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The Quadratic Oil Extraction Oligopoly

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Abstract

Each extractor has a distinct quadratic extraction cost and faces a linear industry demand schedule. We observe that the open loop and closed loop solutions are the same if initial stocks are such that each competitor is extracting in every period in which her competitors are extracting. (oligop_july06.tex)

- key words: oligopoly extractors, closed loop solution
- classification: D43, Q32

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The Quadratic Oil Extraction Oligopoly

1 Introduction

The best hope for understanding oil extraction markets "on paper" would appear to be via oligopoly theory.[1] Here we present a new quadratic revenue, quadratic extraction cost case, each extractor with her own distinct costs, in which open loop and closed loop competition yields the same extraction paths. The sufficiency condition for the solutions to be the same here is simply that the endowments of oil to extractors (the initial finite stocks) must be such that each competitor produces positive quantities in every period.

2 The Model

The inverted market demand schedule is $a-b[q_t^1+q_t^2]$, a and slope b positive. An extractor's current profit, given q_t^1 currently extracted, is $\pi^1(q_t^1,q_t^2)=(a-b[q_t^1+q_t^2])q_t^1-[q_t^1]^2d^1$. We work in discrete time.[2] S_t^i is firm i 's current stock remaining. $S_{t+1}^i=S_t^i-q_t^i$. Each firm starts initially with a fixed endowment, S_0^i. We work here in the text with two extractors for ease of exposition. In Appendix 2 we report on cases with more than two extractors. Each extractor has extraction cost $[q_t^i]^2d^i$ with $d^i>0$. Our focus is on the case of $d^1\neq d^2$.(The case of $d^1=d^2$ is one of the two in Eswaran and Lewis [1985].) Each agent maximizes the

1 Eswaran and Lewis [1985] presented two interesting discrete-time examples in which open loop and closed loop competition among extracting oligopolists yielded the same paths of extraction. For one case competitors had distinct endowments and faced a constant elasticity industry demand schedule. This has recently been re-worked in continuous time by Benchakroun and Long [2005] and has been employed in an interesting exercise. In the other case, each firm had identical endowments and identical quadratic extraction costs and the industry demand schedule was linear. We are then generalizing this last example by allowing for each firm to have distinct quadratic extraction costs and distinct initial stocks.

2 We follow Eswaran and Lewis [1985]. The continuous time treatment of our problem might be simpler to work out because endpoint conditions are quite restrictive. We discuss endpoint conditions below for our discrete time formulation. Levhari and Mirman [1980] is a classic early closed loop oligopoly problem in discrete time.

present value of her profit stream to quantity q^i_{T-1} subject to $q^i_1 + q^i_2 + ... + q^i_{T-1} \leqq S^i_0$. β is the constant discount factor, $0 < \beta < 1$, the same for each extractor. We specify initial stocks so that each competitor solves with $q^i_1 + q^i_2 + ... + q^i_{T-1} = S^i_0$; and these initial stocks are such that each competitor produces positive output in the same period as another competitor is producing.

For the open loop case, each extractor maximizes her present value of profits by choice of a quantity stream, $\{q^i_1, q^i_2, ..., q^i_{T-1}\}$, taking the quantity stream of each competitor as parametric. We can distinguish two cases. (1) "knife-edged" endpoints[3]: in this case the initial quantities are such that in the final period, the q^i_{T-1} $'s$ are such that marginal profit, $mr^i_t - mc^i_t$, for each firm satisfies,

$$[mr^1_{T-1}(q^1_{T-1}, q^2_{T-1}) - mc^1_{T-1}(q^1_{T-1})] = \beta a = [mr^2_{T-1}(q^1_{T-1}, q^2_{T-1}) - mc^2_{T-1}(q^2_{T-1})].$$

This implies that

$$\begin{aligned} [mr^1_{T-2}(q^1_{T-2}, q^2_{T-2}) - mc^1_{T-2}(q^1_{T-2})] &= \beta^2 a = [mr^2_{T-2}(q^1_{T-2}, q^2_{T-2}) - mc^2_{T-2}(q^2_{T-2})], \\ [mr^1_{T-3}(q^1_{T-3}, q^2_{T-3}) - mc^1_{T-3}(q^1_{T-3})] &= \beta^3 a = [mr^2_{T-3}(q^1_{T-3}, q^2_{T-3}) - mc^2_{T-3}(q^2_{T-3})], \\ & \text{and so on, ...} \end{aligned}$$

This backward recursion allows us to solve explicitly for each quantity extracted, namely $\widehat{q}^1_{T-1} = \frac{(1-\beta^1)a[b+2d^2]}{D}$, $\widehat{q}^1_{T-2} = \frac{(1-\beta^2)a[b+2d^2]}{D}$, $\widehat{q}^1_{T-3} = \frac{(1-\beta^3)a[b+2d^2]}{D}$, $\widehat{q}^1_{T-4} = \frac{(1-\beta^4)a[b+2d^2]}{D}$, ... and $\widehat{q}^2_{T-1} = \frac{(1-\beta^1)a[b+2d^1]}{D}$, $\widehat{q}^2_{T-2} = \frac{(1-\beta^2)a[b+2d^1]}{D}$, $\widehat{q}^2_{T-3} = \frac{(1-\beta^3)a[b+2d^1]}{D}$, $\widehat{q}^2_{T-4} = \frac{(1-\beta^4)a[b+2d^1]}{D}$, ... and so on. This knife-edged endpoint solution is too special to merit much attention but the hat

[3] The "knife-edge" terminal condition is central to continuous time dynamic optimization problems. See Gelfand and Fomin [1963, p. 60]. Lozada [1993] discusses terminal condtitions for discrete time problems and compares the "knife-edge" terminal condition with the "general" terminal condition. In brief, dynamic problems end with a very restrictive condition in the "knife-edge" case and end somewhat "ragged" in general. Our analysis focuses on the general case.

q expressions turn out to be useful as components of the solution quantities for the general case.

(2) "general" endpoints: initial quantities are such that in the final period, the q^i_{T-1} $'s$ are such that marginal profit for each firm satisfies,

$$
\begin{aligned}
[mr^1_{T-1}(q^1_{T-1}, q^2_{T-1}) - mc^1_{T-1}(q^1_{T-1})] &> \beta a \\
\text{and } [mr^2_{T-1}(q^1_{T-1}, q^2_{T-1}) - mc^2_{T-1}(q^2_{T-1})] &> \beta a.
\end{aligned}
$$

For this general case, there is not a simple backward recursion yielding the solution values for quantities extracted and this general case is the one which we focus on here.

