PARAMETER VARIABILITY, LEARNING AND INFLATION TARGETING

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Abstract

The present essay analyses the performance of a central bank to achieve the learnability of parameter values that govern the Phillips Curve (PC) equation while trying to stabilize the inflation rate. The present essay shows the ability of policymakers to achieve both objectives simultaneously. Using a Recursive Least Squares (RLS) algorithm to model the learning process performed by the central bank, it will iteratively estimate the parameters, which account for the level of persistence in inflation and the impact of increases in the policy interest rate on the inflation rate in the economy. Various simulation scenarios will confirm that the central's bank estimates, with the addition of sufficient exogenous variation, can progressively converge to their actual mean values. Learnability will also be shown to be robust to the inclusion of time-variability in the structure of parameters that compose the PC equation.

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List of Figures

List of Tables

Contents

1 Introduction

Central banks commonly face a dual problem in an inflation-targeting regime. First, they seek to control, at a stable level, the growth of the price level in the economy. Second, policymakers must also estimate the values of the time-varying parameters that define the decision equations in order to achieve their policy objectives. With additional information, central bankers can gradually improve their assessment of the impact of macroeconomic variables on each other. Ultimately, policymakers are interested in learning the actual value of the parameters to respond proportionally to changes in the inflation rate relative to its target by varying their policy instrument accordingly.

Moreover, central banks must ensure that the models and equations guiding monetary policy decisions are well specified to mitigate potential model uncertainty. In this research, I will assume that the equations are specified correctly. However, policymakers have to operate in an uncertain environment in which the exact value of the parameters that govern these relationships remain unknown. Thus, in addition to their mission of stabilizing selected macroeconomic variables such as the inflation rate, central banks must simultaneously estimate the parameters that weight the impacts of variations in the policy interest rate and the output gap on the observed inflation rate. By identifying these parameters more precisely, the central bank will be able to control the level of inflation adequately. However, these parameters are not directly observed by the central banks, and their estimation may fluctuate over time according to their levels of variability.

The present research will focus on the uncertainty that may lie in the values of the various parameters incorporated in structural macroeconomic equations. Thus, this essay will address whether, in the presence of parameter variability, central banks can learn the value of the parameters that make up the realizations of the inflation rate and simultaneously control inflation, so it remains reasonably close to its target. This essay explores the use of a Phillips Curve (PC) equation by the central bank to define the relationship guiding the readings of the inflation rate with the policy interest rate as well as with past inflation observations. In addition, the model structure will include multiple interest-rate decision rules to numerically assess if the central bank can regulate variations in inflation using a quadratic single-period loss function while gradually learning the value of the parameters under various scenarios. Numerical methods will be used to model the learning process conducted by the central bank. Several exercises will be carried out using Recursive Least Squares (RLS) estimation methods to assess the learning performance of the monetary authority over time.

Following the numerical estimation of the model structure, the central bank will be able to learn the actual value of the parameters while controlling the variance of inflation reasonably well with a feedback interest-rate equation. It will require the interest-rate rule to have a sizeable component that remains exogenous from the monetary policy decision of the central bank. The central bank will then be able to recursively estimate the value of parameters that constitute the PC equation with sufficient exogenous variation. The pace of convergence will vary according to the specification of the loss functions and its resulting interest-rate decision rules. In addition, the central bank will achieve learnability when the values of some parameters incorporate time-variability and remain uncertain to policymakers. In the trade-off between the learning process and controlling the target variable, the monetary authority will be unable to mitigate deviations of the inflation rate from its target entirely. Since this learning process has to be repeated iteratively, the central bank will never entirely stabilize inflation on a given iteration of the recursive algorithm.

Moreover, the essay will demonstrate that the central bank adopts a more cautious optimal stance on its policy rate when facing parameter variability on the coefficient that governs the impact of the interest rate on the level of inflation. In doing so, the central bank will prevent potentially significant costs associated with acute deviations in the inflation rate incurred due to the specification of its loss function. It will also be shown that the results are robust to multiple extensions of the model structure, such as enhanced persistence in the definition of parameters, the introduction of an intercept in the PC equation, and the introduction of variability on multiple parameters that define contemporaneous realizations of inflation. Furthermore, the model structure will be endowed with an IS equation to account for the impact of variations of the output gap on the inflation rate. A robustness exercise will also be conducted to confirm that the choice of parameters does not interfere in the findings that the central bank can achieve learnability and price stability.

Finally, I will test whether the selection of algorithms used by the central bank to estimate the parameter values has a significant impact on its control over the level of inflation and on its learning ability. The simulations will be mostly conducted with an expanding window algorithm. However, in the face of persistence in the law of motion of some parameters, one could be interested in knowing if a rolling window algorithm would follow more closely the actual observations of the parameters. A hypothesis is that the central bank could learn in a better fashion the values of persistent coefficients with a shorten historical sample. Furthermore, a theoretical exercise will show the similitudes between the use of an RLS estimation algorithm and a Kalman filter algorithm to model the learning process of the central bank.

The rest of the essay is structured as follows. Section 2 reviews the literature on the optimal response adopted by central banks in the presence of parameter variability and how policymakers can progressively learn the parameter values to structure their interest rate decisions. Section 3 describes a benchmark example of the model structure as well as the sequence of events of the iterative learning process conducted by the central bank using numerical methods. Section 4 illustrates a model structure departing from the benchmark example, which introduces a feedback interest-rate equation aiming at better stabilizing the inflation rate. Section 5 discusses the impact of introducing parameter variability on the optimal behavior by the central bank. Section 6 details whether findings of learnability hold when varying the values of the parameters in the model. Section 7 covers various extensions to the model structure presented in section 5, which incorporates parameter variability. Section 8 concludes and offers suggestions for further research.

2 Related Research

Several studies have investigated the effects of the introduction of uncertainty on the response of central banks. The novel research of Brainard (1967) showed that multiplicative uncertainty in the parameter values would lead policymakers to adopt a more cautious approach in their interest rate responses to economic disturbances compared to the certainty equivalent alternative. Their optimal choice of policy instrument, computed with a static quadratic loss function, is defined as inversely related to the variance of the impact parameter that guides the response of inflation to the policy interest rate. The presence of multiplicative uncertainty means that changes in interest rates have the potential to increase uncertainty regarding the path of the economy further. Subsequently, policymakers would adopt a more cautious approach when facing an uncertain environment to stabilize inflation optimally. They would become reluctant to change their interest-rate stances sharply to minimize deviations in inflation when multiplicative uncertainty is introduced on the elasticity of inflation to interest rate variations. This is commonly referred to as the “Brainard Uncertainty Principle” in the literature.

Wieland (2003) confirmed the existence of the Brainard Principle as he also found that the optimal response from the central bank to shocks to inflation would require a more cautious policy stance than in the certainty-equivalent scenario. Following an initial shock, the inflation rate would then gradually return towards its target. Wieland (2003) explains that the gradualist approach would remain optimal for various instances of parameter uncertainty induced in a myopic model structure. For example, introducing parameter uncertainty on the intercept of the Phillips Curve, on the transition dynamics of inflation and on the interest-rate impact coefficient would all lead to the adoption of a cautious response to optimally stabilize inflation. However, when incorporating a dynamic structure to the model, the central bank then faces a trade-off. It needs to mitigate current deviations of the inflation rate from its target while simultaneously experimenting to obtain more precise estimates of the parameters in the PC equation to improve future performance. In this instance, the central bank will be inclined to act more decisively compared to its myopic counterpart.

Craine (1979) introduced a dynamic model with random coefficients that incorporates multiple sources of uncertainty. In contrast to Brainard (1967), which used a static single-equation model with a unique random policy coefficient, Craine (1979) found that increased uncertainty would not necessarily lead to a more cautious policy stance adopted by the central bank. He found that the optimal policy responses would depend on the relative uncertainty between the transition and the impact parameters. Increased uncertainty on the impact of the policy rate on the inflation rate would make policymakers more risk-averse, leading to a less aggressive policy behavior. On the other hand, uncertainty about the dynamics of the model, introduced as a positive variance of the transition parameter, would call for a proactive response to mitigate the amplification mechanism of the exogenous shocks over time. Moreover, Söderström (2002) found that, when the central bank attaches some weight to stabilizing output in addition to the level of inflation, the optimal policy becomes more aggressive when uncertain about the value of the coefficient governing the persistence of inflation. However, Söderström (2002) also confirmed the findings of Brainard (1967) that, in the presence of parameter uncertainty on the impact coefficients, the gradual approach would be preferred to mitigate inflation deviations from its target in a strict inflation-targeting regime.

Ferrero, Pietrunti, and Tiseno (2019) instead found that when the planner is uncertain about the parameter values incorporated in a New-Keynesian Phillips Curve design, the optimal monetary policy response, whether to adopt a more gradual or aggressive approach, will depend on the persistence of the shocks that hit the economy. They investigated the impact of introducing uncertainty in the slope of the Phillips curve and in the natural interest rate paired with asymmetric information as the central bank can only observe the shocks with a one-period lag. Ferrero et al. (2019) argue that the optimal behavior under certainty-equivalence would remain optimal when introducing uncertainty regarding the persistence of the technology shock. Policymakers should then react to exogenous shocks as in the full information case. However, when dealing with multiplicative uncertainty on the persistence in the slope of the Phillips curve, the optimal approach would depend on the level of persistence of the cost-push shock. If discovered that the exogenous cost-push shock is highly persistent, policymakers would choose to implement a more cautious approach. When persistence is elevated, there is a higher probability of facing acute welfare losses with the implementation of a misguided aggressive policy response. It would cause a more pronounced divergence of the state variables from their respective targeted equilibrium values.

In the same vein, Kimura and Kurozumi (2007) have illustrated that the presence of uncertainty can cause the central bank to opt for a more aggressive monetary policy. Using Bayesian methods and a hybrid New-Keynesian Phillips curve, they determined the optimal choice of policy stance required to minimize the central bank's loss function while accounting for the given prior distribution of the uncertain coefficients. With the implementation of a micro-founded forward-looking PC equation, Kimura and Kurozumi (2007) confirmed the findings of Söderström (2002) that the appropriate response to uncertainty about the inflation dynamics would lead the central bank to act more aggressively to stabilize inflation following an exogenous shock.

They mention that the difference in results on whether the Brainard Principle held in the presence of uncertainty on inflation dynamics with Wieland (2003) might reside in the fact that they employed a flexible inflation target regime compared to a strict inflation-targeting system. When the central bank attaches a positive weight to output stabilization in its objective function, Kimura and Kurozumi (2007) argue that monetary policy will only limit a fraction of the gap between expected inflation and its target, so that the inflation rate will converge at a slower pace to its target. Thus, the central bank that considers deviations in output will be more affected by uncertainty on the dynamics of inflation and will want to act more decisively in equilibrium. Kimura and Kurozumi (2007) also departs from some findings of Söderström (2002) as the policy stances of the central bank modeled in Kimura and Kurozumi (2007) will not return to a neutral level after responding aggressively initially. Furthermore, contrary to findings made in Söderström (2002), Kimura and Kurozumi (2007) shows that the Brainard Principle would not hold when the structure of aggregate demand remains uncertain. The difference in results mainly resides in the inclusion of a feedback effect in the loss-function uncertainty as well as in the positive correlation between the policy multiplier and the transmission mechanisms of the shocks, according to the authors.

In the learning literature, Wieland (2000) has shown how economic agents would react in a model structure where they faced a trade-off between current control of the targeted variables and experimentation. Experimentation would yield lower contemporaneous payoffs, but it allows one to collect information in an effort to maximize future payoffs. For example, the central bank could deliberately preserve deviations of the inflation rate from its target to learn the parameter values more efficiently. Wieland (2000) found that the experimentation process improved the expected future payoffs of the policymakers. Although acknowledging its existence in the literature, the present research will not account for the experimentation process in the model structure. In this research, the central bank will not able to stabilize inflation entirely. However, it does not consciously allow the inflation rate to deviate from its target in order to experiment and learn the parameter values.

