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CONTINGENT CLAIMS VALUED AND HEDGED BY PRICING AND INVESTING IN A BASIS

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CONTINGENT CLAIMS VALUED AND HEDGED BY PRICING AND INVESTING IN A BASIS

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Contingent claims with payoffs depending on finitely many asset prices are modeled as elements of a separable Hilbert space. Under fairly general conditions, including market completeness, it is shown that one may change measure to a reference measure under which asset prices are Gaussian and for which the family of Hermite polynomials serves as an orthonormal basis. Basis pricing synthesizes claim valuation and basis investment provides static hedging opportunities. For claims written as functions of a single asset price we infer from observed option prices the implicit prices of basis elements and use these to construct the implied equivalent martingale measure density with respect to the reference measure, which in this case is the Black–Scholes geometric Brownian motion model. Data on S & P 500 options from the *Wall Street Journal* are used to illustrate the calculations involved. On this illustrative data set the equivalent martingale measure deviates from the Black–Scholes model by relatively discounting the larger price movements with a compensating premia placed on the smaller movements.

KEY WORDS: European option pricing, Hermite polynomials, Hilbert space, martingale measures, S & P 500 index

1. INTRODUCTION

Many contingent claims have payoffs that depend on finitely many asset prices at finitely many dates. Further these claims define finite mean and variance random variables. They can therefore be viewed as elements of a separable Hilbert space that has a countable orthonormal basis. One could therefore effectively hedge all these claims by statically investing in the basis elements if the latter were traded. The basis therefore provides us with a set of claims that are statically market completing. Even though there may be no direct trading in the basis elements, it is possible that traded claims provide avenues for indirect investment in the basis for at least some practically important subspaces of claims and thereby constitute for these subspaces a static market completing collection of claims.[2]

One may also think of the basis elements as analogous to factors in asset pricing. Typically, one visualizes different variables as factors and we do not consider powers or nonlinear functions of factors. But then the space of claims is also restricted to portfolios that are linear combinations of asset returns. The space of claims considered here includes nonlinear transforms of asset returns and hence the factors involve similar transformations.

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[2] Storing information on the indirect basis investment of a large portfolio of derivative claims can considerably expedite the assessment of one's risk exposure.

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Recently, Bansal and Viswanathan (1993) employ such transforms of factors in an empirical analysis of asset pricing in a context inclusive of both primary assets and options on these.

The role of a Hilbert space basis in the analysis of contingent claims can also usefully be compared to that of the role of pure discount bonds in the analysis of fixed income securities or the role of the paths of a tree for claims written on the nodes of a finite tree as, for example, in the binomial option pricing model. The pure discount bonds or the paths of the tree constitute the basis in these two cases. Every potential claim is a predetermined linear combination of the basis elements, and the valuation problem is completed on pricing the basis. For fixed incomes the basis prices are given by the yield curve, and for claims on the tree the basis prices are given by the yield curve and the risk neutral or equivalent martingale measure. This equivalent martingale measure is a set of security prices and is precisely the futures price of event contingent pure discount bonds that pay a unit face value at maturity only if a particular path is realized. We therefore refer to this measure as the futures price law.[3]

Valuation is completed on determining the yield curve and the futures price law. The technique of extracting the yield curve and the futures price law from the prices of quoted bonds and bond options is now fairly common. For an example involving interest rate contingent claims see Hull and White (1990). This methodology is essentially extended here to the general environment of continuous state spaces. Early attempts at such an extension include Banz and Miller (1978) and Breeden and Litzenberger (1978), who proceeded by discretizing the continuum. Such a discretization approach is likely to become cumbersome in higher dimensions. This paper cuts through to a much neater basis representation that is useful in higher dimensions as well. This new basis representation of general claims and the futures price law comes at the cost of requiring some economic relevance for finite and small second moments.

A Hilbert space basis in general is difficult to construct because it requires an intimate knowledge of the stochastic process of the underlying asset prices. However, under fairly general conditions, including dynamically complete primary asset markets, we show that one can change measure initially to a Gaussian reference measure. In this regard we borrow from the literature on the reference probability approach to filtering theory introduced by Zakai (1969) and further studied by Elliott (1993). For the Gaussian reference measure a basis is provided by the family of Hermite polynomials.[4] The economic and technical assumptions necessary for pricing and hedging claims using such a Hilbert space basis are described in detail. In addition to completeness we require the absence of free lunches in the sense of Stricker (1990). We also require square integrability of the true probability measure with respect to the Gaussian reference measure and Markov price processes. Our use of options as statically completing markets via indirect basis investment extends the approach of earlier literature reviewed in John (1981, 1984) and Amershi (1985). A similar approach was also taken by Jones (1984) to hedge jump discontinuities in asset prices.

[3] We apologize for introducing yet another terminology for this measure. The existing terminology of risk-neutral or equivalent martingale measure, however, stresses the preference and mathematical aspects of the measure. The terminology of state prices provides us with a discounted measure. Dothan (1990) termed this measure the equilibrium price measure, and it is an undiscounted or futures price that is involved. The adjective equilibrium may be dropped as implicitly understood.

[4] Recently Longstaff (1990) employs polynomials to span claims contingent on a single asset price. Implicitly, the reference measure employed in Longstaff (1990) is the negative exponential, and the relevant orthogonal polynomials are the Laguerre polynomials. Longstaff (1990) does not exploit this orthogonality in polynomial representations of claims. Furthermore, the class of stochastic processes for which the negative exponential is an admissible reference measure is unclear.