3 Solving the Closed Loop Problem

In the closed loop case, competition among extractors is re-opened *de novo* at each consecutive period, contingent on each player taking current stock levels as the current state of the system. There is no commitment at period zero to an extraction path as there is with open loop competition. Closed loop competition requires competitive outcomes to be worked out for each period in a backward recursion or by dynamic programming arguments.

In the final period, we have

$$
\begin{aligned}
V^1_{T-1}(q^1_{T-1}, q^2_{T-1}) &= \pi^1(q^1_{T-1}, q^2_{T-1}) \\
\text{and } V^2_{T-1}(q^1_{T-1}, q^2_{T-1}) &= \pi^2(q^1_{T-1}, q^2_{T-1}) \\
\text{with } q^1_{T-1} &= S^1_{T-2} - q^1_{T-2} \text{ and } q^2_{T-1} = S^2_{T-2} - q^2_{T-2}.
\end{aligned}
$$

There are no residual stocks at the termination of extraction for any extractor. Moving one period backwards in time toward the present, we have

$$\begin{aligned} V^1_{T-2}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2}) &= \max_{q^1_{T-2}}\{\pi^1_{T-2}(q^1_{T-2},q^2_{T-2}) \\ &\quad +\beta V^1_{T-1}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2})\} \\ \text{and } V^2_{T-2}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2}) &= \max_{q^2_{T-2}}\{\pi^2_{T-2}(q^1_{T-2},q^2_{T-2}) \\ &\quad +\beta V^2_{T-1}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2})\}. \end{aligned}$$

Assuming differentiability of the V^i_{T-2}'s, the maximizations yield

$$\begin{aligned} mr^1_{T-2}(q^1_{T-2},q^2_{T-2})-mc^1_{T-2}(q^1_{T-2}) &= \beta[mr^1_{T-1}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2}) && (1) \\ &\quad -mc^1_{T-2}(S^1_{T-2}-q^1_{T-2})], \\ mr^2_{T-2}(q^1_{T-2},q^2_{T-2})-mc^2_{T-2}(q^2_{T-2}) &= \beta[mr^2_{T-1}(S^1_{T-2}-q^1_{T-2},S^2_{T-2}-q^2_{T-2}) \\ &\quad -mc^2_{T-2}(S^2_{T-2}-q^2_{T-2})], && (2) \end{aligned}$$

which simplify to:

$$q^i_{T-2}=\frac{1}{1+\beta}\widehat{q}^i_{T-1}+\frac{\beta}{1+\beta}S^i_{T-2} \qquad i=1,2. \tag{3}$$

for $\widehat{q}^i_{T-1}=\frac{(1-\beta)a[b+2d^j]}{D}$, $D=(2b+2d^1)(2b+2d^2)-b^2$. This result in (3) is central because each competitor's current extraction is being represented as independent of the other competitor's current level of stock.[4] It is as if each competitor were extracting from her own stock, independently of the other extractor. The pair of equations in (2) is also fundamental because they are a template for further backward steps in the solution.

The envelope theorem simplifies the "generation" of the equations for proceeding to solve for q^1_{T-3} and q^2_{T-3}. We end up faced with the following system to deal with. Current

[4] This two firm, two period result was first observed by Michael Brolley, an undergraduate research assistant who was checking some of our detailed notes on the Eswaran-Lewis research. We were surprised to get essentially Eswaran-Lewis results for our considerably more general specification of each firm's initial stock and extraction costs.

marginal revenue minus marginal cost for a firm is the appropriately discounted marginal revenue minus marginal cost at the "terminal" period. That is:

$$
\begin{aligned}
mr^1_{T-3}(q^1_{T-3}, q^2_{T-3}) - mc^1_{T-3}(q^1_{T-3}) &= \beta^2[mr^1_{T-1}(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}) \\
&\quad - mc^1_{T-1}(S^1_{T-2} - q^1_{T-2})], \\
mr^2_{T-3}(q^1_{T-3}, q^2_{T-3}) - mc^2_{T-3}(q^2_{T-3}) &= \beta^2[mr^2_{T-1}(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}) \\
&\quad - mc^2_{T-1}(S^2_{T-2} - q^2_{T-2})],
\end{aligned}
$$

The envelope theorem yields the "cancellation" of $mr - mc$ terms for intermediate dates. Using the result in (3), this pair solves as:[5]

$$
\begin{aligned}
q^i_{T-3} &= \frac{1+\beta}{1+\beta+\beta^2}\widehat{q}^i_{T-2} + \frac{\beta^2}{1+\beta+\beta^2}[S_{T-3} - \widehat{q}^i_{T-1}] \qquad i = 1, 2 \qquad (4) \\
\text{for } \widehat{q}^i_{T-2} &= \frac{(1-\beta^2)a[b+2d^j]}{D}.
\end{aligned}
$$

We observe each current extraction a function of the extractor's current stock. This independence property is essential to the sameness of extraction paths under open loop and closed loop competition.[6] Moving one period toward the present, we again consider first order conditions (again invoking the envelope theorem)

$$
\begin{aligned}
mr^1_{T-4}(q^1_{T-4}, q^2_{T-4}) - mc^1_{T-4}(q^1_{T-4}) &= \beta^3[mr^1_{T-1}(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}) \\
&\quad - mc^1_{T-2}(S^1_{T-2} - q^1_{T-2})], \\
mr^2_{T-4}(q^1_{T-4}, q^2_{T-4}) - mc^2_{T-4}(q^2_{T-4}) &= \beta^3[mr^2_{T-1}(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}) \\
&\quad - mc^2_{T-2}(S^2_{T-2} - q^2_{T-2})],
\end{aligned}
$$

[5] Details are provided in Appendix 1.
[6] See Eswaren and Lewis [1985] for details on the sameness of open loop and closed loop solutions.

and using (3) and (4) obtain

$$\begin{aligned} q^i_{T-4} &= \frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\widehat{q}^i_{T-3} + \frac{\beta^3}{1+\beta+\beta^2+\beta^3}[S^i_{T-4} - \widehat{q}^i_{T-1} - \widehat{q}^i_{T-2}] \\ \text{for } i &= 1,2 \text{ and } \widehat{q}^i_{T-3} = \frac{(1-\beta^3)a[b+2d^j]}{D} \ . \end{aligned}$$