In a similar vein, Beck and Wieland (2002) have found, using numerical approximations, that the optimal behavior of the central bank would incorporate a significant level of experimentation to help policymakers learn the parameter values in an uncertain environment. With a model structure which includes time-varying parameters and lagged dependent variables, Beck and Wieland (2002) have shown the convergence of the parameter estimates to their actual values. Also, with the use of a multi-period loss function, he found that the estimates of the impact coefficient would be biased towards zero in the presence of uncertainty, causing policymakers to prioritize gradualism as the optimal decision to mitigate exogenous shocks.

3 Model Environment and Benchmark Example

In the initial model structure, the central bank's decision-making process is characterized by an inflation equation. The relationship between the level of inflation π_t, the policy interest rate R_t, and the output gap, y_t is commonly referred to as the Phillips Curve (PC) specification. Equation (1) of the current model structure defines the PC equation without the addition of an output gap variable y_t. The inflation rate, depicted in equation (1), is negatively impacted by increases in the policy interest rate R_t. As detailed in the Keynesian literature by Clarida, Gali, and Gertler (1999), increases in the policy interest rate will raise the borrowing costs of economic agents and ultimately slow aggregate demand.

The impact parameter φ captures the contemporaneous response of the level of inflation in the economy to changes in the policy interest rate R_t. In addition, the inflation variable, denoted π_t, is also partially explained by an autoregressive term of order one π_{t-1}. This term helps account for the dynamic variations in the inflation rate. Shocks to inflation will have a persistent impact on its future realizations. It accounts for what is known in the literature as "inflation inertia". Studies such as those of Mavroeidis, Plagborg-Moller, and Stock (2014) and Kimura and Kurozumi (2007) have investigated the potential role of a lag component as a factor driving inertia in the inflation rate. In the model, the impact of past realizations of inflation on its current value is governed by the parameter λ. It is set constant for the initial scenarios that do not include parameter variability on the inflation persistence coefficient.

This scenario will serve as a benchmark example for the subsequent exercises. The central bank has to account for an exogenous independent and normally distributed cost-push shock ϵ_t with mean 0 and variance σ_ϵ^2. In this model structure, policymakers are only observing realizations of the nominal interest rate R_t and inflation π_t. Contrary to the structure in Ferrero et al. (2019), the monetary authority does not know the distribution of the exogenous shocks in the economy. Following the backward-looking structure detailed in Söderström (2002), the realizations of inflation are thus found as follows:

$$\pi_t = -\varphi R_t + \lambda \pi_{t-1} + \epsilon_t \tag{1}$$
$$\epsilon_t \sim N(0, \sigma_\epsilon^2).$$

The interest rate decision variable R_t is defined as an autoregressive process of order one with parameter ρ. It is also characterized by an exogenous independent and normally distributed η_t shock of mean 0 and a variance of 1. Thus, the interest rate is defined accordingly:

$$R_t = \rho R_{t-1} + \eta_t,\ 0 < \rho < 1 \tag{2}$$
$$\eta_t \sim N(0, 1).$$

The initial values of the parameters φ, λ and ρ are set to be equal to 0.05, 0.55, and 0.7, respectively, following the findings of Mavroeidis et al. (2014). Details on how these values were determined can be found in Appendix C. Under this framework, the parameter ρ, which refers to the contemporaneous impact of R_t on inflation, is limited to values between 0 and 1 to ensure that the interest rate is both positively autocorrelated and stationary.

With the current framework, the interest rate variable R_t is completely exogenous from monetary policy decisions. Thus, the central bank will not be able to prevent deviations of the inflation rate from its target entirely. It will rather focus on effectively learning the value of parameters φ and λ, which constitute the inflation equation. The central bank can do so since the variation in equation (1) is entirely exogenous. Later on, we will see if the learnability task remains achievable by the central bank with the addition of an endogenous interest-rate reaction function and parameter variability. The realizations of the inflation π_t will be characterized as follows by substituting the interest-rate decision rule (2) in the PC equation (1):

$$\pi_t = -\varphi \cdot \left[\rho R_{t-1} + \eta_t\right] + \lambda\pi_{t-1} + \epsilon_t. \tag{3}$$

The sequence of events of the learning process, conducted by the central bank as it tracks the estimates with a Recursive Least Squares (RLS) algorithm, can be described as follows:

1. The exogenous interest-rate shock η_t is realized initially. With equation (2) of the model structure, the central bank can observe the contemporaneous observation of the interest rate R_t from its realized lagged value R_{t-1}. The persistence parameter ρ is assumed to be exogenous from the decisions of the central bank.

2. The central bank will perform recursive estimations of the inflation equation to progressively learn the values of the parameters φ and λ, which dictate the response of inflation to its past realizations and the interest rate R_t. The central bank will only observe past realizations of the variables R_t and π_t. The central bank cannot observe the true value of the estimated parameters but only learn them over time with the addition of subsequent realizations of inflation and the interest rate. I will denote the central bank's estimates $(\hat{\varphi}_t, \hat{\lambda}_t)$ as their assessment at the beginning of period t using information contained in observations (π_{t-1}, R_{t-1}) ending in the previous period.

3. Following the estimation of the parameters by the central bank, the cost-push shock ϵ_t included in the PC equation is realized. This exogenous shock will not be accounted for contemporaneously by policymakers since they make their policy instrument decision R_t prior to the realization of the aforementioned shock ϵ_t.

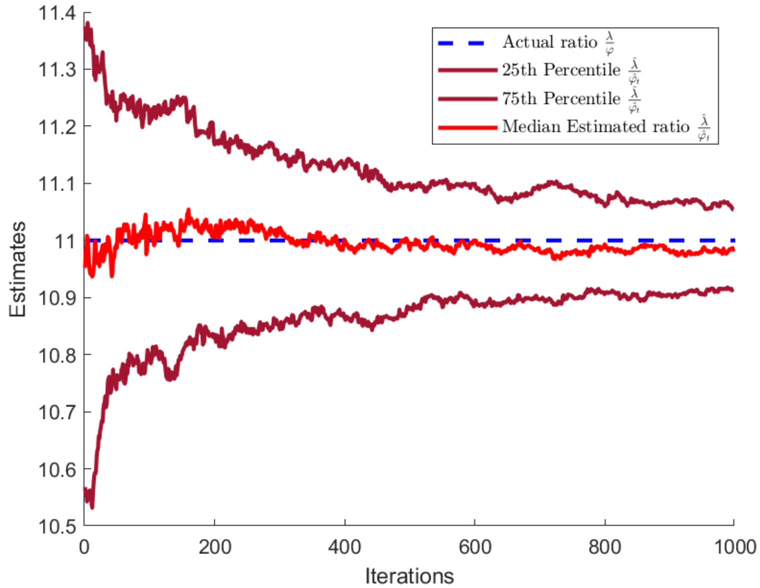
4. The inflation variable π_t is then realized with information contained in its past realization π_{t-1}, the interest rate R_t and all underlying parameter and shock values (φ, λ, and ϵ_t).

In this scenario, the standard deviation of the shock ϵ_t, σ_ϵ is set to take positive values to prevent potential instances where the inflation rate would be equal to zero in the first iterations of the algorithm. The lack of variation in inflation would prevent policymakers from accurately estimating the values of the parameters φ and λ recursively and ultimately converge to their actual values.

Numerical methods were used to perform an estimation of the various models presented in this essay. Models that include a more straightforward structure as in Mendes, Murchison, and Wilkins (2017) can be solved analytically with pen and paper for the variance of inflation and the optimal interest rate response. However, as soon as one departs from this model structure with the addition of a dynamic structure in the interest rate decision and variability in multiple parameters, as it will later be added, it becomes challenging to find the appropriate central bank's response analytically. The RLS algorithm was selected to conduct the simulations following the findings by Evans and Honkapohja (2012) that showed that the equilibrium obtained under rational expectations was stable and that the estimates obtained via an RLS algorithm would converge to their actual value to allow progressive learnability by policymakers.

Figure 1 depicts the median realizations of the ratio of λ/φ performed over 99 Monte-Carlo simulations of the recursive learning algorithm. The estimations were conducted with a positive standard deviation of the cost-push shock ϵ, $\sigma_\epsilon = 0.01$. Section 4 will later clarify why I am using this measure to illustrate the convergence of the parameter estimates. The actual mean values of the parameters φ and λ are set to 0.05 and 0.55, respectively. Details on the simulation procedure can be found in Appendix C relative to the choice of the values. The actual realizations of the ratio are shown in blue and are set to take the value 11 (0.55\0.05). The median estimates of the ratio are shown in red along with the 25th and 75th percentiles of realizations shown in dark red.

Figure 1: Median Estimates with an Exogenous Interest-Rate Rule



These results confirm that the central bank can progressively learn the actual value of the parameters φ and λ. One can see in Figure 1 that the ratio λ/φ, which characterizes the level of aggressiveness deployed by the central bank to control variations in the inflation rate, converges quite rapidly to its actual value. Later on, the definition of the ratio will play a role in setting the optimal policy when decisions by the central bank on the interest rate level R_t are not wholly exogenous. In Figure 1, the central bank is able to continuously estimate the constant actual value of λ/φ within reasonably closed bound following the first observations of the algorithm. Measures of the learning accuracy can be found in Table 2 of Appendix B. In the present case, learnability is achieved because the interest rate variable R_t, defined in equation (2), is entirely exogenous to the central bank since its realizations do not involve an endogenous feedback effect in inflation.

Median realizations of the variance of the inflation rate can be found in Table 1 of Appendix A. In the current scenario, the variance of inflation is slightly positive. The inflation rate in the economy varies due to the addition of exogenous variation in the form of the interest rate shock η_t, and the cost-push shock ϵ_t. In this scenario, the central bank is not actively trying to stabilize the level of inflation entirely to its target of zero. Its main goal resides in learning the parameter values that are part of the equation (1). We will see later on if the central bank can still achieve its goal of learnability with an endogenous feedback interest-rate rule. The realizations of the inflation rate π_t remain centered around the target of zero with the absence of an intercept in the model structure.

4 Feedback Interest-Rate Rule

After noting that the central bank can gradually learn the value of the parameters of the PC equation, the structure of the model will be changed to include a feedback component in the interest rate decision rule. It will allow the central bank to try to control the level of inflation in the economy while continuing to learn the parameter values effectively. The PC equation preserves its initial structure with an autoregressive inflation term π_{t-1} and interest rate component R_t, which accounts for the inflation response to interest rate fluctuations. The structure of the interest-rate decision R_t departs from equation (2). In the current case, the central bank's interest-rate decision incorporates exogenous and endogenous components. First, the exogenous component is still defined as in equation (2) with an autoregressive structure of order one R_{t-1} and an exogenous interest rate shock η_t. The standard deviation of the interest rate shock η_t has to be large enough to yield sufficient exogenous variation. It then allows the central bank to estimate the values of the parameters appropriately. Second, an endogenous factor is introduced as the central bank tries to minimize a single-period quadratic loss function of the inflation rate:

$$L = E_t[\pi_t^2] = E_t[(-\varphi R_t + \lambda \pi_{t-1} + \epsilon_t)^2]. \tag{4}$$

The equation is minimized when the expected value of the inflation rate π_t is equal to zero. It yields the following expression for the endogenous component of the interest rate R_t as $E_t[\epsilon_t] = 0$:

$$R_t = \arg\min E_t[\pi_t^2] \tag{5}$$

$$\frac{\partial L}{\partial R_t} = 2 \cdot E_t[(-\varphi R_t + \lambda \pi_{t-1} + \epsilon_t) \cdot -\varphi] = 0 \tag{6}$$

$$R_t = \frac{\hat{\lambda}_t}{\hat{\varphi}_t} \pi_{t-1}. \tag{7}$$

As described previously, the ratio of parameters λ/φ refers to the degree to which the central bank will respond as it sets its policy rate R_t to changes in past inflation π_{t-1} in order to control contemporaneous realizations of inflation π_t ultimately. The variables of the ratio include a t subscript indicating that the central bank is estimating their value at time t with information available until the end of the previous period. The endogenous component defined in equation (7) will be combined with the exogenous factor defined in equation (2) as the central bank will continue to try estimating the values of λ and φ. The interest-rate equation R_t then cumulates the two expressions in equation (9). As detailed previously, the central bank will estimate $\hat{\lambda}$ and $\hat{\varphi}$ to set its policy rate. The model structure will take the following form:

$$\pi_t = -\varphi R_t + \lambda \pi_{t-1} + \epsilon_t \tag{8}$$

$$R_t = \rho R_{t-1} + \eta_t + \frac{\hat{\lambda}_t}{\hat{\varphi}_t} \pi_{t-1}, \; 0 < \rho < 1 \tag{9}$$

$$\epsilon_t \sim N(0, \sigma_\epsilon^2), \; \eta_t \sim N(0, 1).$$

The algorithm subsequently generates inflation realizations by substituting equation (9) in the inflation equation (8). This yields the law of motion of inflation:

$$\pi_t = -\varphi \cdot \left[\rho R_{t-1} + \eta_t + \frac{\hat{\lambda}_t}{\hat{\varphi}_t} \pi_{t-1} \right] + \lambda \pi_{t-1} + \epsilon_t. \tag{10}$$

Notice that the inflation rate now depends on both the actual parameter values as well as the estimates performed by the central bank. The targeted value π_t remains centered around zero without the addition of an intercept in the inflation equation in the model structure. Policymakers still assume that the parameter values φ and λ are uncertain as they estimate them over time.