Relying on the static completeness of European call and put options for the subspace of claims with payoffs written as a function of a single price, as observed in Breeden and Litzenberger (1978) and Bick (1982) and demonstrated by Green and Jarrow (1987) and Nachman (1988), we infer empirically the prices of basis elements for this subspace. From these basis elements one may infer empirically the futures price law and its density with respect to the reference measure. Previous attempts at this include Banz and Miller (1978), Breeden and Litzenberger (1978), and Jarrow (1986). More recently, Longstaff (1992) used option price data to construct the futures price law as a histogram. An illustrative application of our general theory to S & P 500 options reveals that for the dates studied the empirical futures price law discounted larger price movements and placed a premia on smaller price movements relative to the Black–Scholes lognormal reference measure.

Section 1 presents the general Hilbert space theory of the basis risks. The reference measure space is introduced in Section 2 for which Section 3 presents the explicit basis. Applications to the subspace of claims with payoffs depending on a single price are presented in Section 4. Section 5 concludes.

2. THE BASIS RISKS

We suppose that all processes are defined on the fixed probability space $(\Omega, \mathcal{F}, P)$ for time $t \in [0, T]$. Suppose that $\{\mathcal{F}_t\}$ is a right-continuous filtration of sub-σ-fields of $\mathcal{F}$ with each $\mathcal{F}_t$ containing all the null sets of $\mathcal{F}$. We restrict attention to the securities market model of Jarrow and Madan (1991).

ASSUMPTION 2.1. *We suppose that the filtration $\mathcal{F}_t$ and the probability space $(\Omega, \mathcal{F}, P)$ are generated by a d-dimensional standard Brownian motion $w = [w(t), t \in [0, T]]$ initialized at zero.*

Consider an economy that trades in a finite set of d primary assets continuously over time. Let $S = [S(t), t \in [0, T]]$, where $S(t) = (S^1(t), S^2(t), \ldots, S^d(t))$, be the strictly positive d-dimensional stochastic process of the prices of the primary assets. Suppose there is also available a money market account accumulating at instantaneous interest rates given by the process $r = [r(t), t \in [0, T]]$ with associated accumulation factor $B(t) = \exp \int_0^t r(u)\, du)$. We suppose that r is also a positive process.

The focus of this paper is on valuing and hedging a wide class of contingent claims written as functions of the primary asset prices at various points of time. We restrict attention to a fixed finite set τ^* of times relevant for describing the contingent claims of interest. Let $\tau^* = \{t_0, t_1, \ldots, t_M\}$, where $0 = t_0 < t_1 < \cdots < t_M = T$. For example, the set τ^* could include the times at which trading closes on each day and payments may be viewed as coming due at these times, though the value of the payment may depend on intraday prices. In general, a typical contingent claim consists of an M-dimensional vector $\Psi = (\psi^1, \ldots, \psi^M)$ of functions. Each ψ^j is a function of the jd variables $S^i(t_k)$ for $k = 1, \ldots, j$ and $i = 1, \ldots, d$ and specifies the rule for payment at time t_j on the contract for $j = 1, \ldots, M$. We denote by $\mathcal{C}$ the set of all such contingent claims. This is a fairly large set that includes all European options as well as options with exercise prices that are determined by the prices of other assets at maturity or the same asset at a prespecified time before maturity or both. Many contracts likely to be written in practice fall into this category, including Asian options, the large and growing market for asset swaps, and options with automatic reset features for the exercise price.

DEFINITION 2.1. For each contingent claim $\Psi \in \mathscr{C}$ define the *terminal payoff* $\Pi(\Psi)$ at time T by accumulating earlier payoffs at the money market account, specifically,

$$\Pi(\Psi) = \sum_{j=0}^{M} \frac{B(T)}{B(t_j)} \psi_j.$$

The space of contingent claims that we wish to value and hedge is a subset of the terminal payoffs associated with the elements of $\mathscr{C}$. We restrict attention here to claims whose final payoffs have finite means and variances.

DEFINITION 2.2. Let $\mathscr{H}$ be the Hilbert space of all contingent claims $\Psi \in \mathscr{C}$ such that $\Pi(\Psi) \in L^2[(\Omega,\mathscr{F},P)]$ with inner product

$$\langle \Psi, \Psi' \rangle = E^P[\Pi(\Psi)\Pi(\Psi')].$$

We consider the pricing and hedging of claims in $\mathscr{H}$ by arbitrage in a complete markets environment. The traditional approach to hedging and pricing these claims uses self-financing dynamic trading strategies that continuously rebalance portfolios of the primary assets and the money market account to replicate these claims. In the absence of arbitrage opportunities, the market values of the claims are then given by the initial investment in the associated dynamic hedge.[5] The markets accomplishing the hedge are the infinitely many spot markets in the primary assets and the money market account available over the time continuum. With completeness and continuous readjustment of portfolios this is a rich enough collection of markets that captures all of $\mathscr{H}$. We focus here on static hedging strategies that require no rebalancing and invest instead in a suitable, though infinite, spanning subset of $\mathscr{H}$ immediately.