(In Appendix 1 we fill in the details of solving for q^i_{T-4}.) We observe the independence property present when we extend the backward recursion from the future toward the present. For $T-t$ we get[7]

$$\begin{aligned} q^i_{T-t} &= \frac{1+\beta+\beta^2+...+\beta^{T-t-2}}{1+\beta+\beta^2+\beta^3+...+\beta^{T-t-1}}\widehat{q}^i_{T-t-1} \\ &+ \frac{\beta^{T-t-1}}{1+\beta+\beta^2+\beta^3+...+\beta^{T-t-1}}[S^i_{T-t} - \widehat{q}^i_{T-1} - \widehat{q}^i_{T-2} - ... - \widehat{q}^i_{T-t-2}] \end{aligned}$$

for $i = 1, 2$.[8]

We illustrate with a numerical example. We take extractor 1 to be high cost with $d^1 = 0.3$ and $d^2 = 0.2$. The inverse demand schedule is $10 - 0.1\{q^1_t + q^2_t\}$. The discount factor, β is 0.8. For three periods of positive extraction, we turn to our formulas above and obtain $\widehat{q}^1_T = 0$, $\widehat{q}^1_{T-1} = 1.0$, $\widehat{q}^1_{T-2} = 1.8$, and $\widehat{q}^1_{T-3} = 2.44$, with endowment $S^1_{T-3} = 5.24$; and $\widehat{q}^2_T = 0$, $\widehat{q}^2_{T-1} = 1.4$, $\widehat{q}^2_{T-2} = 2.52$ and $\widehat{q}^2_{T-3} = 3.416$, with endowment $S^2_{T-3} = 7.336$. We now consider the closed loop problem with endowments $S^1_{T-3} = 5.0$ and $S^2_{T-3} = 7.0$, each slightly less than the endowments above, namely those selected to have a special end point condition. Using our formulas above for the q^i $'s$, we obtain, $q^1_{T-3} = 2.377$, $q^1_{T-2} = 1.7$, and $q^1_{T-1}0.913$; and $q^2_{T-3} = 3.326$, $q^2_{T-2} = 2.41$, and $q^2_{T-1} = 1.264$. These values are identical with those which solve the corresponding open loop problem (the one with commitment at the intial period by each player).

[7] There is a check on these derivations. We know the exact expessions for the $\widehat{q}^i_{T-t}$. We can replace the corresponding expressions without hats with these with hats and verify that in each case the right hand sides for our "formulae" match the left hand sides.

[8] There is a very similar set of formulas for the case of a single monopoly extractor with quadatic extraction costs and facing a linear demand schedule.

The final step is to substitute for the $q's$ in the present value profit functions in a backward recursion to obtain, $V^1_{T-1}(S^1_{T-1}, S^2_{T-1})$, $V^2_{T-1}(S^1_{T-1}, S^2_{T-1})$, $V^1_{T-2}(S^1_{T-2}, S^2_{T-2})$, $V^2_{T-2}(S^1_{T-2}, S^2_{T-2})$, $V^1_{T-3}(S^1_{T-3}, S^2_{T-3})$, $V^2_{T-3}(S^1_{T-3}, S^2_{T-3})$, and so on.

We can express each q^i_{T-t} as $\lambda^i_t + \mu^i_t S^i_{T-t}$. (We have solved for the λ^i_t and μ^i_t explicitly above. Note that $\lambda^i_1 = 0$ and $\mu^i_1 = 1$.) We display the coefficients of our equation in cases 1 through 4 and the nth, as they appear in the equation of the form,[9] $V^1_{T-t}(S^1_{T-t}, S^2_{T-t}) = A^1_t + B^1_t S^1_{T-t} + C^1_t S^2_{T-t} + D^1_t (S^1_{T-t})^2 + E^1_t S^1_{T-t} S^2_{T-t}$. That is:

$$
\begin{aligned}
&A^1_1 = \lambda^1_1(a - (b + d^1)\lambda^1_1 - b\lambda^2_1)\\
&A^1_2 = \{\lambda^1_2(a - (b + d^1)\lambda^1_2 - b\lambda^2_2)\\
&+\beta(\lambda^1_1 - \mu_1\lambda^1_2)(a - (b + d^1)(\lambda^1_1 - \mu_1\lambda^1_2) - b(\lambda^2_1 - \mu_1\lambda^2_2))\\
&A^1_3 = \{\lambda^1_3(a - (b + d^1)\lambda^1_3 - b\lambda^2_3) + \beta(\lambda^1_2 - \mu_2\lambda^1_3)(a - (b + d^1)(\lambda^1_2 - \mu_2\lambda^1_3) - b(\lambda^2_2 - \mu_2\lambda^2_3))\\
&+\beta^2(\lambda^1_1 - \mu_1\lambda^1_2 - \mu_1(1 - \mu_2)\lambda^1_3)(a - (b + d^1)(\lambda^1_1 - \mu_1\lambda^1_2 - \mu_1(1 - \mu_2)\lambda^1_3)\\
&-b(\lambda^2_1 - \mu_1\lambda^2_2 - \mu_1(1 - \mu_2)\lambda^2_3)\}\\
&A^1_4 = \{\lambda^1_4(a - (b + d^1)\lambda^1_4 - b\lambda^2_4) + \beta(\lambda^1_3 - \mu_3\lambda^1_4)(a - (b + d^1)(\lambda^1_3 - \mu_3\lambda^1_4) - b(\lambda^2_3 - \mu_3\lambda^2_4))\\
&+\beta^2(\lambda^1_2 - \mu_2\lambda^1_3 - \mu_2(1 - \mu_3)\lambda^1_4)(a - (b + d^1)(\lambda^1_2 - \mu_2\lambda^1_3 - \mu_2(1 - \mu_3)\lambda^1_4)\\
&-b(\lambda^2_2 - \mu_2\lambda^2_3 - \mu_2(1 - \mu_3)\lambda^2_4) + \beta^3(\lambda^1_1 - \mu_1\lambda^1_2 - \mu_1(1 - \mu_2)\lambda^1_3 - \mu_1(1 - \mu_2)(1 - \mu_3)\lambda^1_4)\\
&\times(a - (b + d^1)(\lambda^1_1 - \mu_1\lambda^1_2 - \mu_1(1 - \mu_2)\lambda^1_3 - \mu_1(1 - \mu_2)(1 - \mu_3)\lambda^1_4) - b(\lambda^2_1 - \mu_1\lambda^2_2\\
&-\mu_1(1 - \mu_2)\lambda^2_3 - \mu_1(1 - \mu_2)(1 - \mu_3)\lambda^2_4)\}\\
&...
\end{aligned}
$$