Figure 2 depicts in red the median estimated ratio of λ/φ performed by the central bank with the feedback interest rule R_t defined in equation (9) over 99 Monte-Carlo simulations of the recursive learning algorithm. The 25th and 75th percentile measures of the estimates are depicted in dark red. The blue dashed line illustrates the actual value of the ratio λ/φ. To allow comparison with other exercises, the standard deviation of the shock ϵ_t, σ_ϵ remains equal to 0.01.

Figure 2: Median Estimates with a Feedback Interest-Rate Rule

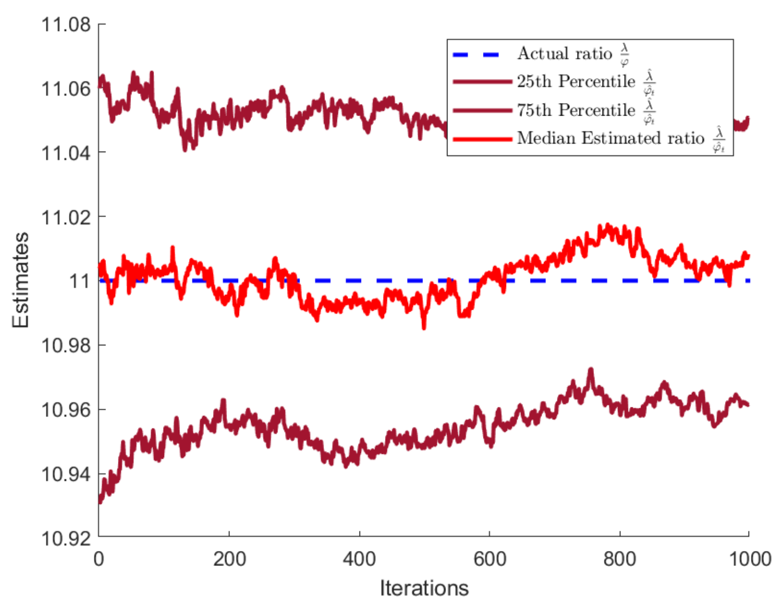


Figure 2 confirms that, with the addition of the endogenous feedback component in the interest-rate decision equation R_t, the central bank is still able to progressively learn the actual value of the parameters of φ and λ. The estimates $\hat{\varphi}_t$ and $\hat{\lambda}_t$ converge at a similar rate than during the previous exercise as they reach close to their respective true values within the first few iterations of the algorithm and remain close to the objective for the whole sample. Quantitative evaluation of the learning accuracy can be found in Table 2 of Appendix B.

Moreover, the variation in inflation can be explained by the presence of exogenous shocks η_t and ϵ_t in the interest-rate decision rule and in the PC equations, respectively. Following the addition of the endogenous component in the interest rate decision rule, the central bank is better able to control inflation variations than in the previous case where the interest rate was determined entirely exogenously from its monetary decisions. As shown in Table 1 of Appendix A, the median variance of inflation is hampered compared to its initial level found in the benchmark example. With the introduction of the endogenous interest-rate component, the central bank seems more apt to control variations in the inflation rate. In contrast with the model structure illustrated in Beck and Wieland (2002), the central bank is not operating deliberately in allowing a positive level of inflation to improve learnability. In the current set-up, it does not constitute a deliberate choice made by the central bank as it tries, unsuccessfully, to mitigate deviations in the inflation rate entirely.

Furthermore, even though the interest rate decision process performed by the central bank is not entirely exogenous, it is better able to pin down the value of the parameters during the iterative process. A measure quantifying the squared difference between the estimates and the actual value of the ratio λ/φ, serving as a proxy of the central bank's convergence ability, can be found in Table 2 of Appendix B. It may seem counter-intuitive at first as one could think that the central bank would have a harder time learning the parameter values accurately while controlling the level of inflation simultaneously. However, it seems that by having a better grasp of inflation variations, the central bank can better forecast the parameter values over time. The policymakers do not appear to be facing a potential tradeoff between learning the values of the parameters and effectively stabilizing the inflation rate.

5 Introduction of Parameter Variability

Following the understanding that the central bank can learn the parameter values progressively with the inclusion of a feedback interest rate equation, the structure of the model will now incorporate parameter variability. The parameter φ_t will be defined as a random variable of mean $\bar{\varphi}$ endowed with an independent and normally distributed shock ν_t with mean 0 and variance σ_ν^2. In the simulation exercise, the mean value $\bar{\varphi}$ is set to 0.05. The actual value of the parameter φ_t will vary in time as the central bank continues to estimate its now time-evolving value recursively. In previous exercises, its mean value was set constant to φ. The time-varying parameter φ_t is now defined as follows:

$$\varphi_t = \bar{\varphi} + \nu_t \tag{11}$$

$$\nu_t \sim N(0, \sigma_\nu^2).$$

The PC equation preserves its initial structure with an autoregressive component π_{t-1} and an interest-rate component R_t, which accounts for the inflation response to fluctuations in the level of the interest rate. The value of the inflation persistence coefficient λ remains constant in this scenario. The model structure will now take the following form as the central bank continues to try estimating the values of both parameters φ and λ while the true structure of the model now includes parameter variability:

$$\pi_t = -\varphi_t R_t + \lambda \pi_{t-1} + \epsilon_t \tag{12}$$

$$R_t = \rho R_{t-1} + \eta_t + \frac{\hat{\lambda}_t}{\hat{\varphi}_t} \pi_{t-1}, \ 0 < \rho < 1 \tag{13}$$

$$\epsilon_t \sim N(0, \sigma_\epsilon^2), \ \eta_t. \sim N(0, 1).$$

The inflation realizations are then defined as follows at each iteration cycle of the algorithm by substituting equations (11) and (13) in equation (12):

$$\pi_t = -(\bar{\varphi} + \nu_t) \cdot \left[\rho R_{t-1} + \eta_t + \frac{\hat{\lambda}_t}{\hat{\varphi}_t} \pi_{t-1} \right] + \lambda \pi_{t-1} + \epsilon_t. \tag{14}$$

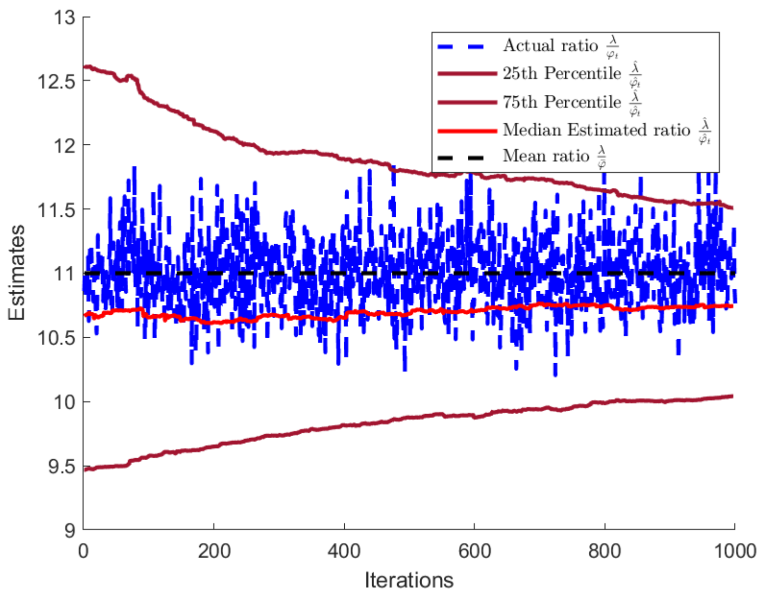
In this scenario, the impact parameter φ_t will vary over time in the sample. The standard deviation of the shock ν_t, σ_ν is set to 0.01. As detailed in Söderström (2002), the realizations of the shocks to the parameters are drawn from the same distribution, which disregards potential issues concerning simultaneous experimentation and price stabilization performed by the central bank. Furthermore, this level of variability in the impact parameter was chosen to prevent the true realizations of the parameter φ_t to become negative and to potentially lead to an inversion of the relationship between inflation and the policy interest rate. Under this alternative context, increases in the interest rate R_t would counter-intuitively lead to increases in the inflation rate. Also, as demonstrated in Mendes et al. (2017), potential policy errors associated with an inadequate level of the nominal interest R_t would be multiplied by the newly added shock ν_t. The additional term $\nu_t \rho R_{t-1}$ in the PC equation could cause multiplicative excess volatility in inflation if the parameter φ_t is not properly estimated. In theory, policymakers would subsequently adopt a wait-and-see approach to avoid causing undesirable volatility to the inflation rate.

In the present model, as detailed in Batini, Martin, and Salmon (1999), policymakers that follow the feedback interest rate rule can offset the effects of a shock to the inflation rate entirely within a given period. The expected deviation of future inflation from its target at time t would be equal to zero. In doing so, the central bank would prevent inflation variations from persisting over multiple periods. Beck and Wieland (2002) and Mendes et al. (2017) have included the standard deviation of the interest rate shock σ_ν as part of the denominator of the endogenous component in the feedback interest rate decision, defined in equation (7), when endowing the structure of the impact parameter with uncertainty. However, the difference in construction with the present essay lies in the fact that the current model is structured with a static one-period loss function instead of using a multi-period loss function. In a multi-period loss function with an uncertain environment, central banks have to account for the persistence of the effects of past exogenous shocks on the contemporaneous realization of the inflation rate when intervening on the interest rate. Thus, they have to account for the potential propagation of inflation deviations on multiple periods.

In this scenario, the central bank will once more estimate recursively the values of the parameters φ_t and λ while simultaneously regulating the level of inflation to be as close as possible to its targeted value. In an effort of comparison with other exercises, the standard deviation of the shock ϵ_t, σ_ϵ is set to 0.01. The sequence of events can be restated for the specific structure with parameter variability to remind the reader of the sequence of events. First, the exogenous shocks ϵ_t, η_t, and ν_t are realized simultaneously at the beginning of period t. The true values of the parameter φ_t and λ are realized following the initial realizations of the exogenous shocks present in the model structure. Second, as defined in previous exercises, the central bank can observe past realizations of π_{t-1} and R_{t-1}. It can subsequently construct the estimates $\hat{\lambda}$ and $\hat{\varphi}_t$ using a RLS algorithm. Following the interest rate decision, the inflation rate is realized as well in period t. The whole process is then repeated iteratively.

Figure 3 depicts the median realizations of the ratio of parameters λ/φ performed by the central bank with the feedback interest rule R_t and the inclusion of parameter variability on φ_t over 99 Monte-Carlo simulations of the recursive learning algorithm. Figure 3 confirms that, while using a feedback interest rate rule R_t and accounting for the inclusion of parameter variability on the impact coefficient φ_t, the central bank is still able to progressively learn the mean value of the parameters $\bar{\varphi}$ and λ. The time-varying true values of the parameters λ/φ_t are depicted with a blue line in Figure 3. The mean value of the ratio $\lambda/\bar{\varphi}$ is shown in black in the graph.