ASSUMPTION 2.2. *Suppose the primary asset price process S is a d-dimensional semimartingale satisfying*

$$S(t,\omega) = S(0) + \int_0^t \hat{S}(u,\omega)\mu(u,\omega)\, du + \int_0^t \hat{S}(u,\omega)\sigma(u,\omega)\, dw(u)$$

for all $(t,\omega) \in [0, T] \times \Omega$, where $\hat{S}(u,\omega)$ is the $d \times d$ diagonal matrix with $S(u,\omega)$ on the diagonal, σ is an adapted nonsingular matrix valued process for which, for all $i = 1, \ldots, d$ and $j = 1, \ldots, d$,

$$\int_0^T \sigma_{ij}^2(u,\omega)\, du < +\infty \quad \text{a.e. } P,$$

and for all $i = 1, \ldots,$ d

$$E\left[\int_0^T \left|\mu_i(u,\omega)\right| du\right] < \infty.$$

[5] For further details on this approach to valuing and hedging contingent claims see Harrison and Pliska (1981), Heath, Jarrow, and Morton (1992), and Jarrow and Madan (1991).

We suppose the absence of arbitrage opportunities and, in particular, the existence of an equivalent martingale measure or futures price law. The latter is a stronger condition than just the absence of arbitrage opportunities as demonstrated in Back and Pliska (1991). Specifically one requires that limits of certain classes of trading strategies taken in an appropriate topology not result in a nonnegative and nonzero cash flow. Such hypotheses are termed the absence of free lunches. For further details see Stricker (1990), Delbaen (1992), and Lakner (1993).

ASSUMPTION 2.3. *There exists a probability measure Q on $(\Omega,\mathcal{F},P)$ equivalent to P such that under Q the processes $BS^{-1} = [B(t)^{-1}S(t), t \in [0, T]]$ are martingales.*

Assumptions 2.2 and 2.3 taken together imply that markets are dynamically complete with respect to the primary assets (Jarrow and Madan 1991, Chatelain and Stricker 1994). The claims in $\mathcal{H}$ are therefore dynamically redundant. This does not preclude their static usefulness, and the Hilbert space basis identifies a minimal statically spanning collection of claims. This basis synthesizes the futures price law, claim valuation, and hedging. In the context of dynamically complete markets, one may define the arbitrage-free value of any claim in $\mathcal{H}$.

Assumption 2.2 also rules out jump discontinuities in the sample paths of asset prices, and this is an important qualification to the specific methods proposed here. The general Hilbert space methodology could be extended to include jump processes, but the reduction to a Gaussian reference measure will probably not be available in this more general context. It is this reduction to a Gaussian reference measure that essentially motivates the use of Assumption 2.2.

DEFINITION 2.3. For each $\Psi \in \mathcal{H}$ define the *arbitrage-free value* $V(\Psi)$ of Ψ by

$$V(\Psi) = E^Q[B(T)^{-1}\Pi(\Psi)].$$

The equivalent martingale measure or the futures price law has by definition a density $\Lambda = dQ/dP$ that is integrable with respect to P, i.e., in $L^1[(\Omega,\mathcal{F},P)]$. We make the stronger assumption of square integrability of this density. This hypothesis is equivalent to continuity of the pricing operator $V(\Psi)$ in the topology induced by the Hilbert space norm on $\mathcal{H}$. For an associated formulation of the required no-free-lunch hypothesis, see Stricker (1990). We shall comment later on the nature of this hypothesis and the associated hedging strategies.

ASSUMPTION 2.4. *The density Λ of the futures price law with respect to P is in $L^2[(\Omega,\mathcal{F},P)]$.*

Under Assumption 2.4 one may value all the elements of $\mathcal{H}$ by pricing a Hilbert space basis for $\mathcal{H}$, and one may hedge all the elements of $\mathcal{H}$ by investing statically in the basis elements. We first present the general structure of such valuation and hedging strategies.

DEFINITION 2.4. A set $\mathcal{B} = \{\Psi_\alpha \mid \alpha \in A\}$ is called an *orthonormal basis* for the Hilbert space $\mathcal{H}$ if $\mathcal{B}$ is an orthonormal set (i.e., $\langle\Psi_\alpha,\Psi_\alpha\rangle = 1$ for all $\alpha \in A$ and $\langle\Psi_\alpha,\Psi_{\alpha'}\rangle = 0$ for all $\alpha \neq \alpha'$ in A) and if, for all Ψ in $\mathcal{H}$,

$$\Psi = \sum_{\alpha\in A} a_\alpha\Psi_\alpha,$$

where $a_\alpha = \langle \Psi, \Psi_\alpha \rangle$ is nonzero for only countably many terms in the sum and the equality represents convergence in the Hilbert space norm of the finite partial sums.