$$
\begin{aligned}
&A^1_n = \{\lambda^1_n(a - (b + d^1)\lambda^1_n - b\lambda^2_n) + \beta(\lambda^1_{n-1} - \mu_{n-1}\lambda^1_n)(a - (b + d^1)(\lambda^1_{n-1} - \mu_{n-1}\lambda^1_n)\\
&-b(\lambda^2_{n-1} - \mu_{n-1}\lambda^2_n)) + ... + \beta^{n-1}(\lambda^1_1 - \mu_1\lambda^1_2 - \mu_1(1 - \mu_2)\lambda^1_3 - ... - \mu_1(1 - \mu_2) \times ...\\
&\times(1 - \mu_{n-1})\lambda^1_n)(a - (b + d^1)(\lambda^1_1 - \mu_1\lambda^1_2 - \mu_1(1 - \mu_2)\lambda^1_3 - ... - \mu_1(1 - \mu_2) \times ...\\
&\times(1 - \mu_{n-1})\lambda^1_n) - b(\lambda^2_1 - \mu_1\lambda^2_2 - \mu_1(1 - \mu_2)\lambda^2_3 - ... - \mu_1(1 - \mu_2)...(1 - \mu_{n-1})\lambda^2_n)\};
\end{aligned}
$$

$$
\begin{aligned}
&B^1_1 S^1_{T-1} = \{\mu_1(a - 2(b + d^1)\lambda^1_1 - b\lambda^2_1)\}S^1_{T-1}\\
&B^1_2 S^1_{T-2} = \{\mu_2(a - 2(b + d^1)\lambda^1_2 - b\lambda^2_2) + \beta\mu_1(1 - \mu_2)(a - 2(b + d^1)(\lambda^1_1 - \mu_1\lambda^1_2)\\
&-b(\lambda^2_1 - \mu_1\lambda^2_2))\}S^1_{T-2}\\
&B^1_3 S^1_{T-3} = \{\mu_3(a - 2(b + d^1)\lambda^1_3 - b\lambda^2_3) + \beta\mu_2(1 - \mu_3)(a - 2(b + d^1)(\lambda^1_2 - \mu_2\lambda^1_3)\\
&-b(\lambda^2_2 - \mu_2\lambda^2_3)) + \beta^2\mu_1(1 - \mu_2)(1 - \mu_3)(a - 2(b + d^1)(\lambda^1_1 - \mu_1\lambda^1_2 - \mu_1(1 - \mu_2)\lambda^1_3)\\
&-b(\lambda^2_1 - \mu_1\lambda^2_2 - \mu_1(1 - \mu_2)\lambda^2_3))\}S^1_{T-3}\\
&B^1_4 S^1_{T-4} = \{\mu_4(a - 2(b + d^1)\lambda^1_4 - b\lambda^2_4)
\end{aligned}
$$

[9] These value functions satisfy the Bellman equation: $V^i_t(S^1_t, S^2_t) = \max_{q^i_t}\{\pi^i(q^1_t, q^2_t) + \beta V^i_{t+1}(S^1_t - q^1_t, S^2_t - q^2_t)\}$.

$+\beta\mu_3(1-\mu_4)(a-2(b+d^1)(\lambda_3^1-\mu_3\lambda_4^1)-b(\lambda_3^2-\mu_3\lambda_4^2))+\beta^2\mu_2(1-\mu_3)(1-\mu_4)$
$\times(a-2(b+d^1)(\lambda_2^1-\mu_2\lambda_3^1-\mu_2(1-\mu_3)\lambda_4^1)-b(\lambda_2^2-\mu_2\lambda_3^2-\mu_1(1-\mu_2)(1-\mu_3)\lambda_4^1))$
$+\beta^3\mu_1(1-\mu_2)(1-\mu_3)(1-\mu_4)(a-2(b+d^1)(\lambda_1^1-\mu_1\lambda_2^1-\mu_1(1-\mu_2)\lambda_3^1$
$-\mu_1(1-\mu_2)(1-\mu_3)\lambda_4^1)-b(\lambda_1^2-\mu_1\lambda_2^2-\mu_1(1-\mu_2)\lambda_3^2-\mu_1(1-\mu_2)(1-\mu_3)\lambda_4^2))\}S_{T-4}^1$
...

$B_n^1S_{T-n}^1=\{\mu_n(a-2(b+d^1)\lambda_n^1-b\lambda_n^2)+\beta\mu_{n-1}(1-\mu_n)(a-2(b+d^1)(\lambda_{n-1}^1-\mu_{n-1}\lambda_n^1)$
$-b(\lambda_{n-1}^2-\mu_{n-1}\lambda_n^2))+...+\beta^{n-1}\mu_1(1-\mu_2)\times...\times(1-\mu_n)(a-2(b+d^1)(\lambda_1^1-\mu_1\lambda_2^1$
$-\mu_1(1-\mu_2)\lambda_3^1-...-\mu_1(1-\mu_2)...(1-\mu_{n-1})\lambda_n^1)-b(\lambda_1^2-\mu_1\lambda_2^2-\mu_1(1-\mu_2)\lambda_3^2$
$-...-\mu_1(1-\mu_2)...(1-\mu_{n-1})\lambda_n^2))\}S_{T-n}^1;$

$C_1^1S_{T-1}^2=-b\{\mu_1\}\lambda_1^1S_{T-1}^2$
$C_2^1S_{T-2}^2=-b\{\mu_2\lambda_2^1+\beta\mu_1(1-\mu_2)(\lambda_1^1-\mu_1\lambda_2^1)\}S_{T-2}^2$
$C_3^1S_{T-3}^2=-b\{\mu_3\lambda_3^1+\beta\mu_2(1-\mu_3)(\lambda_2^1-\mu_2\lambda_3^1)+\beta^2\mu_1(1-\mu_2)(1-\mu_3)$
$\times(\lambda_1^1-\mu_1\lambda_2^1-\mu_1(1-\mu_2)\lambda_3^1)\}S_{T-3}^2$
$C_4^1S_{T-4}^2=-\{b\mu_4\lambda_4^1+\beta\mu_3(1-\mu_4)(\lambda_3^1-\mu_3\lambda_4^1)+\beta^2\mu_2(1-\mu_3)(1-\mu_4)(\lambda_2^1-\mu_2\lambda_3^1$
$-\mu_2(1-\mu_3)\lambda_4^1)+\beta^3\mu_1(1-\mu_2)(1-\mu_3)(1-\mu_4)(\lambda_1^1-\mu_1\lambda_2^1-\mu_1(1-\mu_2)\lambda_3^1$
$-\mu_1(1-\mu_2)(1-\mu_3)\lambda_4^1)\}S_{T-4}^2$
...

