}{\gamma-\alpha} f(\tau)}{F(\phi) + \int_\phi^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)dc} > 0.$$

$$\frac{\partial \beta_r(\phi,\tau)}{\partial \phi} = -\frac{\frac{\gamma-\phi}{\gamma-\alpha} f(\phi)\int_\tau^{\bar{c}} \frac{\gamma-c}{\gamma-\alpha} f(c)dc}{\left[\int_\phi^{\bar{c}} \frac{\gamma-c}{\gamma-\alpha} f(c)dc\right]^2} < 0 \quad ; \quad \frac{\partial \beta_r(\phi,\tau)}{\partial \tau} = \frac{\frac{\gamma-\tau}{\gamma-\alpha} f(\tau)}{\int_\phi^{\bar{c}} \frac{\gamma-c}{\gamma-\alpha} f(c)dc} > 0.$$

□

[18] I is a cutoff strategy if $I = [\underline{c}, \tau]$ for some $\tau \in [\underline{c}, \bar{c}]$.

A.4 Proof of Proposition 1

Proof. We first show that $\beta_{v_0}(\phi,\tau)=\frac{v_0}{R}=\beta_r(\phi,\tau)$ if and only if $\{\phi,\tau\}=\{\phi_B^*,\tau_B^*\}$ defined in (12) and (13). The condition $\beta_{v_0}(\phi,\tau)=\frac{v_0}{R}$ can be rewritten as:

$$F(\phi)+\int_\phi^\tau \frac{c-\alpha}{\gamma-\alpha}f(c)\,dc=\frac{v_0}{R}\left[F(\phi)+\int_\phi^{\bar{c}}\frac{c-\alpha}{\gamma-\alpha}f(c)\,dc\right]. \tag{27}$$

Similarly, $\beta_r(\phi,\tau)=\frac{v_0}{R}$ can be rewritten as:

$$F(\tau)-\left[F(\phi)+\int_\phi^\tau \frac{c-\alpha}{\gamma-\alpha}f(c)\,dc\right]=\frac{v_0}{R}-\left[\frac{v_0}{R}F(\phi)+\frac{v_0}{R}\int_\phi^{\bar{c}}\frac{c-\alpha}{\gamma-\alpha}f(c)\,dc\right]. \tag{28}$$

Substituting (27) into (28) yields

$$F(\tau)=\frac{v_0}{R},$$

implying $\tau=\tau_B^*$ in (12). Next, from (27) we have

$$F(\phi)-\int_{\underline{c}}^\phi \frac{c-\alpha}{\gamma-\alpha}f(c)\,dc+\int_{\underline{c}}^\tau \frac{c-\alpha}{\gamma-\alpha}f(c)\,dc=\frac{v_0}{R}\left[F(\phi)-\int_{\underline{c}}^\phi \frac{c-\alpha}{\gamma-\alpha}f(c)\,dc+\int_{\underline{c}}^{\bar{c}}\frac{c-\alpha}{\gamma-\alpha}f(c)\,dc\right]$$

$$\iff \int_{\underline{c}}^\phi(\gamma-c)f(c)\,dc+\int_{\underline{c}}^\tau(c-\alpha)f(c)\,dc=\frac{v_0}{R}\left[\int_{\underline{c}}^\phi(\gamma-c)f(c)\,dc+E[c]-\alpha\right]$$

$$\iff \underbrace{\int_{\underline{c}}^\phi(\gamma-c)f(c)\,dc}_{V_B(\hat{\pi}^\phi,[\underline{c},\tau])}=\frac{v_0/R}{1-v_0/R}\left[E[c]-\alpha-\frac{\int_{\underline{c}}^\tau cf(c)\,dc-F(\tau)\alpha}{v_0/R}\right]. \tag{29}$$

The left-hand side of (29) is the buyer welfare under the DC equilibrium $\{\phi,\tau\}$. Since it is strictly increasing in ϕ, whereas the right-hand side of (29) is strictly decreasing in τ. Hence, there exists a unique pair (ϕ,τ) satisfying (29). Substituting $\tau=\tau_B^*$ (where $F(\tau_B^*)=\frac{v_0}{R}$) gives u_B^* in (13), and the corresponding ϕ is ϕ_B^*.