THEOREM 2.1. *Every Hilbert space contains an orthonormal basis for itself.*

Proof. See Dunford and Schwartz (1988, Theorem 12, p. 252). □

THEOREM 2.2. *Assumptions 2.1–2.4 imply that the valuation operator V satisfies*

$$V(\Psi) = \sum_{\alpha \in A} a_\alpha(\Psi) V(\Psi_\alpha).$$

Proof. Let $\Psi \in \mathcal{H}$ and let α_n enumerate the countable subset of A for which $\langle \Psi, \Psi_\alpha \rangle \neq 0$. By definition of V we have

$$V(\Psi) = E^Q[B(T)^{-1}\Pi(\Psi)] = E^P[\Lambda B(T)^{-1}\Pi(\Psi)].$$

On the other hand,

$$\begin{aligned}\sum_{k=1}^{n} a_{\alpha_k} V(\Psi_{\alpha_k}) &= \sum_{k=1}^{n} a_{\alpha_k} E^Q[B(T)^{-1}\Pi(\Psi_{\alpha_k})] \\ &= \sum_{k=1}^{n} a_{\alpha_k} E^P[\Lambda B(T)^{-1}\Pi(\Psi_{\alpha_k})] \\ &= E^P[\Lambda B(T)^{-1} \sum_{k=1}^{n} a_{\alpha_k}\Pi(\Psi_{\alpha_k})],\end{aligned}$$

and so

$$V(\Psi) - \sum_{k=1}^{n} a_{\alpha_k} V(\Psi_{\alpha_k}) = E^P[\Lambda B(T)^{-1}(\Pi(\Psi) - \sum_{k=1}^{n} a_{\alpha_k}\Pi(\Psi_{\alpha_k}))].$$

It follows from Assumption 2.4 and the boundedness of $B(T)^{-1}$ that $\Lambda B(T)^{-1} \in L^2[(\Omega, \mathcal{F}, P)]$. Applying the Cauchy–Schwarz inequality we have that

$$\left| V(\Psi) - \sum_{k=1}^{n} a_{\alpha_k} V(\Psi_{\alpha_k}) \right| \leq \|\Lambda B(T)^{-1}\| \left\| \Pi(\Psi) - \sum_{k=1}^{n} a_{\alpha_k}\Pi(\Psi_{\alpha_k}) \right\|.$$

The result follows as the right-hand side converges to zero as n tends to infinity by definition of the Hilbert space basis. □

THEOREM 2.3. *The futures price law density Λ is*

$$\Lambda = \sum_{\alpha \in A} V(\Psi_\alpha)\Psi_\alpha.$$

Proof. The futures price for T delivery of any claim Ψ coming due at T is

$$V(\Psi) = E^Q[\Psi] = E^P[\Lambda\Psi] = \langle\Lambda,\Psi\rangle.$$

Define

$$\Gamma = \sum_{\alpha\in A} V(\Psi_\alpha)\Psi_\alpha.$$

By Theorem 2.2 we have $V(\Psi) = \langle\Psi, \Gamma\rangle$ and hence Λ must equal Γ. □

By Theorem 2.2, once one has found and priced a basis, all other claims may be priced in terms of the prices of the basis elements. A useful analogy may be made between the role of basis elements in pricing $\mathcal{H}$ and the role of zero-coupon bonds in the pricing of fixed income securities or the role of the paths of the tree in the binomial option pricing model. They both form a basis for the respective class of securities. Typical valuation exercises involve backing out the prices of the pure discount bonds and the paths of the tree from quoted bond and option prices. This identifies the yield curve and the futures price law, which may then be used to value all other claims. A similar strategy will be invoked here to back out the prices for the Hilbert space basis. Theorem 2.3 shows how one may recover completely the futures price law from these basis prices.

Current practice treats observed market prices as free of error and backs out basis prices. More generally one should permit observed prices to be noisy and the extraction of basis prices is then a nonlinear filtering problem. We leave this filtering problem for future research and follow here the current practice of directly backing out basis prices.

One can easily visualize how quoted bond prices encode pure discount bond prices and option prices encode state prices or discounted futures prices and why one may therefore back out the yield curve and prices of the paths of the tree. In contrast, the encoding here is in a Hilbert space sense in that the difference between the claim and, for all practical purposes, the approximating basis representation is an arbitrarily small mean and variance random variable. This representation is obtained at the cost of requiring that such low-second-moment random variables are of little economic significance to investors. This is precisely the economic content of the assumption of square integrability of the futures price law density and delivers in the process Theorems 2.2 and 2.3 and the relevance of the futures price law density expressed in terms of the basis with basis prices as coefficients.

In general, the basis for an arbitrary Hilbert space can be very large, and there is no operational advantage in describing or pricing the basis. Under certain conditions, described below, a countable basis can be constructed and effectively employed to facilitate pricing and hedging strategies. One need merely store the basis representation of claims and the claim representation of basis elements to execute simultaneously the valuation and hedging strategy. The hedging may be done statically or dynamically.

The static hedging strategy using basis investment requires that one purchase $\langle\Psi, \Psi_\alpha\rangle$ units of Ψ_α. Only countably many of these requirements are nonzero, and one may approximate the hedge by investing in a suitable partial sum of assets. An intuition of the relevant basis components can be obtained by comparing the complexity of the claim payoffs with that of the basis elements. For example, when we later introduce polynomial basis elements it is important to evaluate the degree of nonlinearity or oscillation in the claim payoff to assess whether one is going to need many higher-order terms for an adequate basis representation of the claim. As stated earlier, it is important to note that the approximate hedge is a Hilbert space approximation in that the difference between the hedge cash flow and the cash flow that is to be hedged is a random variable with an arbitrarily small mean and variance. In particular, the hedge cash flow may leave one exceedingly short in contingencies that have appropriately small probabilities so as not to disturb the second moment. In the terminology of financial hedging, there is present in these hedges "basis" risk or positive deviation between the hedge and the hedged. The quality of the hedge is therefore dependent on the validity of the underlying probability model.