$C_n^1S_{T-n}^2=-\{b\mu_n\lambda_n^1+\beta\mu_{n-1}(1-\mu_n)(\lambda_{n-1}^1-\mu_{n-1}\lambda_n^1)+...+\beta^{n-1}\mu_1(1-\mu_2)$
$\times...\times(1-\mu_n)(\lambda_1^1-\mu_1\lambda_2^1-\mu_1(1-\mu_2)\lambda_3^1-...-\mu_1(1-\mu_2)(1-\mu_{n-1})\lambda_n^1)\}S_{T-n}^2;$

$D_1^1(S_{T-1}^1)^2=-(b+d^1)\{(\mu_1)^2\}(S_{T-1}^1)^2$
$D_2^1(S_{T-2}^1)^2=-(b+d^1)\{(\mu_2)^2+\beta(\mu_1(1-\mu_2))^2\}(S_{T-2}^1)^2$
$D_3^1(S_{T-3}^1)^2=-(b+d^1)\{(\mu_3)^2+\beta(\mu_2(1-\mu_3))^2+\beta^2(\mu_1(1-\mu_2)(1-\mu_3))^2\}(S_{T-3}^1)^2$
$D_4^1(S_{T-4}^1)^2=-(b+d^1)\{(\mu_4)^2+\beta(\mu_3(1-\mu_4))^2+\beta^2(\mu_2(1-\mu_3)(1-\mu_4))^2$
$+\beta^3(\mu_1(1-\mu_2)(1-\mu_3)(1-\mu_4))^2\}(S_4^1)^2$
...

$D_n^1(S_{T-n}^1)^2=-(b+d^1)\{(\mu_n)^2+\beta(\mu_{n-1}(1-\mu_n))^2+...+\beta^{n-1}(\mu_1(1-\mu_2)$
$\times...\times(1-\mu_n))^2\}(S_{T-n}^1)^2;$

$E_1^1S_{T-1}^1S_{T-1}^2=-b\{(\mu_1)^2\}S_{T-1}^1S_{T-1}^2$
$E_2^1S_{T-2}^1S_{T-2}^2=-b\{(\mu_2)^2+\beta(\mu_1(1-\mu_2))^2\}S_{T-2}^1S_{T-2}^2$
$E_3^1S_{T-3}^1S_{T-3}^2=-b\{(\mu_3)^2+\beta(\mu_2(1-\mu_3))^2+\beta^2(\mu_1(1-\mu_2)(1-\mu_3))^2\}S_{T-3}^1S_{T-3}^2$
$E_4^1S_{T-4}^1S_{T-4}^2=-b\{(\mu_4)^2+\beta(\mu_3(1-\mu_4))^2+\beta^2(\mu_2(1-\mu_3)(1-\mu_4))^2$
$+\beta^3(\mu_1(1-\mu_2)(1-\mu_3)(1-\mu_4))^2\}S_{T-4}^1S_{T-4}^2$
...

$E_n^1S_{T-n}^1S_{T-n}^2=-b\{(\mu_n)^2+\beta(\mu_{n-1}(1-\mu_n))^2+...+\beta^{n-1}(\mu_1(1-\mu_2)$
$\times...\times(1-\mu_n))^2\}S_{T-n}^1S_{T-n}^2$

4 Concluding Remarks

Our initial scrutiny of end-point conditions for the quadratic oligopoly exhaustible resource extraction problem led us to the surprising discovery of the well-behavedness of the closed loop version with each extractor with distinct quadratic extraction costs and distinct initial holdings of stock to extract. The well-behavedness extends to the closed loop and open loop solutions being the same provided each extractor is doing positive extraction when each of her competitors is doing positive extractions, a seemingly weak requirement. We know that such well-behavedness is present when industry demand is specified as constant elasticity and each firm has no cost of extraction but future research will reveal if these two cases exhaust the list of oligopoly extraction problems in which open loop and closed loop problems exhibit identical extraction paths.

Appendix 1: Calculations for obtaining q^i_{T-4} with 2 firms

Profit for firm 1 is

$$\begin{aligned}
\pi^1_{T-4} &= \{a - b[q^1_{T-4} + q^2_{T-4}]\}q^1_{T-4} - d^1[q^1_{T-4}]^2 \\
&+\beta\left[\{a - b[q^1_{T-3} + q^2_{T-3}]\}q^1_{T-3} - d^1[q^1_{T-3}]^2\right] \\
&+\beta^2\left[\{a - b[q^1_{T-2} + q^2_{T-2}]\}q^1_{T-2} - d^1[q^1_{T-2}]^2\right] \\
&+\beta^3\left[\{a - b[S^1_{T-2} - q^1_{T-2} + S^2_{T-2} - q^2_{T-2}]\}[S^1_{T-2} - q^1_{T-2}] - d^1[S^1_{T-2} - q^1_{T-2}]^2\right].
\end{aligned}$$

There is an analogous profit statement for firm 2, with the chief difference the presence of cost parameter d^2 in place of d^1. For these profit statements, we have explicit substitutions for q^1_{T-3}, q^1_{T-2}, and q^2_{T-3} and q^2_{T-2}, expressions already obtained earlier in the backward recursion, namely

$$\begin{aligned}
q^i_{T-2} &= \frac{1}{1+\beta}\widehat{q}^i_{T-1} + \frac{\beta}{1+\beta}S^i_{T-2} \qquad i = 1,2 \\
\text{and } q^i_{T-3} &= \frac{1+\beta}{1+\beta+\beta^2}\widehat{q}^i_{T-2} + \frac{\beta^2}{1+\beta+\beta^2}[S_{T-3} - \widehat{q}^i_{T-1}] \qquad i = 1,2.
\end{aligned}$$

When we make the substitutions and solve for $\frac{\partial \pi^1_{T-4}}{\partial q^1_{T-4}} = 0$. Exploiting the envelope theorem, we get

$$\begin{aligned}
a - [2b + 2d^1]q^1_{T-4} - bq^2_{T-4} &= \beta^3\{a - \frac{[2b + 2d^1]}{1+\beta+\beta^2}[S^1_{T-4} - q^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}]\} \\
&-\beta^3\{\frac{b}{1+\beta+\beta^2}[S^2_{T-4} - q^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}]\}
\end{aligned}$$

and for the analogous first order condition for firm 2 in

$$\begin{aligned}
a - [2b + 2d^2]q^2_{T-4} - bq^1_{T-4} &= \beta^3\{a - \frac{[2b + 2d^2]}{1+\beta+\beta^2}[S^2_{T-4} - q^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}]\} \\
&-\beta^3\{\frac{b}{1+\beta+\beta^2}[S^1_{T-4} - q^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}]\}.
\end{aligned}$$