Unattainability of $u_B>u_B^*$. Suppose, for contradiction, that some $\hat{u}_B>u_B^*$ is attainable. By Lemma 4, it must be attainable by a DC equilibrium $\{\hat{\phi},\hat{\tau}\}$ such that

$$\int_{\underline{c}}^{\hat{\phi}}(\gamma-c)f(c)\,dc=\hat{u}_B.$$

Then $\hat{u}_B>u_B^*$ implies $\hat{\phi}>\Phi(u_B^*)$. By Lemma 5, condition (SIC-v_0) implies

$$\beta_{v_0}(\hat{\phi},\hat{\tau})\le\frac{v_0}{R}=\beta_{v_0}(\phi_B^*,\tau_B^*)<\beta_{v_0}(\hat{\phi},\tau_B^*)\;\Rightarrow\;\hat{\tau}<\tau_B^*.$$

However, from (SIC-r),

$$\beta_r(\hat{\phi},\hat{\tau}) \geq \frac{v_0}{R} = \beta_r(\phi_B^*,\tau_B^*) > \beta_r(\hat{\phi},\tau_B^*) \;\Rightarrow\; \hat{\tau} > \tau_B^*.$$

This contradiction implies that no $\hat{u}_B > u_B^*$ is attainable.

Boundary condition. The pair $\{\Phi(u_B), T(u_B)\}$ forms a DC equilibrium that attains $(u_B, \bar{U}_S(u_B))$ for every $u_B \in [0, u_B^*]$.[19] Since $\Phi(0) = \underline{c}$,

$$\beta_{v_0}\Big(\Phi(0), T(0)\Big) = \frac{v_0}{R} \iff \frac{\int_{\underline{c}}^{T(0)} \frac{c-\alpha}{\gamma-\alpha} f(c)\,dc}{\int_{\underline{c}}^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)\,dc} = \frac{v_0}{R} \iff \int_{\underline{c}}^{T(0)} (c-\alpha) f(c)\,dc = \frac{v_0}{R}\Big(E[c]-\alpha\Big),$$

so $T(0) = \tau_S^*$ in (12). We show that $T' < 0$ in (33) below. □

A.5 Proof of Proposition 2

Proof. **Necessity.** Let $\{\pi, I\}$ attain $\big(u_B, \bar{U}_S(u_B)\big)$ on the Pareto frontier. By Lemma 6, I must be a cutoff strategy $[\underline{c}, \tau]$ and $\pi_{v_0}(c) = \frac{c-\alpha}{\gamma-\alpha}$ for all $c > \tau$. Hence buyer welfare equals

$$\begin{aligned} &\int_{\underline{c}}^{\tau} [\alpha + \pi_{v_0}(c)(\gamma-\alpha) - c] f(c) d = u_B = \int_{\underline{c}}^{\Phi(u_B)} (\gamma - c) f(c) dc, \\ \iff &\int_{\underline{c}}^{\tau} \pi_{v_0}(c) f(c) dc = F(\Phi(u_B)) + \int_{\Phi(u_B)}^{\tau} \frac{c-\alpha}{\gamma-\alpha} f(c) dc, \end{aligned} \tag{30}$$

the seller's profit is

$$\begin{aligned} &\int_{\underline{c}}^{\tau} \left[R - \pi_{v_0}(c)(R - v_0)\right] f(c) dc + \int_{\tau}^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} v_0 f(c) dc \\ =&F(\tau)R - (R - v_0) \underbrace{\left[F(\Phi(u_B)) + \int_{\Phi(u_B)}^{\tau} \frac{c-\alpha}{\gamma-\alpha} f(c) dc\right]}_{=\int_{\underline{c}}^{\tau} \pi_{v_0}(c) f(c) dc \text{ from } (30)} + \int_{\tau}^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} v_0 f(c) dc \\ =&F(\Phi(u_B)) v_0 + R \int_{\Phi(u_B)}^{\tau} \left[1 - \frac{c-\alpha}{\gamma-\alpha}\right] f(c) dc + \int_{\Phi(u_B)}^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} v_0 f(c) dc \\ =&\hat{V}_S\left(\Phi(u_B), \tau\right) \qquad \text{from } (10) \end{aligned}$$

Since $\bar{U}_S(u_B) = \hat{V}_S\Big(\Phi(u_B), T(u_B)\Big)$ and $\hat{V}_S(\phi, \tau)$ is strictly increasing in τ (see Footnote 14), we must have $\tau = T(u_B)$. This establishes (1)-(3) as necessary.

[19] Note that $\Phi(u_B) < T(u_B)$ is required, which follows from $\frac{v_0}{R} < \beta_r(\Phi(u_B), T(u_B))$.