For the hedge to be reasonable and less dependent on the validity of the underlying probability model we need the convergence of hedge cash flows to the hedged cash flow in the L^∞ or maximum absolute deviation sense. The associated no-free-lunch hypothesis is in the sense of Delbaen (1992) with an associated L^1 futures price law density.[6] Unfortunately the L^∞ space does not have a basis, and for this stronger sense of hedging the synthesizing methods of this paper fail. The square integrability of the futures price law is therefore an important economic restriction that is implicit in interpreting the derived basis prices and futures price laws.

For the purposes of analyzing the structure of $\mathcal{H}$ we may restrict attention to the embedded discrete time stochastic processes $\rho_j = B(t_j)$ and $P^i_j = S^i(t_j)$ for $j = 0, 1, \ldots, M$ and $i = 1, \ldots, d$. The final payoffs are then functions of the finitely many entities $\rho_j^{-1} P^i_j$, and the resulting Hilbert space is separable (Dunford and Schwartz 1988, p. 169, Exercise 6). It follows (Royden 1968, Proposition 27, p. 212) that $\mathcal{H}$ has a countable basis. Pricing and hedging can then be effectively synthesized in terms of this countable basis. In the next section we proceed to construct such a basis for the general securities model introduced here.

3. THE REFERENCE MEASURE SPACE

A basis for a separable Hilbert space can in general be constructed from any countable maximal linearly independent set in the space by using the procedures of Gram–Schmidt orthogonalization (Hochstadt 1973). Apart from the choice of the initial dense set, the orthogonalization depends critically on the probability measure P that defines the inner product for $\mathcal{H}$. An effective basis cannot be constructed without a detailed knowledge of the measure P. The approach we follow is suggested by the reference probability method used first in the theory of nonlinear filtering by Zakai (1969) and more recently by Elliott (1993). We shall in fact change measure to a Gaussian reference measure $\overline{\mathrm{P}}$ in a discrete time context, and for this purpose we follow the approach of Elliott (1993) closely.

For the study of $\mathcal{H}$, as noted before, we may restrict attention to the discrete time discounted price processes $X^i_j = \rho_j^{-1} P^i_j$. The securities market model implies that

$$X^i_j = X^i_{j-1} + \int_{t_{j-1}}^{t_j} S^i(u, \omega)(\mu(u,\omega) - r(u,\omega))\, du + \int_{t_{j-1}}^{t_j} S^i(u,\omega) \sum_{k=1}^{d} \sigma_{ik}(u,\omega)\, dw_k(u).$$

[6]We are indebted to David Heath for alerting us to these qualifications on the nature of the related hedging strategies.

This general model is restricted further in this section to that of a Markov process. This may be done by supposing that the drift and diffusion coefficients as well as the interest rate depend just on the vector of current asset prices. One can, however, be a little more general and allow for an exogenous vector Markov process of state variables as well.

ASSUMPTION 3.1. *There exists an m-dimensional Markov process $\xi = [\xi(t), t \in [0, T]]$ satisfying*

$$\xi(t) = \xi(0) + \int_0^t g(u, \xi(u))\, du + \int_0^t \theta(u, \xi(u))\, dB(u),$$

where g is a K-dimensional vector-valued function, θ is an $m \times q$ matrix-valued function on $[0, T] \times \mathbb{R}^m$, and B is q-dimensional standard Brownian motion possibly correlated with w. Furthermore the coefficients μ and σ satisfy

$$\mu(u,\omega) = \mu(u, S(u), \xi(u))$$

and

$$\sigma(u,\omega) = \sigma(u, S(u), \xi(u)).$$

Let $y_j = (\ln P_j^1, \ldots, \ln P_j^d)$, $x_j = \xi(t_j)$ for $j = 1, \ldots, M$ and consider a discrete time approximation to the continuous time process $[\ln S, \xi]$ constructed for the time points τ^*. For example, one could use a higher-order approximation as constructed by Mihlshtein (1974). This resulting approximation provides functions F_j: $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and H_j: $\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$x_j - x_{j-1} = F_j(x_{j-1}, \sqrt{t_j - t_{j-1}}\, w_j)$$

and

$$y_j - y_{j-1} = H_j(y_{j-1}, x_{j-1}, \sqrt{t_j - t_{j-1}}\, b_j),$$

where w_j, b_j are sequences of random variables, independent across j, that are distributed as multivariate normals with mean 0 and variance–covariance matrices of I_m and I_d respectively. Let the density of w_j be $\psi(w)$ and that of b_j be $\phi(b)$.

We suppose that the nonlinear functions H_j may be inverted to recover b_j from the returns $y_j - y_{j-1}$. This is a complete markets hypothesis for the discrete time model.

ASSUMPTION 3.2. *There exists a function G: $\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$, such that $b_j = G_j(y_{j-1}, x_{j-1}, y_j - y_{j-1})$ when $y_j - y_{j-1} = H_j(y_{j-1}, x_{j-1}, b_j)$. Moreover, the $d \times d$ matrices $\partial H_j/\partial b_j$ and $\partial G_j/\partial(y_j - y_{j-1})$ are nonsingular.*

Let $\mathcal{G}_j = \sigma\{y_0, \ldots, y_{j-1}, x_0, \ldots, x_{j-1}\}$ be the σ-field generated by the past in the discrete time model. Denote by $\tilde{P}$ the probability measure of the discrete time model with $\tilde{\Omega}$ the corresponding event space. The continuous time model $(\Omega, \mathcal{F}, P)$ is then approximated by $(\tilde{\Omega},\mathcal{G},\tilde{P})$. We now construct a change of measure from $\tilde{P}$ to P_0 on $\mathcal{G}$ in the following way.