We solve for q^1_{T-4} and q^2_{T-4} in these two linear equations to get

$$q^i_{T-4} = \frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\widehat{q}^i_{T-3} + \frac{\beta^3}{1+\beta+\beta^2+\beta^3}[S^i_{T-4} - \widehat{q}^i_{T-1} - \widehat{q}^i_{T-2}] \qquad i = 1,2.$$

We emphasize that each extractor's current quantity extracted is being expressed as a linear function of her own current stock alone, even though each extractor has distinct extraction costs. Below we provide detail for solving for the q^i_{T-t} $'s$ for the 3 firm case. This detail "works" as well for the two firm case above. One can readily see the "transition" from the backward recursion from period $T-t$ to period $T-t-1$.

Appendix 2: More than Two Firms

Moving backward two periods, the first order conditions for profit maximization over the two end periods for each of the three firms are

$$\begin{aligned} a-[2b+2d^1]q^1_{T-2}-bq^2_{T-2}-bq^3_{T-2} &= \beta\{a-[2b+2d^1](S^1_{T-2}-q^1_{T-2})-b(S^2_{T-2}-q^2_{T-2}) \\ &\quad -b(S^3_{T-2}-q^3_{T-2}), \\ a-bq^1_{T-2}-[2b+2d^2]q^2_{T-2}-bq^3_{T-2} &= \beta\{a-b(S^1_{T-2}-q^1_{T-2})-[2b+2d^2](S^2_{T-2}-q^2_{T-2}) \\ &\quad -b(S^3_{T-2}-q^3_{T-2}), \\ a-bq^1_{T-2}-bq^2_{T-2}-[2b+2d^3]q^3_{T-2} &= \beta\{a-b(S^1_{T-2}-q^1_{T-2})-b(S^2_{T-2}-q^2_{T-2}) \\ &\quad -[2b+2d^3](S^3_{T-2}-q^3_{T-2}). \end{aligned}$$

This is three linear equations in q^1_{T-2}, q^2_{T-2} and q^3_{T-2} :

$$\begin{bmatrix} [2b+2d^1] & b & b \\ b & [2b+2d^2] & b \\ b & b & [2b+2d^3] \end{bmatrix} \begin{bmatrix} q^1_{T-2} \\ q^2_{T-2} \\ q^3_{T-2} \end{bmatrix}$$

$$= \frac{1}{1+\beta} \begin{bmatrix} (1-\beta)a+\beta[2b+2d^1]S^1_{T-2}+\beta bS^2_{T-2}+\beta bS^3_{T-2} \\ (1-\beta)a+\beta bS^1_{T-2}+\beta[2b+2d^2]S^2_{T-2}+\beta bS^3_{T-2} \\ (1-\beta)a+\beta bS^1_{T-2}+\beta bS^2_{T-2}+\beta[2b+2d^3]S^3_{T-2} \end{bmatrix}.$$

The solutions are

$$\begin{aligned} q^1_{T-2} &= \left[\frac{1}{1+\beta}\right]\widehat{q}^1_{T-1} + \left[\frac{\beta}{(1+\beta)}\right]S^1_{T-2} \\ q^2_{T-2} &= \left[\frac{1}{1+\beta}\right]\widehat{q}^2_{T-1} + \left[\frac{\beta}{(1+\beta)}\right]S^2_{T-2} \\ q^3_{T-2} &= \left[\frac{1}{1+\beta}\right]\widehat{q}^3_{T-1} + \left[\frac{\beta}{(1+\beta)}\right]S^3_{T-2} \end{aligned}$$

for $\widehat{q}^1_{T-1} = \left[\frac{1-\beta}{\Delta}\right]a\{[2b+2d^2][2b+2d^3]-b[2b+2d^2]-b[2b+2d^3]+b^2\}$, $\widehat{q}^2_{T-1} = \left[\frac{1-\beta}{\Delta}\right]a\{[2b+2d^1][2b+2d^3]-b[2b+2d^1]-b[2b+2d^3]+b^2\}$, $\widehat{q}^3_{T-1} = \left[\frac{1-\beta}{\Delta}\right]a\{[2b+2d^1][2b+2d^2]-b[2b+2d^1]-b[2b+2d^2]+b^2\}$, and $\Delta = \{[2b+2d^1][2b+2d^2][2b+2d^3]-b^2[2b+2d^1]-b^2[2b+2d^2]-b^2[2b+2d^3]+2b^3\}$. These solutions or "extraction rules" have the identical form as those for the two firm case.

What we are dealing with generically is a system of the form

$$\begin{bmatrix} k^1 & b & b \\ b & k^2 & b \\ b & b & k^3 \end{bmatrix}\begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} = \begin{bmatrix} A + k^1S^1 + bS^2 + bS^3 \\ A + bS^1 + k^2S^2 + bS^3 \\ A + bS^1 + bS^2 + k^3S^3 \end{bmatrix},$$

for k^i $'s$, $b's$, $A's$ and S^i $'s$ positive scalars. The presence of the $A's$ lead to the solution for the $\widehat{q}^i$ part of our solutions above. We are however interested in when the solution q^i depends on S^i alone. This leaves us to focus our attention on the reduced system

$$\begin{bmatrix} k^1 & b & b \\ b & k^2 & b \\ b & b & k^3 \end{bmatrix}\begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} = \begin{bmatrix} k^1S^1 + bS^2 + bS^3 \\ bS^1 + k^2S^2 + bS^3 \\ bS^1 + bS^2 + k^3S^3 \end{bmatrix}.$$

This system is fundamental to our result that q^i solves in terms of S^i alone. We verify that this 3 equation system solves with

$$q^i = \{k^1(k^2k^3 - [b]^2) - b(bk^3 - [b]^2) + b([b]^2 - bk^2)\}S^i, \quad i = 1, 2, 3.$$

We now indicate how a proof by induction on the size of our system of equations establishes that each firm's current extraction, q^i can be expressed as a function of its own current stock S^i alone. We illustrate the induction step of moving from an $(n-1)\times(n-1)$ system, for which the result is assumed true, to an $n \times n$ system. We consider now the corresponding 4×4 system in terms of 3×3 subsystems (this illustrates the key step in an induction proof).