Figure 3: Median Estimates with Variability on the Impact Parameter φ_t



As mentioned above, one can notice that the median value of the ratio λ/φ is still learnable over time by the central bank as the estimates of the ratio converge to its mean value $\lambda/\bar{\varphi}$. In this structure, since the normal distribution is symmetric, the median value of the actual ratio corresponds to its mean. The estimates converge to its mean value despite the φ_t, not including a time-persistency component. Later on, the model structure will include persistence in the impact coefficient.

The simulations with added parameter variability show that the estimates of the ratio λ/φ converge at a slower pace to its respective real value than in previous cases. The efficiency of the convergence algorithm can also be seen in the gradual narrowing of the range between the 25th and 75th percentiles of observations. A quantitative measure of convergence efficiency can be found in Table 2 of Appendix B. It shows that the convergence process is not as efficient as in previous cases. Graphically, one can see, especially in the initial periods, that the inter-quartile range is more extensive than in previous instances that did not include parameter variability in their structures.

Moreover, the response measured by the ratio of λ/φ indicates that the central bank would act more cautiously following the introduction of the endogenous term in its decision function. It can be seen that, compared to the median response in Figures 1 and 2, the optimal response of the central bank to variations in the inflation rate would be more gradual. Graphically, one can see that the red median line remains below the mean value of $\lambda/\bar{\varphi}$ depicted in black for the entire sample. In this context, the central bank prefers to act cautiously instead of reacting decisively and risking drastic variations in the inflation rate as a consequence. As shown in Figure 3, the ratio slowly converges towards its true value following the initial underestimation of its actual value by the central bank. This finding would suggest a confirmation of the Brainard (1967) effect where central banks would adopt a more cautious approach in equilibrium when facing uncertainty in the transition parameter φ_t.

The inclusion of variability on the impact coefficient has slightly affected the central bank's ability to control the variance of the inflation rate in the economy. Readers are directed to Table 1 of Appendix A for a quantitative comparison of the central bank's efficiency in mitigating deviations in the inflation rate. The median estimates of the variance are slightly more elevated in the present scenario compared to the previous case that included an endogenous feedback component in the interest rate decision equation but omitted parameter variability on φ_t. A pronounced variance of inflation would indicate that the central bank cannot stabilize inflation as efficiently while trying to learn the parameter values newly endowed with time-variability.

6 Parametrization

This section will serve as a robustness exercise to test whether the choice of parameter values has an impact on the learnability of parameter values. A natural exercise is to increase the variability of the interest-rate exogenous shock ν_t, defined in equation (11), while preserving the standard deviation of the cost-push shock ϵ_t constant. Intuitively, if the observations of the parameter φ_t are more volatile around its mean value, it should become more challenging for the central bank to predict its actual value overtime correctly. To test this assumption, the standard deviation of the shock ν_t, σ_ν was increased above its original value of 0.01 to either 0.02 or 0.05. Figure 4 illustrates the median estimates of the ratio λ/φ_t in red, blue, green for increasing values of the standard deviation σ_ν to 0.01, 0.02 and 0.05 respectively. The value of the standard deviation of the cost-push shock σ_ϵ was set constant to 0.01 in all scenarios.

Figure 4: Median Estimates with Increasing Variability on the Impact Parameter φ_t

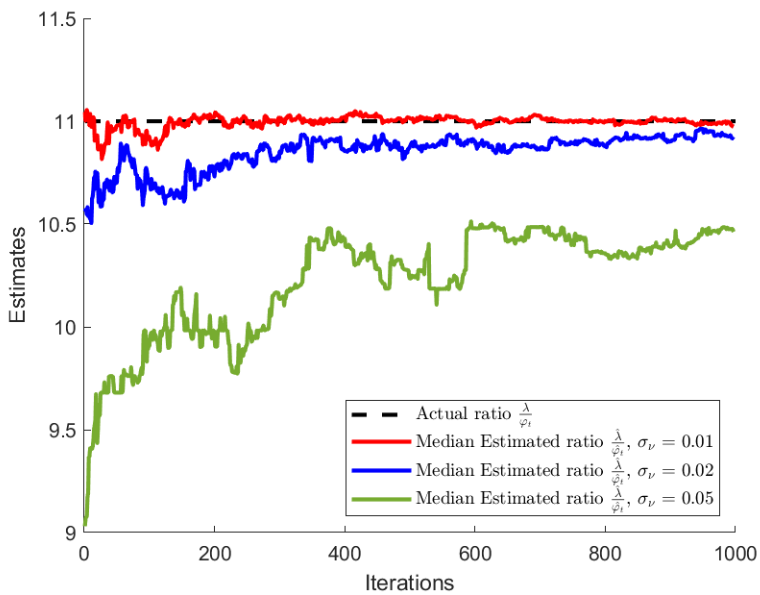
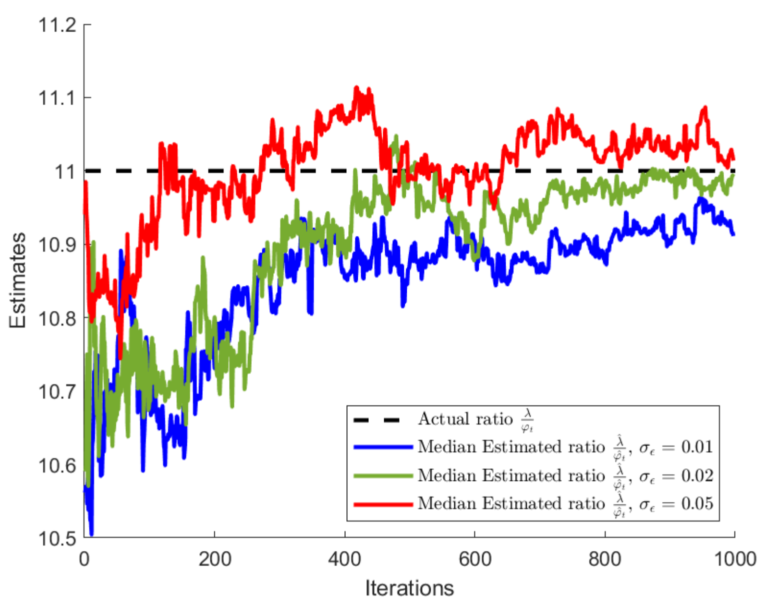


Figure 4 confirms that the rate of convergence is slower when the parameter φ_t fluctuates more around its mean value with the growing values of the standard deviation of the impact parameter shock ν_t. The starting point estimates are also lower. It reinforces the previous findings that the gradual approach would be optimal in the face of considerable variability in the impact parameter. As economic intuition would predict, policymakers have more difficulty correctly estimating the values of the parameters when they are highly variable. Nevertheless, the central bank can still simultaneously control the level of inflation relatively close to its target of zero in all three cases.

Another potential exercise to validate the findings of learnability is to investigate variations in the standard deviation of the cost-push shocks ϵ_t in the inflation equation (14) while keeping constant the value of the standard deviation of the impact parameter shock ν_t. The standard deviation of ϵ_t, σ_ϵ is increased from 0.01 to either 0.02 and 0.05. Figure 5 confirms that the central bank seems to be more apt to learn the parameter values when the variability of the cost-push shock ϵ_t increases. With heightened exogenous variation in the PC equation, the central bank can accurately estimate the value of the parameters faster as its estimates converge more quickly within reasonable bounds of the ratio λ/φ. More pronounced volatility in the cost-push shock ϵ_t increases the variability in past observations of inflation, which is a regressor in the interest rate decision equation. The heightened exogenous variation subsequently helps the central bank to achieve learnability when recursively estimating the parameter values. Moreover, the central bank can still bring the inflation rate close to its targeted value. The value of the standard deviation of the impact parameter σ_ν in all scenarios was set constant to 0.01.

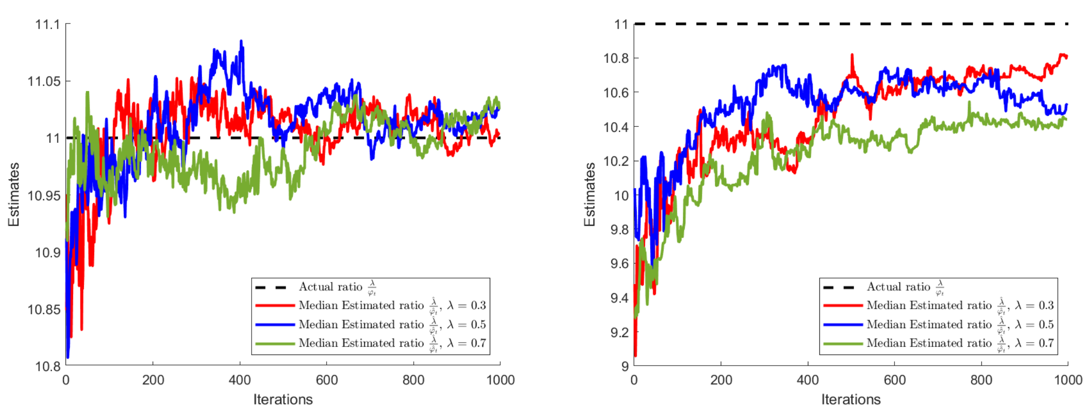
Figure 5: Median Estimates with Variability on the Cost-Push Shock ϵ_t



Moreover, the choice of the value for the parameter λ, which describes the persistence level in the inflation rate, should not affect the previous findings of learnability and price stability achieved by the central bank. However, the optimal response might vary according to the persistence factor of inflation, λ. For example, with a less persistent law of motion of inflation, the realizations of inflation will become more influenced by the contemporaneous realizations of the exogenous shock ϵ_t. As seen previously, increases in the variability of the cost-push shock do not affect learnability of the parameter values. In the previous parametrization exercise, increasing the standard deviation of the cost-push shock ϵ_t led to a reduction in the time required to obtain convergence of the estimates.

Figure 6 describes two estimation exercises that incorporate different levels of parameter variability while varying the value of the persistence coefficient λ. Panel A of Figure 6 illustrates the instance of low parameter variability where σ_ϵ and σ_ν are set to 0.01. Panel B of Figure 6 rather illustrates the instance with increased variability with more pronounced values for the standard deviations of the shocks in the model as σ_ϵ and σ_ν are set to 0.05.

Figure 6: Median Estimates with Increasing Persistence λ in Inflation



The current scenario is conducted to confirm whether the realizations of the learning process are in line with the findings in Ferrero et al. (2019). In Ferrero et al. (2019), the persistence of inflation is modeled as the relative persistence in the cost-push shock ϵ_t. An auto-regressive process of order one is defined to capture the persistence factor in the shock. However, in the context of the present essay, it would have been challenging to implement a similar model structure. It would have required a learning process of the cost-push shock conducted by the central bank.

In the present case, the persistence in inflation is instead modeled with variations in the persistence parameter λ in the PC equation. The red, blue, and green lines display estimates of the ratio λ/φ_t with increasing values of the parameter λ from 0.3 to 0.5 and 0.7, respectively. As previously mentioned, the level of aggressiveness of a central bank in response to variations in the inflation rate is visually represented by the initial position of the estimated ratio compared to the actual value. For example, if the estimated ratio converges from above, the central bank will display an aggressive response on its interest rate to fluctuations in inflation, as defined in equation (7).

As shown in Ferrero et al. (2019), within an environment that is endowed with a lesser degree of parameter variability, the central bank will adopt a more gradual approach if it recognizes the exogenous cost-push shock to be less persistent. The illustration in panel A of Figure 6 replicates that behavior as the estimates computed with a higher degree of persistence shown in green converges to the actual value from below. However, the other estimates which incorporate a lesser degree of persistence are converging from below as well and are not significantly different in magnitude.