Define the process γ_j by

$$\gamma_j = \frac{\phi(y_j - y_{j-1})}{\phi(b_j)} \left[\left| \det \frac{\partial G_j(y_{j-1}, x_{j-1}, y_j - y_{j-1})}{\partial(y_j - y_{j-1})} \right| \right]^{-1}$$

and the process Λ_j by

$$\Lambda_j = \prod_{i=1}^{j} \gamma_i.$$

The new probability measure P is defined by setting the Radon–Nikodym derivative $dP_0/d\tilde{P}$ equal to Λ_j.

THEOREM 3.1. *Under P the random variables $y_j - y_{j-1}$ are independent multivariate normal with zero mean and covariance matrix I_d.*

Proof. Consider the $\mathcal{G}_j$ conditional probability distribution function for the returns $y_j - y_{j-1}$ under P. This is given by

$$\begin{aligned} P(y_j - y_{j-1} \leq z \mid \mathcal{G}_j) &= E[I(y_j - y_{j-1} \leq z) \mid \mathcal{G}_j] \\ &= \frac{E[\Lambda_j I(y_j - y_{j-1} \leq z) \mid \mathcal{G}_j]}{E[\Lambda_j \mid \mathcal{G}_j]} \\ &= \frac{E[\gamma_j I(y_j - y_{j-1} \leq z) \mid \mathcal{G}_j]}{E[\gamma_j \mid \mathcal{G}_j]}. \end{aligned}$$

Substituting for γ_j we obtain

$$E[\gamma_j \mid \mathcal{G}_j] = \int_{\mathbb{R}^d} \frac{\phi(y_j - y_{j-1})}{\phi(b_j)} \left[\left| \det \frac{\partial G_j(y_{j-1}, x_{j-1}, y_j - y_{j-1})}{\partial(y_j - y_{j-1})} \right| \right]^{-1} \phi(b_j)\, db_j.$$

Since

$$db_j = \frac{\partial G_j(y_{j-1}, x_{j-1}, y_j - y_{j-1})}{\partial(y_j - y_{j-1})}\, d(y_j - y_{j-1}),$$

it follows that

$$E[\gamma_j \mid \mathcal{G}_j] = \int_{\mathbb{R}^d} \phi(y_j - y_{j-1})\, d(y_j - y_{j-1}) = 1$$

and

$$P(y_j - y_{j-1} \leq z \mid \mathcal{G}_j) = E[\gamma_j I(y_j - y_{j-1} \leq z) \mid \mathcal{G}_j]$$
$$= \int_{\mathbb{R}^d} \frac{\phi(y_j - y_{j-1})}{\phi(b_j)} \cdot \left| \det \frac{\partial G_j(y_{j-1}, x_{j-1}, y_j - y_{j-1})}{\partial(y_j - y_{j-1})} \right|^{-1}$$
$$I(y_j - y_{j-1} \leq z)\phi(b_j)\, db_j$$
$$= \int_{\mathbb{R}^d} \phi(y_j - y_{j-1}) I(y_j - y_{j-1} \leq z)\, d(y_j - y_{j-1}).$$

□

Hence the asset log returns are independently identically distributed as multivariate normals with zero means, unit variances, and zero covariances under P. However, the Hilbert space with respect to the reference measure P and the approximation to the true measure $\tilde{P}$ are not necessarily the same. For this to be the case we require an assumption on the boundedness of the density Λ_M.

ASSUMPTION 3.3. *The density $dP/d\tilde{P} = \Lambda_M$ is uniformly bounded above and below by Δ and δ respectively.*

Under Assumptions 3.1–3.3 the Hilbert space $\mathcal{K} = L^2[(\tilde{\Omega},\mathcal{G},P)]$ is the same as the space $L^2[(\tilde{\Omega},\mathcal{G},\tilde{P})]$, and the latter is an approximation for $L^2[(\Omega,\mathcal{F},P)] = \mathcal{H}$. Furthermore, the Hilbert space $\mathcal{K}$ is that of a Gaussian random process by construction. The Hilbert space $\mathcal{K}$ has a well-known basis given by the Hermite polynomials that we shall use to price all of $\mathcal{H}$. We therefore begin with the space $\mathcal{K}$, and the first step is to construct $\tilde{P}$ starting from P by proceeding in an inverse manner.

Define

$$\gamma_j = \frac{\phi(b_j)}{\phi(y_j - y_{j-1})} \cdot \left| \det \frac{\partial H_j(y_{j-1}, x_{j-1}, b_j)}{\partial b_j} \right|^{-1}$$

and

$$\bar{\Lambda}_j = \prod_{i=1}^{j} \bar{\gamma}_i.$$

$\tilde{P}$ is obtained by setting the Radon–Nikodym derivative $d\tilde{P}/dP$ equal to $\bar{\Lambda}_j$. A proof similar to that of Theorem 3.1 shows that under $\tilde{P}$ thus defined, the b_j are independent and identically distributed with zero mean, unit variance, and zero covariance.