$$\begin{bmatrix} k^1 & b & b & b \\ b & k^2 & b & b \\ b & b & k^3 & b \\ b & b & b & k^4 \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \\ q^3 \\ q^4 \end{bmatrix} = \begin{bmatrix} k^1S^1 + bS^2 + bS^3 + bS4 \\ bS^1 + k^2S^2 + bS^3 + bS^4 \\ bS^1 + bS^2 + k^3S^3 + bS^4 \\ bS^1 + bS^2 + bS^3 + k^4S^4 \end{bmatrix}.$$

The solution for q^1 for the above system can be written

$$\frac{k^1S^1}{D} \times \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix}$$

$$+\frac{bS^2}{D} \times \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \times \det \begin{bmatrix} bS^1 + k^2S^2 + bS^3 + bS^4 & b & b \\ bS^1 + bS^2 + k^3S^3 + bS^4 & k^3 & b \\ bS^1 + bS^2 + bS^3 + k^4S^4 & b & k^4 \end{bmatrix}$$

$$+\frac{bS^3}{D} \times \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \times \det \begin{bmatrix} k^2 & bS^1 + k^2S^2 + bS^3 + bS^4 & b \\ b & bS^1 + bS^2 + k^3S^3 + bS^4 & b \\ b & bS^1 + bS^2 + bS^3 + k^4S^4 & k^4 \end{bmatrix}$$

$$+\frac{bS4}{D} \times \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \times \det \begin{bmatrix} k^2 & b & bS^1 + k^2S^2 + bS^3 + bS^4 \\ b & k^3 & bS^1 + bS^2 + k^3S^3 + bS^4 \\ b & b & bS^1 + bS^2 + bS^3 + k^4S^4 \end{bmatrix}$$

$$
\begin{aligned}
= \quad & \frac{k^1 S^1}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} \\
& -\frac{b}{D} \left\{ \det \begin{bmatrix} bS^1 & b & b \\ bS^1 & k^3 & b \\ bS^1 & b & k^4 \end{bmatrix} + \det \begin{bmatrix} k^2 & bS^1 & b \\ b & bS^1 & b \\ b & bS^1 & k^4 \end{bmatrix} + \det \begin{bmatrix} k^2 & b & bS^1 \\ b & k^3 & bS^1 \\ b & b & bS^1 \end{bmatrix} \right\} \\
& +\frac{bS^2}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \det \begin{bmatrix} k^2 S^2 + bS^3 + bS^4 & b & b \\ bS^2 + k^3 S^3 + bS^4 & k^3 & b \\ bS^2 + bS^3 + k^4 S^4 & b & k^4 \end{bmatrix} \\
& +\frac{bS^3}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \det \begin{bmatrix} k^2 & k^2 S^2 + bS^3 + bS^4 & b \\ b & bS^2 + k^3 S^3 + bS^4 & b \\ b & bS^2 + bS^3 + k^4 S^4 & k^4 \end{bmatrix} \\
& +\frac{bS^4}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix} - \frac{b}{D} \det \begin{bmatrix} k^2 & b & k^2 S^2 + bS^3 + bS^4 \\ b & k^3 & bS^2 + k^3 S^3 + bS^4 \\ b & b & bS^2 + bS^3 + k^4 S^4 \end{bmatrix}
\end{aligned}
$$

$$= \frac{k^1 S^1}{D} \det \begin{bmatrix} k^2 & b & b \\ b & k^3 & b \\ b & b & k^4 \end{bmatrix}$$

$$- \frac{b}{D} \left\{ \det \begin{bmatrix} bS^1 & b & b \\ bS^1 & k^3 & b \\ bS^1 & b & k^4 \end{bmatrix} + \det \begin{bmatrix} k^2 & bS^1 & b \\ b & bS^1 & b \\ b & bS^1 & k^4 \end{bmatrix} + \det \begin{bmatrix} k^2 & b & bS^1 \\ b & k^3 & bS^1 \\ b & b & bS^1 \end{bmatrix} \right\},$$

since the last six terms cancel each other pairwise,

for $D = \det \begin{bmatrix} k^1 & b & b & b \\ b & k^2 & b & b \\ b & b & k^3 & b \\ b & b & b & k^4 \end{bmatrix}$, leaving the solution for q^1 simply in terms of S^1. ■

The induction step is the observation that the three terms on the right hand side in the last three pairs of terms in the penultimate large expression are each the essentials for a solution for q^i in a 3×3 system. Hence we have established that if the result is true for a 2×2 sytem, and for a $(n-1) \times (n-1)$ system, it is true for an $n \times n$ system. (In fact we established it true for a 2×2 sytem, and then made use of its validity for a 3×3 system in establishing the result for a 4×4 system. We simply illustrated the key step in a complete induction proof.) We have established that for any finite number of firms, each firm's current quantity extracted can be expressed as a function of its own current stock alone. This is the key step in characterizing the closed loop solution. (Given our calculations it is obvious that the open loop solution is the same.) Besides drawing on the quadratic nature of revenue and extraction cost for each firm, the key property in inferring that the open loop and closed loop solutions are the same is that in the solutions, each firm ends up extracting over the same number of periods as each of its competitors. This requires that each firm's endowment of stock must be "right" in order that our demonstration of the sameness of the open and closed loop solutions is valid. Hence sufficient conditions for the open loop and closed loop solutions to be the same are (a) quadratic forms for revenue and extraction cost per firm and (b) "appropriate" endowments of stock for each firm at the initial date.