Sufficiency. Now suppose $\{\pi, I\}$ satisfies (1)-(3). Then $I = [\underline{c}, T(u_B)]$, buyer welfare equals u_B by (3), and by the computation above the seller's profit equals $\hat{V}_S(\Phi(u_B), T(u_B)) = \bar{U}_S(u_B)$. (BIC') holds by (2)-(3). It remains to verify (SIC-r) and (SIC-v_0).

(SIC-r) for the DC equilibrium $\left\{\Phi(u_B), T(u_B)\right\}$ implies

$$\frac{v_0}{R} \leq \frac{\int_{\Phi(u_B)}^{T(u_B)} \left[1 - \frac{c-\alpha}{\gamma-\alpha}\right] f(c)dc}{\int_{\Phi(u_B)}^{\bar{c}} \left[1 - \frac{c-\alpha}{\gamma-\alpha}\right] f(c)dc} = \frac{F(T(u_B)) - F(\Phi(u_B)) - \int_{\Phi(u_B)}^{T(u_B)} \frac{c-\alpha}{\gamma-\alpha} f(c)dc}{1 - F(\Phi(u_B)) - \int_{\Phi(u_B)}^{T(u_B)} \frac{c-\alpha}{\gamma-\alpha} f(c)dc - \int_{T(u_B)}^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)dc}$$
$$= \frac{F(T(u_B)) - \int_{\underline{c}}^{T(u_B)} \pi_{v_0}(c) f(c)dc}{1 - \int_{\underline{c}}^{T(u_B)} \pi_{v_0}(c) f(c)dc - \int_{T(u_B)}^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)dc} = \frac{\int_{\underline{c}}^{T(u_B)} [1 - \pi_{v_0}(c)] f(c)dc}{\int_{\underline{c}}^{\bar{c}} [1 - \pi_{v_0}(c)] f(c)dc}, \tag{31}$$

where (31) follows from (30). This implies that (SIC-r) holds for $\{\pi, I\}$. Next, (SIC-v_0) binds for the DC equilibrium $\left\{\Phi(u_B), T(u_B)\right\}$, which implies that

$$\frac{v_0}{R} = \frac{F(\Phi(u_B)) + \int_{\Phi(u_B)}^{T(u_B)} \frac{c-\alpha}{\gamma-\alpha} f(c)dc}{F(\Phi(u_B)) + \int_{\Phi(u_B)}^{T(u_B)} \frac{c-\alpha}{\gamma-\alpha} f(c)dc + \int_{T(u_B)}^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)dc}$$
$$= \frac{\int_{\underline{c}}^{T(u_B)} \pi_{v_0}(c) f(c)dc}{\int_{\underline{c}}^{T(u_B)} \pi_{v_0}(c) f(c)dc + \int_{T(u_B)}^{\bar{c}} \frac{c-\alpha}{\gamma-\alpha} f(c)dc} = \frac{\int_{\underline{c}}^{T(u_B)} \pi_{v_0}(c) f(c)dc}{\int_{\underline{c}}^{\bar{c}} \pi_{v_0}(c) f(c)dc}, \tag{32}$$

where (32) follows from (30). Thus, (SIC-v_0) also binds for $\{\pi, I\}$. Thus all three conditions are sufficient, and the proposition follows. □

A.6 Proof of Proposition 3

Proof. Applying the implicit function theorem on (11),

$$
\begin{aligned}
T'(u_B) &= -\left[\frac{\partial \beta_{v_0}(\phi,\tau)}{\partial \phi} \times \frac{\partial \Phi(u_B)}{\partial u_B}\right] \div \left.\frac{\partial \beta_{v_0}(\phi,\tau)}{\partial \tau}\right|_{\phi=\Phi(u_B),\tau=T(u_B)} \\
&= -\left[\frac{f(\phi)\frac{\gamma-\phi}{\gamma-\alpha}\int_\tau^{\bar c}\frac{c-\alpha}{\gamma-\alpha}f(c)dc}{\left(F(\phi)+\int_\phi^{\bar c}\frac{c-\alpha}{\gamma-\alpha}f(c)dc\right)^2} \times \frac{1}{(\gamma-\phi)f(\phi)}\right] \div \left.\frac{\frac{\tau-\alpha}{\gamma-\alpha}f(\tau)}{F(\phi)+\int_\phi^{\bar c}\frac{c-\alpha}{\gamma-\alpha}f(c)dc}\right|_{\phi=\Phi(u_B),\tau=T(u_B)} \\
&= -\frac{\int_{T(u_B)}^{\bar c}\frac{c-\alpha}{\gamma-\alpha}f(c)dc}{\left[F(\Phi(u_B))+\int_{\Phi(u_B)}^{\bar c}\frac{c-\alpha}{\gamma-\alpha}f(c)dc\right][T(u_B)-\alpha]\,f(T(u_B))} \\
&= -\left[1-\beta_{v_0}\left(\Phi(u_B),T(u_B)\right)\right] \times \frac{1}{[T(u_B)-\alpha]\,f(T(u_B))} \\
&= -\left[1-\frac{v_0}{R}\right] \times \frac{1}{[T(u_B)-\alpha]\,f(T(u_B))} \;<\; 0
\end{aligned}
\tag{33}
$$