THEOREM 3.2. *Under Assumptions 3.1–3.3 the market value of an arbitrary claim Ψ in $\mathcal{H}$ may be approximated by*

$$V(\Psi) \approx E[B(T)^{-1}\Lambda(T)\bar{\Lambda}_M\Pi(\Psi)].$$

Proof. This is the result of two successive changes of measure:

$$\begin{aligned} V(\Psi) &= E^{Q}[B(T)^{-1}\Pi(\Psi)] \\ &= E^{P}[\Lambda(T)B(T)^{-1}\Pi(\Psi)] \approx \tilde{E}[\Lambda(T)B(T)^{-1}\Pi(\Psi)] \\ &= E[\bar{\Lambda}_M \Lambda(T)B(T)^{-1}\Pi(\Psi)]. \end{aligned}$$

□

The results of Section 1 may now be applied to the reference Hilbert space $\mathcal{H}$ with market prices of arbitrary claims represented in terms of the market prices of the basis claims. Static hedging strategies in the Hilbert space sense may also be implemented. The next section introduces the basis for $\mathcal{H}$.

4. THE HERMITE POLYNOMIAL BASIS FOR THE REFERENCE SPACE

The Hilbert space $\mathcal{H}$ may be viewed as the space of functions defined on $\mathbb{R}^{Md}$, with the coordinates representing log asset returns $z_j = y_j - y_{j-1}$ that are square integrable with respect to P. With $z = (z_1, \ldots, z_M)$, the measure P is defined by the density

$$P(dz) = e^{-z^T z/2}\, dz/(2\pi)^{Md/2}.$$

A basis for $\mathcal{H}$ may be constructed using Hermite polynomials, and for this we follow Rozanov (1982).

DEFINITION 4.1. A polynomial $\phi(z)$ of degree p in the variables z is said to be *a Hermite polynomial* if it is orthogonal to all polynomials of degree $q < p$ in the space $\mathcal{H}$. Let $H_p(z)$ denote the closed linear span of all Hermite polynomials of degree p.

The spaces $H_p(z)$ for $p \geq 0$ are orthogonal by construction and $\mathcal{H}$ is their direct sum. Hence we may write

$$\mathcal{H} = \sum_{p=0}^{\infty} \oplus\, H_p(z).$$

Furthermore, if z_1 and z_2 are two components of z with no common variables, then the spaces $H_p(z_1)$ and $H_p(z_2)$ are orthogonal. In particular, for every Ψ in $\mathcal{H}$, let Ψ_p be the projection of Ψ onto $H_p(z)$. Then we may write $\Psi = \sum_{p=0}^{\infty} \Psi_p$.

The structure of H_p may be further described in terms of specific Hermite polynomials. Let $\phi_p(x)$ be the Hermite polynomial of order p in a single variable. Specifically, let $n(x)$ be the density function of the standard normal variate and let

$$\Phi_p(x) = (-1)^p \frac{\partial^p n(x)}{\partial x^p} \frac{1}{n(x)}.$$

The first few Hermite polynomials are

$$\Phi_0(x) = 1; \quad \Phi_1(x) = x; \quad \Phi_2(x) = x^2 - 1; \quad \Phi_3(x) = x^3 - 3x; \cdots.$$

These polynomials have to be normalized to unit variance to form an orthonormal system, so define

$$\phi_p(x) = \Phi_p(x)/\sqrt{p!}.$$

DEFINITION 4.2. A complete orthonormal set or basis for $\mathcal{H}$ is given by the polynomials

$$B(p, z) = \phi_{p_1}(z_1)\phi_{p_2}(z_2) \cdots \phi_{p_k}(z_k) \cdots \phi_{p_{Md}}(z_{Md}),$$

where $p = (p_1, \ldots, p_{Md})$, $p_k = 0, 1, 2, \cdots$ for $k = 1, \cdots, Md$, and the degree of the Hermite polynomial $B(p, z)$ is $h = p_1 + \cdots + p_{Md}$.

THEOREM 4.1. *An arbitrary claim Ψ in $\mathcal{H}$ may be written as*

$$\Psi(z) = \sum_p a(p)B(p, z),$$

where $p = (p_1, \ldots, p_{Md})$ and $p_k = 0, 1, \cdots$ for $k = 1, \ldots, Md$. The coefficients $a(p)$ are obtained by the inner product.

$$a(p) = \int_{\mathbb{R}^{Md}} \Psi(z)B(p, z)\bar{P}(dz).$$

The market value of Ψ is approximated by

$$V(\Psi) \approx \sum_p a(p)\pi(p),$$

where $p = (p_1, \ldots, p_{Md})$, $p_k = 0, 1, \cdots$ for $k = 1, \ldots, Md$, and $\pi(p)$ is the implicit market price of $B(p, z)$. The futures price law density is given by

$$\Lambda(z) = \sum_p \pi(p)B(p, z).$$

Proof. This is a direct consequence of Theorems 2.1, 2.2, and 2.3 applied to the specific space $\mathcal{H}$. □

The fundamental risks that are to be priced in valuing a space of contingent claims are therefore given by the Hermite polynomial risks of various orders in the returns. Every model for contingent claim valuation implicitly does this fundamental evaluation. These polynomials represent risks of increasing complexity in a larger number of variables. The next section makes an application of this theory to the case of a single asset and time point. This includes European calls and puts, and we also present an analysis of these assets in terms of the basis.