In addition, one can find in panel B of Figure 6 that, in contrast with findings in Ferrero et al. (2019), the central bank will adopt a slightly more gradual approach within a highly variable environment associated with pronounced persistence in inflation to stabilize the level of inflation. As shown in panel B of Figure 6, regardless of the level of persistence in inflation, the optimal response of the central bank will be to adopt a cautious approach with pronounced parameter variability. Moreover, one can note that the optimal responses shown in panel B of Figure 6 will display an enhanced caution effect compared to the reactions shown in panel A of Figure 6, regardless of the level of persistence in the inflation rate.

The difference in results with Ferrero et al. (2019) may reside in the fact that the present central bank uses a myopic rule as it tries to control inflation contemporaneously. An aggressive approach would be required to prevent deviations of inflation from its target to persist in time. Moreover, in instances where the shock is not persistent, an aggressive approach might not be suitable if it has the potential to cause large avoidable swings in the inflation rate, leading to losses incurred as part of the central bank's loss function defined in equation (4).

The selection of parameter values for other coefficients in the model structure presented in section 5 will not impact the learning process and the stabilization effort on the inflation rate conducted by the central bank. For example, varying the mean value of the impact parameter φ, $\bar{\varphi}$ will not affect the previous findings. As long as its mean value remains positive, the relationship between the interest rate variable R_t and the corresponding inflation rate will remain intact. Also, changing the value of the parameter ρ, which measures the interest rate persistence over time, as defined in equation (2), will not affect the results found previously. As long as the parameter ρ displays a sufficient degree of persistence and remains below 1, it should not affect the previous findings.

7 Extensions

7.1 Intercept

Some extensions to the problem detailed in section 5 can be introduced to enhance the practicality of the model structure with parameter variability. For example, it might be interesting to add an intercept α to the construction of the inflation equation. This addition would reflect a typical inflation-targeting regime where a central bank sets a target for the desired rate of inflation in the economy. Currently, the inflation targeted inflation rate is commonly established at a rate of 2 percent annually. In the model, it would, therefore, be possible to add an intercept α and to set the inflation target to 2 percent. The new PC equation takes the following form:

$$\pi_t = \alpha - \varphi_t R_t + \lambda \pi_{t-1} + \epsilon_t. \tag{15}$$

In section 5, the targeted rate of inflation is implicitly set to 0. Therefore, in the initial simulations, the central bank seeks to maintain the inflation rate at 0 percent to the best of its ability. Thus, the sole change from the previous cases would have the central bank stabilizing the level of inflation around a new threshold of 2 percent. The mean value of the interest rate would evolve according to the choice of the inflation target in the economy. The optimal interest-rate rule can be found by solving the central bank's loss function that penalizes variations in the inflation rate relative to its target π^*:

$$L = E_t[(\pi_t - \pi^*)^2] = E_t[(\alpha - \varphi_t R_t + \lambda \pi_{t-1} + \epsilon_t - \pi^*)^2]. \tag{16}$$

As mentioned in Mendes et al. (2017), a quadratic loss function implies that the central bank will be more concerned with significant deviations of the inflation rate from its target and that incurred losses are symmetric around the target. One can derive the optimal interest rate R_t minimizing the loss function with the inflation target defined as $\pi^* = 2$ and $E_t[\epsilon_t] = 0$:

$$\frac{\partial L}{\partial R_t} = 2 \cdot E_t[(\alpha - \varphi_t R_t + \lambda \pi_{t-1} + \epsilon_t - \pi^*) \cdot -\varphi] = 0 \tag{17}$$

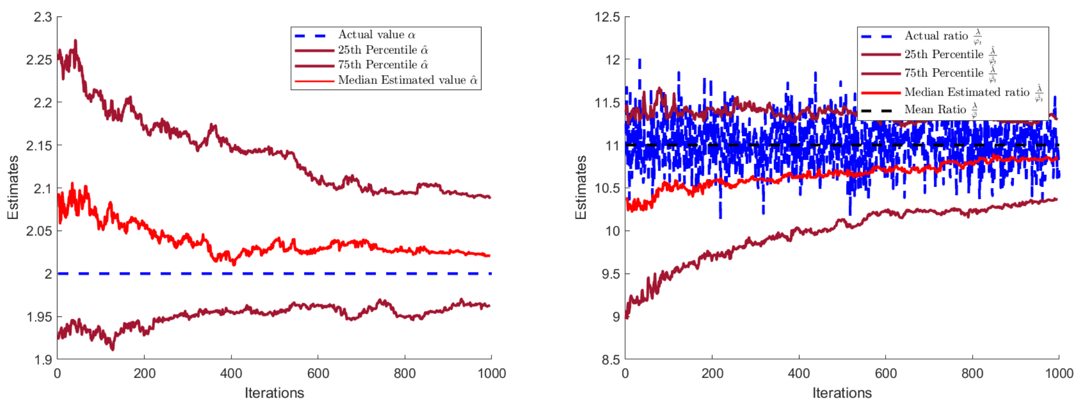
$$R_t = \frac{(\hat{\alpha}_t - \pi^*) + \hat{\lambda}_t \pi_{t-1}}{\hat{\varphi}_t}. \tag{18}$$

The central bank will, therefore, have to recursively estimate the value of the intercept α in addition to the two standard parameters (φ_t, λ) in the PC equation. The inflation realizations will take the following form by substituting equations (11) and (18) in equation (15):

$$\pi_t = \alpha - (\bar{\varphi} + \nu_t) \cdot \left[\rho R_{t-1} + \eta_t + \frac{(\hat{\alpha}_t - \pi^*) + \hat{\lambda}_t \pi_{t-1}}{\hat{\varphi}_t}\right] + \lambda \pi_{t-1} + \epsilon_t. \tag{19}$$

The standard deviation of the interest rate shock η_t was increased to 2 instead of remaining equal to 1 as in previous scenarios to ensure sufficient exogenous variation. Intuitively, the introduction of the time-varying estimates of $\hat{\alpha}_t$ induces addition endogenous variation as the central bank has to compute the difference between its estimates of the intercept and the inflation target in order to adopt its policy interest rate. To offset such a change, it emerges that one requires more exogenous variation in the interest rate structure to ensure learnability. Subsequently, policymakers will be able to gradually estimate the value of the parameters that characterize the PC equation defined in equation (15) and prevent deviations of the inflation rate from its target of 2 percent effectively. Intuitively, the realizations of the inflation rate are now centered around its mean of 2 percent.

Figure 7: Median Estimates with an Intercept α



Panel A of Figure 7 illustrates the median estimates of the intercept α_t, while panel B of Figure 7 shows the median estimates of the ratio λ/φ_t in red computed over 99 Monte-Carlo iterations of the RLS algorithm. Panel A of Figure 7 shows that the central bank can learn progressively the value of the intercept α when its values are set to 2 with the progressive narrowing of the interquartile range shown in dark red. The learning process of the intercept value is conducted concomitantly with the estimations of the ratio $\lambda/\bar{\varphi}$ as in the previous scenarios. In addition, one can see in Table 1 of Appendix A the similarities in the magnitude of the variance of inflation relative to the other cases as the central bank can still mitigate deviations of the inflation rate reasonably well. In conclusion, the addition of the intercept value does not interfere in the learnability process and previous findings.

7.2 Slowly-Evolving Impact Parameter Variability

Following the confirmation that the central bank can learn the value of the parameters, even though the structure of the model incorporates variability in the impact parameter φ_t, it would be interesting to see whether changing the form of the randomness in the impact parameter φ_t can affect the central bank's optimal response. Thus, the parameter φ_t will now take the form of a stationary autoregressive process of order one with an independent and normally distributed exogenous shock ν_t with mean 0 and variance of σ_ν^2. Its mean value $\bar{\varphi}$ will remain set to the previous value of 0.05. The new structure of the time-varying parameter φ_t will take the following form:

$$(\varphi_t - \bar{\varphi}) = \gamma \cdot (\varphi_{t-1} - \bar{\varphi}) + \nu_t \tag{20}$$
$$\nu_t \sim N(0, \sigma_\nu^2).$$

The autoregressive nature of the parameter φ_t will allow its values to have a slowly-evolving law of motion. Thus, the present value of the impact parameter will incorporate a persistent effect of past realizations of exogenous shocks. By carefully selecting the standard deviation of the exogenous shock ν_t and the persistence parameter γ, one can ensure that the parameter value and its estimate remain stationary and within a reasonably close interval of its mean value. For this exercise, the value of the parameters γ and $\sigma_n u$ are set to 0.8 and 0.01, respectively. This selection helps to prevent extreme and unrealistic fluctuations in the value of the impact parameter. As previously stated, the parameter value of φ_t, if not properly defined, could become negative for a sizeable part of the sample. In turn, it would mean that increases in the policy interest rate are positively correlated with hikes in the inflation rate. Moreover, both inflation and the interest-rate decision equations will remain in their previously-defined structures as detailed in equations (12) and (13).

In addition, the estimation procedure of the central bank will vary from an RLS algorithm performed with an expanding window to a rolling window estimation, where the central bank estimates the parameter values over a selected portion of the sample instead of the entire historical series. By not relying on extended historical samples and computing estimates with a fixed window size, policymakers would avoid potential structural breaks in the data.

In the current exercise, the rolling window size is set to 50 observations and remains constant over the entire iteration process. With the slowly-evolving structure of the impact parameter φ_t, the central bank should be able to estimate the parameter values better as it only considers the more recent observations. The latest observations of the algorithm capture the most up-to-date information about the contemporaneous realization of the parameter. It allows the central bank to gain insight into its present value and progressively improve its estimates. To facilitate the estimation of the parameter φ_t by the central bank, equation (20) can be rewritten:

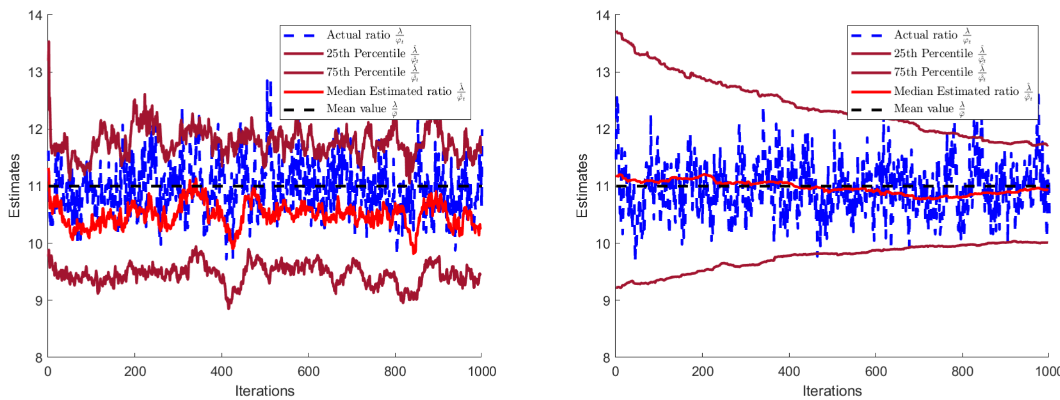
$$\varphi_t = (1-\gamma)\cdot\bar{\varphi} + \gamma\varphi_{t-1} + \nu_t \tag{21}$$
$$\nu_t \sim N(0, \sigma_\nu^2).$$

By substituting equation (21) in combination with equation (13) in the PC equation (12), one can find the realizations of inflation:

$$\pi_t = -[(1-\gamma)\cdot\bar{\varphi} + \gamma\varphi_{t-1} + \nu_t]\cdot\left[\rho R_{t-1} + \eta_t + \frac{\hat{\lambda}_t\pi_{t-1}}{\hat{\varphi}_t}\right] + \lambda\pi_{t-1} + \epsilon_t. \tag{22}$$

Panel A of Figure 8 shows the median estimates of the ratio λ/φ_t computed with the rolling window algorithm of a fixed window size of 50 observations. Panel B of Figure 8 details the estimates of the same ratio λ/φ_t, but its estimates were computed using the original expanding algorithm. The estimations were conducted without the addition of an intercept in the inflation equation.