Since $\bar U_S(u_B) = F\big(T(u_B)\big)R$,

$$
\bar U_S'(u_B) = f\big(T(u_B)\big)T'(u_B)R = -\frac{\left[1-\frac{v_0}{R}\right]R}{T(u_B)-\alpha} = -\frac{R-v_0}{T(u_B)-\alpha}
$$

Thus, $\bar U_S'(u_B) < -1 \iff R+\alpha-v_0 > T(u_B)$. Note that $R+\alpha = \int_r^{\bar v} v d\chi(v)$, which implies the second statement of the proposition. Since $\tau_S^* \geq T(u_B) \geq \tau_B^*$ for all $u_B \in [0,u_B^*]$, $R+\alpha-v_0 > \tau_S^*$ implies that $\bar U_S'(u_B) < -1$ for all $u_B \in [0,u_B^*]$, whereas $R+\alpha-v_0 \leq \tau_B^*$ implies that $\bar U_S'(u_B) \geq -1$ for all $u_B \in [0,u_B^*]$. This implies the two bullet points in the proposition. □

A.7 Proof of Proposition 4

Proof. We know that $\beta_{v_0}\big(\Phi(u_B),T(u_B)\big) = \frac{v_0}{R} < \beta_r\big(\Phi(u_B),T(u_B)\big)$ for all $u_B < u_B^*$. From Lemma 5, both $\beta_{v_0}\big(\Phi(u_B),\cdot\big)$ and $\beta_r\big(\Phi(u_B),\cdot\big)$ are strictly increasing. Define $\underline{T}(u_B)$ by $\beta_r\big(\Phi(u_B),\underline{T}(u_B)\big) = \frac{v_0}{R}$. This means for all $\tau \in [\underline{T}(u_B),T(u_B)]$, we have $\beta_{v_0}\big(\Phi(u_B),\tau\big) \leq \frac{v_0}{R} \leq \beta_r\big(\Phi(u_B),\tau\big)$, thereby implying that $\{\Phi(u_B),\tau\}$ is a DC equilibrium. $\beta_r\big(\Phi(u_B),\underline{T}(u_B)\big) = \frac{v_0}{R}$ implies that $\hat V_S\big(\Phi(u_B),\underline{T}(u_B)\big) = v_0$. Since $\hat V_S$ increases continuously in τ, for all $u_S \in [v_0,\bar U_S(u_B)]$, there exists $\tau \in [\underline{T}(u_B),T(u_B)]$ such that $\hat V_S(\Phi(u_B),\tau) = u_S$. The buyer welfare under the DC equilibrium $\{\Phi(u_B),\tau\}$ is u_B for any τ. This proves the result. □

A.8 Proof of Proposition 5

For a distribution J , let $E_J[c]$ denote its mean, and define its Lorenz curve by:

$$L_J(q) := \frac{\int_0^q J^{-1}(z)dz}{\int_0^1 J^{-1}(z)dz} = \frac{\int_{\underline{c}}^{J^{-1}(q)} cdJ(c)}{E_J[c]} \qquad \text{for } q \in [0,1], \tag{34}$$

where the second equality uses the change of variable of $z = J(c)$. The following property follows from Theorem 3.A.10 of Shaked and Shanthikumar (2007).

Theorem 0. *If F is a mean-preserving spread of G, then $L_F(q) \leq L_G(q)$ for all $q \in [0,1]$.*

Proof of Proposition 5. Assume that F is a mean-preserving spread of G. For distribution $J \in \{F, G\}$, let $\tau_B^{J*} = J^{-1}\left(\frac{v_0}{R}\right)$. From (13),

$$u_B^{J*} = \frac{v_0/R}{1 - v_0/R}\left(E_J[c] - \frac{\int_{\underline{c}}^{\tau_B^{J*}} cdJ(c)}{v_0/R}\right) = \frac{E_J[c]}{1 - v_0/R}\left[\frac{v_0}{R} - L_J\left(\frac{v_0}{R}\right)\right].$$

Since $E_F[c] = E_G[c]$ and $L_F\left(\frac{v_0}{R}\right) \leq L_G\left(\frac{v_0}{R}\right)$ by Theorem 0, we obtain $u_B^{F*} \geq u_B^{G*}$.