In general, the systems to solve in terms of periods in the backward recursion for the three firm case are in the form

$$\begin{aligned}
& mr^1_{T-t}(q^1_{T-t}, q^2_{T-t}, q^3_{T-t}) - mc^1_{T-t}(q^1_{T-t}) \\
= \; & \beta^{t-1}[mr^1_{T-1}(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}, S^3_{T-2} - q^3_{T-2}) - mc^1_{T-2}(S^1_{T-2} - q^1_{T-2})] \\
& mr^2_{T-t}(q^1_{T-t}, q^2_{T-t}, q^3_{T-t}) - mc^2_{T-t}(q^2_{T-t}) \\
= \; & \beta^{t-1}[mr^2_{T-1}(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}, S^3_{T-2} - q^3_{T-2}) - mc^2_{T-2}(S^2_{T-2} - q^2_{T-2})] \\
& mr^3_{T-t}(q^1_{T-t}, q^2_{T-t}, q^3_{T-t}) - mc^3_{T-t}(q^3_{T-t}) \\
= \; & \beta^{t-1}[mr^3_{T-1}(S^1_{T-2} - q^1_{T-2}, S^2_{T-2} - q^2_{T-2}, S^3_{T-2} - q^3_{T-2}) - mc^3_{T-2}(S^3_{T-2} - q^3_{T-2})].
\end{aligned}$$

The non-mechanical step is substituting for $(S^i_{T-2} - q^i_{T-2})$ each time one moves backwards in the recursion. Crucial here is the fact that the matrix algebra is essentially the same for each period in the sequence. Hence our induction proof sketched above holds for any date for n firms. For the 3 firm case, we have for $t = 3$, the system

$$\begin{bmatrix} a \\ a \\ a \end{bmatrix} - \begin{bmatrix} [2b+2d^1] & b & b \\ b & [2b+2d^2] & b \\ b & b & [2b+2d^3] \end{bmatrix} \begin{bmatrix} q^1_{T-3} \\ q^2_{T-3} \\ q^3_{T-3} \end{bmatrix}$$

$$= \beta^2 \begin{bmatrix} a - \frac{[2b+2d^1]}{1+\beta}[S^1_{T-3} - q^1_{T-3} - \widehat{q}^1_{T-1}] - \frac{b}{1+\beta}[S^2_{T-3} - q^2_{T-3} - \widehat{q}^2_{T-1}] \\ -\frac{b}{1+\beta}[S^3_{T-3} - q^3_{T-3} - \widehat{q}^3_{T-1}] \\ a - \frac{b}{1+\beta}[S^1_{T-3} - q^1_{T-3} - \widehat{q}^1_{T-1}] - \frac{[2b+2d^2]}{1+\beta}[S^2_{T-3} - q^2_{T-3} - \widehat{q}^2_{T-1}] \\ -\frac{b}{1+\beta}[S^3_{T-3} - q^3_{T-3} - \widehat{q}^3_{T-1}] \\ a - \frac{b}{1+\beta}[S^1_{T-3} - q^1_{T-3} - \widehat{q}^1_{T-1}] - \frac{b}{1+\beta}[S^2_{T-3} - q^2_{T-3} - \widehat{q}^2_{T-1}] \\ -\frac{[2b+2d^3]}{1+\beta}[S^3_{T-3} - q^3_{T-3} - \widehat{q}^3_{T-1}] \end{bmatrix}$$

leading to solutions

$$
\begin{aligned}
q^1_{T-3} &= \left[\frac{(1+\beta)}{1+\beta+\beta^2}\right]\widehat{q}^1_{T-2} + \left[\frac{\beta^2}{1+\beta+\beta^2}\right][S^1_{T-2} - \widehat{q}^1_{T-1}] \\
q^2_{T-3} &= \left[\frac{(1+\beta)}{1+\beta+\beta^2}\right]\widehat{q}^2_{T-2} + \left[\frac{\beta^2}{1+\beta+\beta^2}\right][S^2_{T-2} - \widehat{q}^2_{T-1}] \\
q^3_{T-3} &= \left[\frac{(1+\beta)}{1+\beta+\beta^2}\right]\widehat{q}^3_{T-2} + \left[\frac{\beta^2}{1+\beta+\beta^2}\right][S^3_{T-2} - \widehat{q}^3_{T-1}]
\end{aligned}
$$

For $t = 4$, we have

$$
\begin{bmatrix} a \\ a \\ a \end{bmatrix} - \begin{bmatrix} [2b+2d^1] & b & b \\ b & [2b+2d^2] & b \\ b & b & [2b+2d^3] \end{bmatrix} \begin{bmatrix} q^1_{T-4} \\ q^2_{T-4} \\ q^3_{T-4} \end{bmatrix}
$$

$$
= \beta^3 \begin{bmatrix}
a - \frac{[2b+2d^1]}{1+\beta+\beta^2}[S^1_{T-4} - q^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}] - \frac{b}{1+\beta+\beta^2}[S^2_{T-4} - q^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}] \\
- \frac{b}{1+\beta+\beta^2}[S^3_{T-4} - q^3_{T-4} - \widehat{q}^3_{T-1} - \widehat{q}^3_{T-2}] \\
a - \frac{b}{1+\beta+\beta^2}[S^1_{T-4} - q^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}] - \frac{[2b+2d^2]}{1+\beta+\beta^2}[S^2_{T-4} - q^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}] \\
- \frac{b}{1+\beta+\beta^2}[S^3_{T-4} - q^3_{T-4} - \widehat{q}^3_{T-1} - \widehat{q}^3_{T-2}] \\
a - \frac{b}{1+\beta+\beta^2}[S^1_{T-4} - q^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}] - \frac{b}{1+\beta+\beta^2}[S^2_{T-4} - q^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}] \\
- \frac{[2b+2d^3]}{1+\beta+\beta^2}[S^3_{T-4} - q^3_{T-4} - \widehat{q}^3_{T-1} - \widehat{q}^3_{T-2}]
\end{bmatrix}
$$

leading to solutions

$$
\begin{aligned}
q^1_{T-4} &= \left[\frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\right]\widehat{q}^1_{T-3} + \left[\frac{\beta^3}{1+\beta+\beta^2+\beta^3}\right][S^1_{T-4} - \widehat{q}^1_{T-1} - \widehat{q}^1_{T-2}] \\
q^2_{T-4} &= \left[\frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\right]\widehat{q}^2_{T-3} + \left[\frac{\beta^3}{1+\beta+\beta^2+\beta^3}\right][S^2_{T-4} - \widehat{q}^2_{T-1} - \widehat{q}^2_{T-2}] \\
q^3_{T-4} &= \left[\frac{1+\beta+\beta^2}{1+\beta+\beta^2+\beta^3}\right]\widehat{q}^3_{T-3} + \left[\frac{\beta^3}{1+\beta+\beta^2+\beta^3}\right][S^3_{T-4} - \widehat{q}^3_{T-1} - \widehat{q}^3_{T-2}]
\end{aligned}
$$

and so on for additional "terms" in the backward recursion. It is easy to see how the system of equations changes with each step backwards. The central result is of course that the expressions for the solved $q's$ end up as linear functions of each own stock alone for each step back in the recursion. For arbitrary date $T-t$, an induction proof would establish the validity of the "general term", given say M instead of 3 firms.

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