Figure 8: Median Estimates with Persistence in the Impact Parameter φ_t



Notice that the new structure of the parameter φ_t gives a slowly-evolving form to the ratio λ/φ. The central bank's estimates of the actual ratio are well specified as it can learn the parameter values progressively with the addition of persistence in the impact parameter. As shown in panel A of Figure 8, the central bank, which is using a rolling window to estimate the ratio in this case, converges to the mean value faster than in the scenario computed with an expanding window algorithm. The rolling window algorithm seems to help remove some historical observations that might not be informative on the contemporaneous value of the ratio λ/φ_t. By factoring information closer in time, the central bank can perform better-informed decisions on its optimal interest rate level with a persistent impact parameter φ_t. The rolling window estimation procedure seems to capture the autoregressive nature of the ratio better as it was induced in the impact parameter φ_t. The convergence interval does, however, narrow to similar values at the end of the sample when calculating the estimates with either choice of algorithm. It would indicate that some past observations still contain, at the margin, some information about the current value of the ratio λ/φ_t. In both instances, the central bank is once again able to control variations in inflation in a comparable fashion than previous scenarios, as shown in Table 1 of Appendix A.

7.3 Introduction of an IS Equation

It would be interesting to aim towards the structure of the Neo-Keynesian model after having noted the effective learning of the parameter values in the previous models. As conducted in Söderström (2002), which includes a backward-looking IS equation, the current model structure could introduce an IS equation where the output gap variable y_t reacts to variations in the real interest rate. Thus, by extension, the inflation rate would no longer respond directly to changes in the nominal interest rate, as was the case in previous simulation scenarios. The inflation rate would instead respond to changes in the output gap, which values are dictated in part by the realizations of the real interest rate r_t. The real interest rate r_t is defined as the difference between the nominal interest rate R_t, and the inflation rate π_t.

The consolidated model structure, which includes an IS equation, should not affect any of the findings. In this present case, the interest-rate decision equation can be modeled as a purely exogenous process, as defined in equation (2). The output gap y_t would react negatively to variations in the real interest rate. One can then couple the IS equation with an independent and normally distributed demand shock ψ of mean 0 and variance σ^2_ψ. The PC equation would then incorporate the output gap variable, an autoregressive term of order one of the inflation rate π_{t-1} with an exogenous shock $\tilde{\epsilon}_t$ as previously defined. The augmented model, with the addition of the IS equation, would take the following form:

$$R_t = \rho R_{t-1} + \eta_t, \ 0 < \rho < 1 \tag{23}$$

$$y_t = -\delta(R_t - \pi_t) + \psi_t \tag{24}$$

$$\pi_t = \lambda_y y_t + \lambda_\pi \pi_{t-1} + \tilde{\epsilon}_t \tag{25}$$

$$\tilde{\epsilon}_t \sim N(0, \sigma^2_{\tilde{\epsilon}_t}), \ \eta_t \sim N(0,1), \ \psi_t \sim N(0, \sigma^2_\psi).$$

One can rearrange the model by substituting the output gap equation (24) in the PC equation

(25) and isolate the contemporaneous inflation term π_t:

$$\pi_t = \lambda_y \cdot \Big[-\delta(R_t - \pi_t) + \psi_t \Big] + \lambda_\pi \pi_{t-1} + \tilde{\epsilon}_t \tag{26}$$

$$\pi_t = \frac{1}{1 - \lambda_y \delta} \cdot \Big[-\lambda_y \delta R_t + \lambda_\pi \pi_{t-1} + \lambda_y \psi_t + \tilde{\epsilon}_t \Big]. \tag{27}$$

One could then recover the initial structure of the inflation equation (28) by specifying the value of certain parameters in equation (27):

$$\pi_t = -\varphi R_t + \lambda \pi_{t-1} + \epsilon_t \tag{28}$$

$$\varphi = \frac{\lambda_y \delta}{1 - \lambda_y \delta}, \quad \lambda = \frac{\lambda_\pi}{1 - \lambda_y \delta}, \quad \epsilon_t = \frac{\lambda_y \psi_t + \tilde{\epsilon}_t}{1 - \lambda_y \delta}. \tag{29}$$

The model structure would not fundamentally differ from the initial version with consolidated parameter values. Consequently, the addition of an IS equation to the model structure would not alter in any form the previous findings indicating that the central bank can progressively learn the parameter values even in the presence of parameter variability. Policymakers would still be able to regulate the level of inflation, as seen in the Benchmark example.

7.4 Variability on the Transition Parameter

After noting that the central bank is progressively able to learn the parameter values with the inclusion of parameter variability on the impact coefficient φ_t, the structure of the model shown in Section 5 can incorporate another source of variability. One can test whether the inclusion of parameter variability on the parameter λ_t, which accounts for the relative persistence of the inflation rate over time, would encourage a different reaction from the central bank than the variability induced on the impact coefficient. Craine (1979) and Söderström (2002), among others, have found that the optimal response would vary whether parameter uncertainty was induced on the measurement coefficient or on the transition parameter in the Phillips curve equation. According to their findings, the central bank should opt for a more aggressive approach with the introduction of parameter variability affecting the law of motion of the inflation variable. One can test these findings by introducing variability on the parameter λ_t. The stochastic process of the parameter λ_t can then be defined in accordance with the previous definition of φ_t in Section 5 as a random variable with a mean value of $\bar{\lambda}$ coupled with an independent and normally distributed exogenous shock ξ_t with mean 0 and variance σ_{ξ}^2. In previous exercises, the parameter was kept constant to its mean value $\bar{\lambda}$. The time-varying parameter λ_t is now defined as follows:

$$\lambda_t = \bar{\lambda} + \xi_t \tag{30}$$
$$\xi_t \sim N(0, \sigma_{\xi}^2).$$

The model structure will now take the following form as the central bank continues to try estimating the values of both parameters φ_t and λ_t as their respective values vary over time:

$$\pi_t = -\varphi_t R_t + \lambda_t \pi_{t-1} + \epsilon_t \tag{31}$$
$$R_t = \rho R_{t-1} + \eta_t + \frac{\hat{\lambda}_t}{\hat{\varphi}_t}\pi_{t-1}, \ 0 < \rho < 1 \tag{32}$$
$$\varphi_t = \bar{\varphi} + \nu_t \tag{33}$$
$$\epsilon_t \sim N(0, \sigma_{\epsilon}^2), \ \eta_t \sim N(0, 1), \ \nu_t \sim N(0, \sigma_{\nu}^2).$$

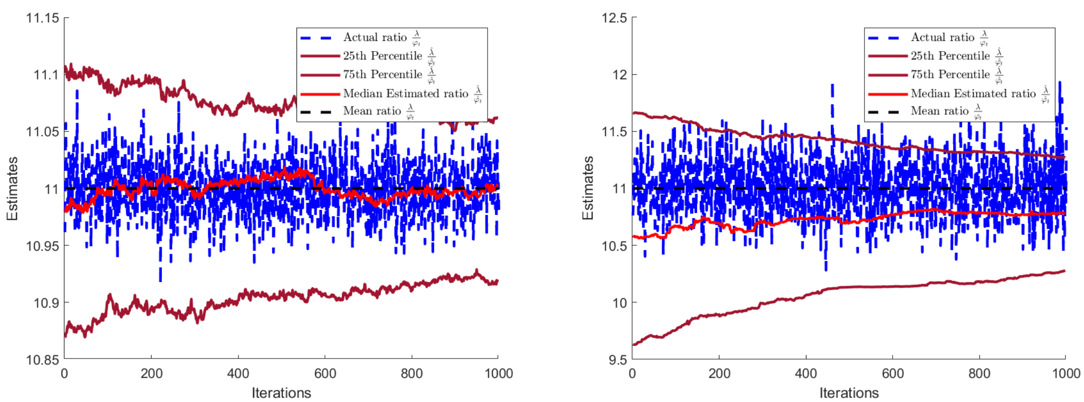
The inflation realizations are then defined as follows at each iteration cycle of the algorithm by substituting equations (30), (32), and (33) in equation (31):

$$\pi_t = -(\bar{\varphi} + \nu_t) \cdot \left[\rho R_{t-1} + \eta_t + \frac{\hat{\lambda}_t}{\hat{\varphi}_t}\pi_{t-1}\right] + (\bar{\lambda} + \xi_t) \cdot \pi_{t-1} + \epsilon_t. \tag{34}$$

With the introduction of parameter variability in the persistence component of inflation, the central bank should, in theory, act more decisively in the initial period to mitigate potential propagation effects from the persistence in inflation, in accordance with Craine (1979). Panel A of

Figure 9 shows the median estimates of the ratio λ_t/φ_t where the standard deviation of the shock to the impact parameter σ_ν is set to 0 to measure the effect of the introduction of variability on the persistence parameter λ_t separately. The standard deviation of the shock ξ_t is set to 0.01. Panel B of Figure 9 introduces a scenario where both parameters λ_t and φ_t are time-varying as their respective shocks ξ_t and ν_t are endowed with positive standard deviations. Both standard deviations are set to 0.01 in this case. The standard deviation of the cost-push shock, σ_ϵ is set to take a constant value of 0.01 for both panels of Figure 9.

Figure 9: Median Estimates with Variability on the Transition Parameter λ_t



Panel A of Figure 9 shows that the central bank would react in a slightly cautious fashion since the starting point of the estimates is converging to the actual ratio from below. By doing so, the central bank would not cause additional volatility in the inflation rate. Panel B of Figure 9 shows the estimates of the reaction coefficients when introducing variability on the shocks ξ_t and ν_t. Contrary to the findings of Kimura and Kurozumi (2007), in the context of the present essay, the central bank would prefer to act cautiously when facing heightened variability on both parameters. The rate to which the estimates of the ratio converge to their actual values is also slower since the central bank has to account for variability on the pair of parameters. In addition, the central bank is less able to control inflation as the model structure includes a more significant share of exogenous variation with a positive standard deviation induced on both shocks ξ_t and ν_t. Table 1 of Appendix A can be consulted for an overview of the median variance of inflation in both instances.

7.5 Loss Function Equation

One can also test the learning process of the central bank when it has to consider multiple objectives at once instead of uniquely targeting the level of inflation. In this instance, the central bank attaches a certain weight on smoothing the variation of the interest rate realizations R_t. It would try to minimize a single-period quadratic loss function of the inflation rate and the interest rate deviations:

$$L = E_t[\pi_t^2 + \omega(\Delta R_t)^2] = E_t[(-\varphi R_t + \lambda \pi_{t-1} + \epsilon_t)^2 + \omega(R_t - R_{t-1})^2]. \tag{35}$$

As defined in Mendes et al. (2017), monetary authorities are interested in smoothing interest rate variation over time to allow agents to adapt instead of forcing sudden and unanticipated changes in the policy interest rate. Central banks are also afraid of having to reverse their previous monetary policy decision in the event of a changing economic situation. Thus, they fear a loss of credibility among economic agents if they need to reverse their monetary policy stance. According to Mendes et al. (2017), few central banks have ventured to reverse their policy stances under the current inflation-targeting regimes. The optimal interest-rate decision can then be found by minimizing the latest loss function (35) by R_t with $E_t[\epsilon_t] = 0$ and it is determined as follows:

$$R_t = \arg\min E_t[\pi_t^2 + \omega(\Delta R_t)^2]. \tag{36}$$

The first-order condition yields the following result:

$$\frac{\partial L}{\partial R_t} = E_t[2\pi_t \frac{\partial \pi_t}{\partial R_t} + 2\omega(R_t - R_{t-1})] = 0. \tag{37}$$

With the initial specification of the Phillips Curve equation (1), one can find the optimal interest-rate response by the central bank:

$$R_t = \frac{\omega}{\hat{\varphi}_t^{\,2} + \omega} R_{t-1} + \frac{\hat{\varphi}_t \hat{\lambda}}{\hat{\varphi}_t^{\,2} + \omega} \pi_{t-1}. \tag{38}$$