Fix any $u_B \in [0, u_B^{G*}]$. For a given distribution J, we include a superscript "J" for $\Phi(\cdot)$, $T(\cdot)$, and $\beta_{v_0}(\cdot)$ to denote the distribution that these functions are defined with respect to. $\int_{\underline{c}}^{\Phi^J(u_B)}(\gamma - c)dJ(c) = u_B$ implies that $J\left(\Phi^J(u_B)\right) = \frac{u_B}{\gamma} + \int_{\underline{c}}^{\Phi(u_B)} \frac{c}{\gamma}dJ(c)$. Substitute this into $\beta_{v_0}^J\left(\Phi^J(u_B), T^J(u_B)\right) = \frac{v_0}{R}$ implies that

$$\frac{v_0}{R} = \frac{\frac{u_B}{\gamma} + \int_{\underline{c}}^{T^J(u_B)} \frac{c}{\gamma}dJ(c)}{\frac{u_B}{\gamma} + \int_{\underline{c}}^{\bar{c}} \frac{c}{\gamma}dJ(c)} = \frac{u_B + \int_{\underline{c}}^{T^J(u_B)} cdJ(c)}{u_B + E_J[c]} \quad \text{for } J \in \{F, G\}.$$

Since $E_G[c] = E_F[c]$, which we denote by $E[c]$,

$$\begin{aligned}
&\frac{u_B + \int_{\underline{c}}^{T^G(u_B)} cdG(c)}{u_B + E_G[c]} = \frac{u_B + \int_{\underline{c}}^{T^F(u_B)} cdF(c)}{u_B + E_F[c]} \\
\iff &(u_B + E[c])\, E[c]\left[\frac{\int_{\underline{c}}^{T^F(u_B)} cdF(c)}{E_F[c]} - \frac{\int_{\underline{c}}^{T^G(u_B)} cdG(c)}{E_G[c]}\right] = 0 \\
\implies &L_F\left(F\left(T^F(u_B)\right)\right) = L_G\left(G\left(T^G(u_B)\right)\right) \;\geq\; L_F\left(G\left(T^G(u_B)\right)\right),
\end{aligned}$$

where the last inequality follows from Theorem 0. Since the Lorenz curve is strictly increasing, it follows that $F\left(T^F(u_B)\right) \geq G\left(T^G(u_B)\right)$. By Proposition 3, $\bar{U}_S^J(u_B) = J\left(T^J(u_B)\right)R$. Hence, $\bar{U}_S^F(u_B) \geq \bar{U}_S^G(u_B)$. □

A.9 Proof of Proposition 6

Proof. If there exists a set C_0 that satisfies (14), then, as explained in the main text after the proposition, the signal structure in (15), together with $I = C_+^W \cup C_0$, is an equilibrium that attains W. To prove the converse, suppose that $\{\pi, I\}$ is an equilibrium that attains W. We know that C_+^W must be a subset of I, and $\pi_{v_0}(c) = \frac{W(c)+c-\alpha}{\gamma-\alpha}$ for all $c \in C_+^W$. Let $C_0 = I \setminus C_+^W$, and $N = C \setminus I$. (BIC') implies that $\pi_{v_0}(c) = \frac{c-\alpha}{\gamma-\alpha}$ for all $c \in C_0$, and $\pi_{v_0}(c) \leq \frac{c-\alpha}{\gamma-\alpha}$ for all $c \in N$. Constraint (SIC-v_0) for $\{\pi, I\}$ requires that:

$$\frac{v_0}{R} \geq \frac{\int_{C_+^W} \frac{W(c)+c-\alpha}{\gamma-\alpha} f(c)dc + \int_{C_0} \frac{c-\alpha}{\gamma-\alpha} f(c)dc}{\int_{C_+^W} \frac{W(c)+c-\alpha}{\gamma-\alpha} f(c)dc + \int_{C_0} \frac{c-\alpha}{\gamma-\alpha} f(c)dc + \int_N \pi_{v_0}(c) f(c)dc}$$
$$\geq \frac{\int_{C_+^W} \frac{W(c)+c-\alpha}{\gamma-\alpha} f(c)dc + \int_{C_0} \frac{c-\alpha}{\gamma-\alpha} f(c)dc}{\int_{C_+^W} \frac{W(c)+c-\alpha}{\gamma-\alpha} f(c)dc + \int_{C_0} \frac{c-\alpha}{\gamma-\alpha} f(c)dc + \int_N \frac{c-\alpha}{\gamma-\alpha} f(c)dc}.$$

Thus, C_0 must satisfy the first inequality in (14). Constraint (SIC-r) for $\{\pi, I\}$ requires that:

$$\frac{v_0}{R} \leq \frac{\int_{C_+^W} \left[\frac{\gamma-c-W(c)}{\gamma-\alpha}\right] f(c)dc + \int_{C_0} \frac{\gamma-c}{\gamma-\alpha} f(c)dc}{\int_{C_+^W} \frac{\gamma-c-W(c)}{\gamma-\alpha} f(c)dc + \int_{C_0} \frac{\gamma-c}{\gamma-\alpha} f(c)dc + \int_N \left[1 - \pi_{v_0}(c)\right] f(c)dc}$$
$$\leq \frac{\int_{C_+^W} \frac{\gamma-c-W(c)}{\gamma-\alpha} f(c)dc + \int_{C_0} \frac{\gamma-c}{\gamma-\alpha} f(c)dc}{\int_{C_+^W} \frac{\gamma-c-W(c)}{\gamma-\alpha} f(c)dc + \int_{C_0} \frac{\gamma-c}{\gamma-\alpha} f(c)dc + \int_N \left[1 - \frac{c-\alpha}{\gamma-\alpha}\right] f(c)dc},$$

Thus, C_0 must satisfy the second inequality in (14). Therefore, if $\{\pi, I\}$ is an equilibrium that attains W, there must exist C_0 that satisfies (14).

For the last statement, the "only if" direction follows from Proposition 2. We prove the "if" direction using the first part of the proposition and showing that if $u_B^W \in [0, u_B^*]$ and C_+^W is a subset of $[\underline{c}, T(u_B^W)]$, then (14) holds for $C_0 = [\underline{c}, T(u_B^W)] \setminus C_+^W$. First,

$$\int_{C_+^W} W(c)f(c)dc = u_B^W = \int_{\underline{c}}^{\Phi(u_B^W)} (\gamma - c)f(c)dc \tag{35}$$

(35) implies that

$$\int_{C_+^W} \frac{W(c) + c - \alpha}{\gamma - \alpha} f(c)dc + \int_{C_0} \frac{c - \alpha}{\gamma - \alpha} f(c)dc \;=\; \int_{\underline{c}}^{\Phi(u_B^W)} \frac{\gamma - \alpha}{\gamma - \alpha} f(c)dc + \int_{C_0 \cup C_+^W} \frac{c - \alpha}{\gamma - \alpha} f(c)dc$$
$$= F(\Phi(u_B^W)) - \int_{\underline{c}}^{\Phi(u_B^W)} \frac{c - \alpha}{\gamma - \alpha} f(c)dc + \int_{C_0 \cup C_+^W} \frac{c - \alpha}{\gamma - \alpha} f(c)dc \;=\; F(\Phi(u_B^W)) + \int_{\Phi(u_B^W)}^{T(u_B^W)} \frac{c - \alpha}{\gamma - \alpha} f(c)dc.$$

In turn, (SIC-v_0) in (8) for $\{\Phi(u_B^W), T(u_B^W)\}$ implies the first inequality of (14).

(35) and $C_0 \cup C_+^W = [\underline{c}, T(u_B)]$ also imply that

$$\int_{C_+^W} \frac{\gamma - c - W(c)}{\gamma - \alpha} f(c)dc + \int_{C_0} \frac{\gamma - c}{\gamma - \alpha} f(c)dc$$
$$= \int_{C_+^W} \frac{\gamma - c}{\gamma - \alpha} f(c)dc - \int_{\underline{c}}^{\Phi(u_B^W)} \frac{\gamma - c}{\gamma - \alpha} f(c)dc + \int_{C_0} \frac{\gamma - c}{\gamma - \alpha} f(c)dc = \int_{\Phi(u_B^W)}^{T(u_B^W)} \frac{\gamma - c}{\gamma - \alpha} f(c)dc.$$

In turn, (SIC-r) in (8) for $\{\Phi(u_B^W), T(u_B^W)\}$ implies the second inequality of (14). □

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