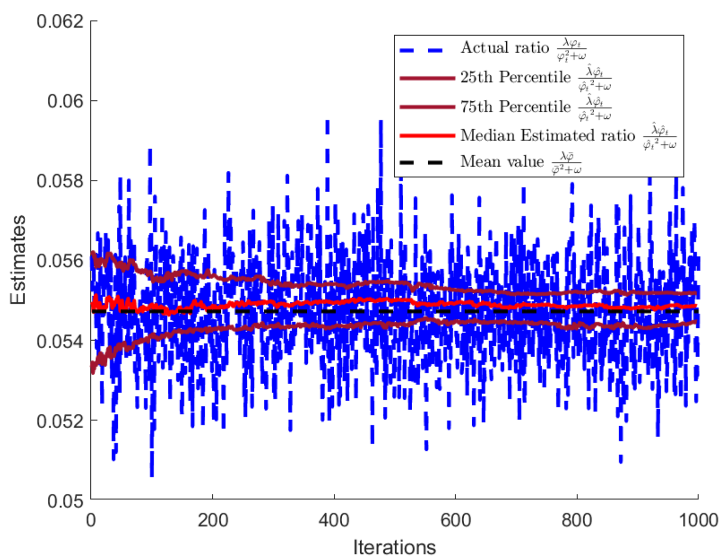
The model structure includes parameter variability in the impact coefficient φ_t as defined in equation (11). The realizations of inflation will take the following form when combining equations (11) and (38) in equation (1):

$$\pi_t = -(\bar{\varphi} + \nu_t) \cdot \left[\frac{\omega}{\hat{\varphi}_t^{\,2} + \omega} R_{t-1} + \frac{\hat{\varphi}_t \hat{\lambda}}{\hat{\varphi}_t^{\,2} + \omega} \pi_{t-1} + \eta_t \right] + \lambda \pi_{t-1} + \epsilon_t. \tag{39}$$

The weight placed on the interest-rate smoothing coefficient ω is set to 0.5 to have a meaningful difference with the simulation exercise shown in section 5. Contrary to the previous exercises, the endogenous component of the interest rate R_t, defined in equation (38), already accounts for autoregressive variation embedded in the structure of the interest rate. The exogenous shock η_t will remain in the inflation equation to preserve sufficient exogenous variation and to ensure learnability. The level of aggressiveness displayed by the central bank in response to variations in past inflation rate is now defined as the coefficient measuring the impact of past realizations of inflation on the policy rate in equation (38). The estimation process was computed with an expanding window. Figure 10 shows the median estimates of the ratio $\lambda\varphi/(\varphi^2 + \omega)$ conducted with the addition of an interest-rate smoothing coefficient in the central bank's loss function.

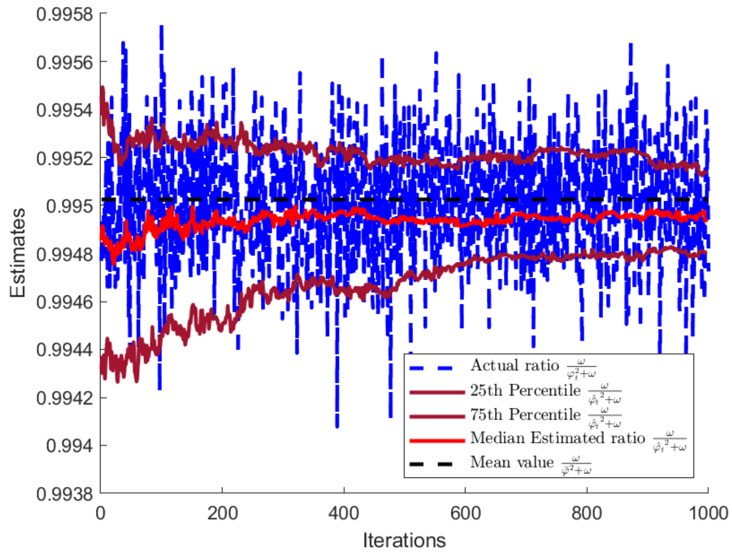
Figure 10 shows that the central bank can once again learn the actual values of parameters with the additional objective of smoothing the interest-rate level of time. The median optimal response depicted in red seems to be slightly aggressive as the estimates are converging to the actual value of the ratio of parameters from above. Table 2 of Appendix B confirms that the addition of the interest-rate smoothing allows the ratio of the estimates to converge very rapidly to its actual value. However, as seen in Table 1 of Appendix A, the variance of the inflation realizations is higher than the results found in other extensions of the model structure. It seems that the central bank has more difficulty in effectively stabilizing the level of inflation compared to previous instances since its loss function penalizes large movements in the interest-rate variable from period to period, which would prevent the central bank from acting more decisively in a given period in response to shocks. Figure 11 shows the median estimates of the ratio $\omega/(\varphi^2 + \omega)$ conducted with an interest rate smoothing coefficient in the central bank's loss function displaying the estimates of the endogenous level of persistence in the interest rate.

Figure 10: Median Estimates with an Interest-Rate Smoothing (IRS) Loss Function



Furthermore, one can see also in Figure 11 that the central bank is able to accurately estimate the level of persistence in the interest rate decision equation while stabilizing the level of inflation. The central bank can still achieve learnability, even though the persistence level in the interest rate is defined endogenously. Previously, the persistence level of the interest rate was determined by the parameter ρ and defined exogenously from the central bank's interest-rate decision.

Figure 11: Median Estimates of Inflation Persistence with an IRS Loss Function



7.6 Kalman Filtering Estimation

As previously shown with the rolling window algorithm, the central bank can effectively learn the value of the parameters. It can be interesting to see if other algorithms, such as the Kalman filter would allow the central bank to update its prior estimates of the ratio λ/φ and thus improve its estimates with incoming new observations of the inflation rate π_t and the policy interest rate R_t. But, with the methodology developed in Duncan and Horn (1972), one can find that the RLS estimation algorithm can be interpreted as a Kalman filter structure. Then, using the Kalman filter yields the same results as the ones found in Section 5 of the essay with the estimations performed with an RLS algorithm. In the present essay, the inflation rate evolves as follows:

$$\pi_t = -\varphi_t R_t + \lambda \pi_{t-1} + \epsilon_t. \tag{40}$$

Using the Evans and Honkapohja (2012) RLS formulas, one can find the estimates of the impact parameter φ_t and of the inflation persistence parameter λ_t. The central bank is running an RLS regression of the inflation rate π_t on the vector of variables z_t, where $z_t = [R_t\ \pi_{t-1}]$. The vector of coefficients is defined as follows:

$$\hat{\beta}_t = [-\hat{\varphi}_t \ \hat{\lambda}_t]'. \tag{41}$$

It yields the following results for the dynamics of the estimates when applying the RLS formulas:

$$\hat{\beta}_t = \hat{\beta}_{t-1} + t^{-1} B_t^{-1} z_{t1} [\pi_t - \hat{\beta}_t z_t] \tag{42}$$

$$B_t = B_{t-1} + t^{-1} [z_t z_t' - B_{t-1}]. \tag{43}$$

B_t denotes the variance-covariance matrix of the vector z_t using data up to period t, which is updated according to equation (43). In equation (42), the forecast error is defined as:

$$F_t = [\pi_t - \hat{\beta}_t z_t]. \tag{44}$$

If the forecast error is relatively limited, the adjustment to the estimates $\hat{\beta}_t$ will evolve slightly in accordance. One can see the parallel with the forecast error computed with the Kalman filter procedure as defined in Duncan and Horn (1972). The Kalman filter state-space structure applied to the model described in section 7.4 would be represented as follows with the vector of actual parameter values $\beta_t = [-\varphi_t \ \lambda_t]'$ and the vector of shocks to parameters $\tau_t = [\nu_t \ \xi_t]'$:

$$\beta_t = T_t \bar{\beta} + \tau_t \tag{45}$$

$$\pi_t = \beta_t z_t + \epsilon_t. \tag{46}$$

Equation (45) illustrates the transition equation of the unobserved parameters in the model describing the evolution of the true values of the parameters. In contrast, equation (46) refers to the measurement equation, which dictates the contemporaneous response of the variables. In the model structure used in Section 7.4, the matrix T_t defined in equation (45) would be equal to the identity matrix as the law of motion of the actual parameters composing the vector β_t, whose definitions can be found in equations (30) and (33), are evolving around their respective mean values. The shocks included in the model are following a normal process:

$$\begin{bmatrix} \tau_t \\ \epsilon_t \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q_t & 0 \\ 0 & H_t \end{bmatrix}\right). \tag{47}$$

The matrices Q_t and H_t describe the variance matrices associated with the shock vectors τ_t and ϵ_t. The starting equation for β_t can be written in this form:

$$\beta_1 = \bar{\beta} + \tau_1. \tag{48}$$

The recursive estimation equations of the Kalman filter are then found as follows for the corresponding model structure defined in previous sections:

$$\hat{\beta}_t = \hat{\beta}_{t-1} + S_{t-1} z_t' D_t^{-1} [\pi_t - \hat{\beta}_t z_t] \tag{49}$$

$$S_t = S_{t-1} - S_{t-1} z_t' D_t^{-1} z_t S_{t-1}. \tag{50}$$

where the matrices S_t and D_t represent the conditional variances of the inflation rate and of the forecast error F_t respectively:

$$\hat{\beta}_1 = \bar{\beta}; \qquad S_t = T_t S_{t-1} T_t' + Q_t;$$

$$S_1 = Q_1; \qquad D_t = H_t + z_t S_t z_t'.$$

Tedious but straight-forward calculations would show that the Kalman filter estimates in its state-space form defined equations (49) and (50) reproduce the estimates found with the RLS algorithm in equations (42) and (43). One can consult Duncan and Horn (1972) for an elaboration of the equivalence in further detail. It then confirms that the estimates attributed to the RLS method performed in this essay coincide with the central bank's estimates measured with a Kalman filter algorithm.

8 Concluding Remarks and Further Research

This essay analyses the learning process of parameter values performed by the central bank. It shows that the monetary authority is able to control the level of inflation close to its target while learning progressively the values that govern the law of motion of inflation. The essay sought to analyze whether the optimal behavior of the central bank was influenced by the introduction of parameter variability in the model. In Section 5, the model structure incorporated parameter variability in the impact parameter measuring the responses of the interest rate level R_t to past realizations of the inflation. Using numerical methods, I performed estimations of a Phillips Curve equation with various interest-rate decision rules that guided the responses by the central bank to exogenous shocks. The median estimates were then computed via 99 Monte-Carlo iterations.

In this essay, I showed that the central bank is able to progressively learn the value of the parameters when the interest-rate decision rule is defined as an entirely exogenous process of order one. In this instance, the central bank has, however, less control over fluctuations in inflation since the law of motion of the interest rate is an entirely exogenous process. Moreover, I demonstrated that the monetary authority could learn the parameter values with the addition of a feedback interest-rate decision, which depended on the optimal rule found by deriving the central bank's quadratic one-period loss function. Intuitively, the central bank was more apt to regulate the level of inflation closer to its targeted value. Subsequently, the model structure was endowed with parameter variability on the parameter, which accounts for the impact of the interest rate on inflation. The simulations showed that the central bank was still able to learn the mean value of the ratio λ/φ_t. The central bank would adopt a more cautious approach in accordance with the Brainard (1967) gradualism principle as the central bank faces heightened parameter uncertainty on the impact coefficient of the Phillips Curve equation. The monetary authority would then prefer a gradual approach to mitigate variations in inflation and, thus, prevent significant losses due to an inappropriate policy response. In addition, the central bank achieved relative price stability even with the introduction of additional exogenous shocks within the structure of φ_t.

Other numerical exercises were also performed using increasing variability in the exogenous shocks and parameters present in the model structure. The standard deviation of the shocks ϵ_t and ν_t, σ_ϵ, and σ_ν were increased to test whether the central bank would change its behavior in the response of shocks to inflation. The simulation procedure presented in Appendix C details the additional values that I used to perform the estimations. These exercises demonstrated an enhanced caution effect from the central bank when setting its monetary policy under elevated parameter variability compared to instances shown in the figures of the essay.

The model structure analyzed in this essay is similar to those presented by Brainard (1967) and Söderström (2002). As in Brainard (1967), the central bank's loss function is modeled in this research by a myopic one-period quadratic function of inflation realizations. As in the structure shown in Söderström (2002), the Phillips Curve equation (1) is defined as a backward-looking function of the inflation rate. Thus, with these similarities in the structure of the model, it is not surprising to find results that confirm the findings that the adoption of a more cautious approach by the central bank would be optimal with the introduction of variability on the value of the impact parameter. These results were detailed in section 5 of the essay. Even though the impact coefficients in Söderström (2002) were defined as the relationship between the real interest rate and the output gap as well as the dependency between the output gap and the inflation, results remain comparable. As shown in Section 7.3, variability on these two relationships can be aggregated into the variability of a single impact parameter, φ_t. The difference would not alter the findings that the central bank can progressively learn the parameter values and stabilize inflation.

On the other hand, some studies such as those of Söderström (2002) and Craine (1979) have found that uncertainty about the dynamics of the inflation rate in the economy would lead to the adoption of a more aggressive policy by the central bank as its optimal decision. As shown in Section 7.4, I found instead that variability on the transition parameter λ would lead once again to a wait-and-see approach to stabilize the level of inflation optimally. This difference in results may lie in the fact that the central bank modeled in the present research only stabilize the one-period deviations of inflation. The central bank would not want to adopt an aggressive approach that could cause significant deviations in inflation contemporaneously in order to stabilize future inflation. In the current model structure with a single-period loss function, the central bank does not consider subsequent periods. It would instead prioritize the use of a cautious approach when facing variability in the key parameters.

In addition, the essay included various extensions of the initial model to test whether the central bank could still learn the parameter values in multiple environments and stabilize inflation. It showed that the central bank was still able to determine the parameter values with the addition of an intercept value α to the Phillips Curve equation. The monetary authority was also able to mitigate fluctuations in the inflation rate while estimating the value of the slowly-evolving parameter φ_t with a rolling window algorithm. Furthermore, the addition of an IS curve, variability in the inflation persistence coefficient λ and the inclusion of an interest-rate smoothing parameter in the central bank's loss function did not alter the main findings of parameter learnability. Also, I showed that the use of a Kalman filter structure yields the same results as the RLS algorithm, which was used in the essay to model the learning process performed by the central bank.

Interesting extensions to the present essay concerns both statistical learning and economic structure. It is well known that the estimated autoregressive coefficients can be biased downward under the RLS algorithm in finite samples. See for examples Kendall (1954), Hurwicz (1950) and Hamilton (1994), among others. In the essay, the autoregressive structure of inflation is defined by the parameter λ. In the present case, it would cause the estimated ratio λ/φ to be biased downward and could serve as an explanation of why the estimates would converge from below the mean true value in most instances. It might be interesting to consider alternate estimators when performing the estimations.

Further analyses could be considered to enrich the economic model structure presented in this essay. A similar learning exercise could be implemented with the inclusion of a law of motion for the output gap variable y_t. As defined in Section 7.3, the model could include a persistence factor in the output gap variable. Moreover, the central bank could depart from a strict inflation-targeting regime and incorporate multiple objectives in its loss function. The model structure would then closely correspond to the structure of a formal Taylor Rule defined in Taylor (1993), where the central bank has to limit variations of inflation and in the output gap from their respective equilibrium levels. It would also be possible to study the learnability with a hybrid New-Keynesian Phillips Curve, as defined in Mavroeidis et al. (2014) illustrating an extension to the law of motion of the inflation rate presented in this essay. The realizations of inflation could be determined by a forward-looking component accounting for the expectation of agents regarding future realizations of the inflation rate in the economy. The central bank would have to forecast future values of the parameters in the PC equation when setting its policy interest rate. Its optimal behavior under parameter variability could be different from the results found in the present essay. As seen in Ferrero et al. (2019), policymakers could choose to deploy an aggressive strategy contemporaneously to mitigate present deviations of the inflation from its target. By doing so, it would prevent the potential unanchoring of inflation expectations and avert a potential inflationary spiral caused by the amplification of the effects of exogenous shocks on the inflation rate over time. Studying a forward-looking price setting and a multi-period loss function as used in Beck and Wieland (2002) and Kimura and Kurozumi (2007), would, however, require more advanced solution methods.

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A Appendix A: Realizations of the Variance of Inflation

One can compute a measure of the variance in inflation to capture how effectively central banks are able to stabilize inflation to their respective targets. The variance measures were computed over the 1000 simulations of a given Monte-Carlo sample. Then, the median, the 25th percentile, and 75th percentile measures were calculated over the 99 variances of the inflation rate in the sample. A comparison of the values in the table should be made cautiously. Scenarios 3, 4, and 5 can be compared together since they formalize the initial cases where the endogenous response and the variability in the parameter φ_t are progressively added to the model structure. The other values found in scenarios 7.1 to 7.5 can be compared to the dispersion measures found in scenario 5 since the standard deviations of the exogenous shocks ϵ_t and ν_t are set constant to 0.01 and remain consistent for all subsequent estimations.

Table 1: Median Realizations of the Variance of Inflation

Scenarios	Median	25th percentile	75th percentile
3. Benchmark Example	0.0157	0.0147	0.0165
4. Feedback Interest Rate Rule	0.0040	0.0039	0.0042
5. Parameter Variability φ_t	0.0041	0.0039	0.0043
7.1. Intercept α_t	0.1512	0.0154	0.1666
7.2.a Persistent Parameter φ_t - Rolling Window	0.0044	0.0042	0.0047
7.2.b Persistent Parameter φ_t - Expanding Window	0.0044	0.0040	0.0047
7.4.a Parameter Variability λ_t	0.0041	0.0039	0.0043
7.4.b Parameter Variability λ_t and ν_t	0.0043	0.0040	0.0045
7.5. Interest Rate Smoothing	0.4122	0.3378	0.5109

B Appendix B: Learning Accuracy

In an effort to quantify the degree of convergence of the various simulations and their respective extensions, it is possible to calculate the squared difference between the estimates and the actual value of the ratio λ/φ at a given observation. Then, one can take the sum of every squared difference for the 1000 observations of the algorithm. Subsequently, it is possible to derive the median, the 25th percentile, and 75th percentile measures of the 99 Monte-Carlo sums of differences as a comparison criterion. The measure of convergence can be summarized in the following form:

$$Difference_t = \sum_{t=1}^{1000} \left[\frac{\hat{\lambda}_t}{\hat{\varphi}_t} - \frac{\lambda}{\bar{\varphi}} \right]^2. \tag{51}$$

The disclaimer regarding the comparison of dispersion measures between scenarios issued for Table 1 of Appendix A also applies here.

Table 2: Median Deviations of the Estimates from the Actual Ratio λ/φ

Scenarios	Median	25th percentile	75th percentile
3. Benchmark Example	34.1309	16.0920	70.2331
4. Feedback Interest Rate Rule	4.0808	1.9731	8.4163
5. Parameter Variability φ_t	828.6741	199.1454	2050.5706
7.1. Intercept α_t	644.3159	325.8248	1502.5398
7.2.a Persistent Parameter φ_t - Rolling Window	3724.0770	2811.9769	4854.5474
7.2.b Persistent Parameter φ_t - Expanding Window	1355.6659	370.2488	4474.6513
7.4.a Parameter Variability λ_t	7.2397	3.0923	22.0720
7.4.b Parameter Variability λ_t and ν_t	746.4991	187.4913	2025.9888

C Appendix C: Simulation Procedure

Simulations were performed following the detailed sequences of events illustrated in the Model Environment and Introductory Benchmark section. The various simulations were estimated with a recursive least squares (RLS) method while varying the standard deviations of the cost-push shock σ_ϵ and the interest rate shock σ_ν to incorporate progressively parameter variability in the model structure. The standard deviations of the exogenous shocks varied in an increasing fashion from 0 to 0.01, 0.02, and 0.05. to assess the impact of including increasing parameter variability on the impact coefficient φ_t and having more sizeable volatility of the cost-push shock ϵ_t in the Phillips curve equation. For the Benchmark Example, the standard deviation of the interest rate shock ν_t is initially set to be equal to 0. The central bank can, therefore, estimate the parameters without considering any potential parameter variability endowed in the equations of the model.

The interest-rate decision equation defined in equation (2) includes an autoregressive process of order one. The coefficient ρ is set to an arbitrary value of 0.7 to induce sufficient persistence in the interest rate variable R_t. In addition, the mean value of the λ_t parameter is set to be constant at a value of 0.55. The value was estimated using an Ordinary Least Squares regression of the U.S. inflation rate computed as variations in the Consumer Price Index (CPI) (CPIAUCSL). To mimic the structure of the Phillips Curve equation defined in equation (1), the inflation rate was regressed on its lag component of order one and on the Fed Funds interest rate (FEDFUNDS), which corresponds to the policy interest rate variable R_t. The sample contained quarterly observations from 1954q3 to 2020q1. The data originates from the U.S. Congressional Budget Office obtained on the Federal Reserve Economic Database (FRED).

The parameter value of the interest-rate decision coefficient φ_t was also estimated from the empirical exercise mentioned previously. Its equilibrium value was estimated to be equal to 0.05. With the inclusion of a negative sign in front of the parameter φ_t in equation (1), one can see that increases in the policy interest rates have historically weakened inflation readings. The initial magnitudes of π_t and R_t were set to be equal to 2 and 1, respectively. The initial value of the inflation variable π_t was chosen to correspond to the current inflation target by the Bank of Canada and the U.S. Federal Reserve. The starting value of the interest rate was selected to be positive to avoid potential realizations of the interest rate falling below the zero lower bound with the introduction of parameter variability later on. A negative reading of the interest rate would lead to a counter-intuitive positive relationship between increases in the policy rate R_t and the inflation rate π_t.

In the simulation exercises, the Recursive Least Squares (RLS) regressions were computed analytically as follows:

$$\pi_t = \beta_1 R_t + \beta_2 \pi_{t-1} + \epsilon_t \tag{52}$$

$$\pi_t = -\varphi R_t + \lambda \pi_{t-1} + \epsilon_t. \tag{53}$$

The regression estimation procedure helps to find linear estimates of β_1 and β_2, defined as $\hat{\beta_1}$ and $\hat{\beta_2}$ respectively. It is then possible to translate the OLS estimates to their respective coefficients in the structure of the PC equation (53). The coefficient $\hat{\beta_2}$ translates directly to the $\hat{\lambda}$ coefficient, which measures the persistence of inflation. One can then add a minus sign to the estimates $\hat{\beta_1}$ in order to derive the value of the impact parameter φ_t. Subsequently, it is possible to match the structure of equation (53) computed with the RLS estimates, which model the negative relationship between the interest rate R_t and the contemporaneous inflation rate π_t. The t subscript indicates that interest rate variable R_t contain information up to the beginning of period t.

In the simulation algorithm, the initial values of π_t and R_t were computed over the smallest sample possible needed to have sufficient degrees of freedom to initiate the iterating process. As an example, the number of degrees of freedom in the Benchmark Example was set to 3 to estimate the values of φ and λ correctly. Once the first three observations were determined with the recursive algorithm, they were discarded from the sample to reduce the impact of the initial periods on the final values of the estimates.

The coefficients defined in equation (52) are then estimated recursively with an RLS algorithm over an additional 1050 observations. The first 50 observations are then removed from the final sample to limit the noise embedded in the very first observations as the central bank is trying to pin down the values of the parameters. In the initial sub-sample, the central bank does not have a sufficiently large historical sample size to estimate the governing parameters in the inflation rate equation (1) accurately. The figures presented in this essay only displayed the last 1000 (1053-3-50) iterations of the recursive algorithm.

The majority of exercises were conducted using an expanding window algorithm as the central bank uses the entire sample at its disposal to perform estimations of the parameters φ and λ. As detailed in section 7.2, the central bank also conducts the RLS estimations using a rolling window algorithm with a fixed window size of 50 observations. This algorithm specification would account for potential breaks in the structure of the economy. The central bank would then prefer to reduce its estimation sample to discard such likely observations. The algorithms, both expanding and rolling windows, were subsequently estimated repeatedly over 99 Monte-Carlo iterations to compute measures of dispersion of the estimated coefficients.