5. APPLICATIONS TO CLAIMS DEPENDING ON A SINGLE PRICE

We consider in this section a basis analysis of the subspace of claims written as functions of the price of a single asset at a single time point termed the maturity. Let S denote the price of the asset at the maturity time t.

ASSUMPTION 5.1. *Suppose that under the Gaussian reference measure P we may write that*

$$S = S_0 e^{\mu t + \sigma\sqrt{t}z - \sigma^2 t/2},$$

where z is a standard normal random variate of zero mean, unit variance, and density $n(z)$.

This assumption is consistent, for example, with time-varying but deterministic drifts and volatilities under the reference measure. More generally, such a reference measure may be employed as long as the ratio of the true density for ln S to a normal with mean μt and variance $\sigma^2 t$ is bounded above and below by Δ and δ, as required by Assumption 3.3.

A complete orthonormal system for the Hilbert space of square-integrable functions of S, and hence z with respect to P, is then given by the Hermite polynomials $\phi_p(z)$ for $p = 0, \ldots, \infty$. Any claim $g(z)$ in this subspace may be expressed in terms of this basis by

$$g(z) = \sum_{k=0}^{\infty} a_k \phi_k(z) \qquad \text{where} \qquad a_k = \int_0^{\infty} g(z)\phi_k(z)n(z)\, dz.$$

The market values are given in accordance with Theorem 4.1 by

$$V[g(z)] = \sum_{k=0}^{\infty} a_k \pi_k,$$

where π_k is the implicit price of $\phi_k(z)$ for all k. Furthermore, the futures price law has the density

$$\lambda(z) = \sum_{k=0}^{\infty} \pi_k \phi_k(z)$$

with respect to the standard normal reference measure on z.

The Hermite polynomial risk prices are not directly observable as these assets are not directly traded, but their prices may be inferred from the prices of traded assets that use them as basis elements. One such class of traded assets is the collection of European call and put options on a stock, with varying exercise prices. In fact, if the continuum of exercise prices is employed for the fixed maturity, then these assets form a market completing set of securities for the subspace of claims considered in this section (Nachman 1988). We proceed by performing a basis analysis of these claims and obtain analytically the coefficients of these claims with respect to the basis.

Let $c(z, S_0, x, \mu, \sigma, t)$ and $p(z, S_0, x, \mu, \sigma, t)$ be the payoffs at maturity t to European call and put options for exercise prices of x when the underlying stock has, under the reference measure, a mean return of μ, a volatility of σ, and the normal random disturbance z. Specifically,

$$c(z, S_0, x, \mu, \sigma, t) = [S_0 e^{\mu t + \sigma\sqrt{t}z - \sigma^2 t/2} - x]^+$$

and

$$p(z, S_0, x, \mu, \sigma, t) = [x - S_0 e^{\mu t + \sigma\sqrt{t}z - \sigma^2 t/2}]^+.$$

Let the coefficients of the call and put options with respect to the basis be respectively $a(k, S_0, x, \mu, \sigma, t)$ and $b(k, S_0, x, \mu, \sigma, t)$.

For explicit expressions for these coefficients we employ the generating function of the Hermite polynomials to obtain generating functions for the European call and put option coefficients. From the generating function for the Hermite polynomials we have that

$$\frac{e^{-(z-u)^2/2}}{\sqrt{2\pi}} = \sum_{k=0}^{\infty} \phi_k(z) n(z) \frac{u^k}{\sqrt{k!}}.$$

Now define

$$\Phi(u, S_0, x, \mu, \sigma, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(z, S_0, x, \mu, \sigma, t) e^{-(z-u)^2/2}\, dz$$

and observe that

$$a(k, S_0, x, \mu, \sigma, t) = \left.\frac{\partial^k \Phi(u, S_0, x, \mu, \sigma, t)}{\partial u^k}\right|_{u=0} \frac{1}{\sqrt{k!}}.$$

Standard integration techniques yield

$$\Phi(u, S_0, x, \mu, \sigma, t) = S_0 e^{\mu t + \sigma\sqrt{t}u} N(d_1(u) - xN(d_2(u)),$$

where

$$d_1(u) = \frac{1}{\sigma\sqrt{t}} \ln \frac{S_0}{x} + \left(\frac{\mu}{\sigma} + \frac{\sigma}{2}\right) \sqrt{t} + u$$

and

$$d_2(u) = d_1(u) - \sigma\sqrt{t}.$$

Analogously define the European put coefficient generator by

$$\Psi(u, S_0, x, \mu, \sigma, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} p(z, S_0, x, \mu, \sigma, t) e^{-(z-u)^2/2}\, dz.$$

From the put-call parity condition we have, by integration,

$$\Phi(u, S_0, x, \mu, \sigma, t) - \Psi(u, S_0, x, \mu, \sigma, t) + x = S_0 e^{\mu t + \sigma\sqrt{t}u}.$$

Taking successive derivatives we obtain the following relations between the European call and put basis coefficients:

$$b(0, S_0, x, \mu, \sigma, t) = a(0, S_0, x, \mu, \sigma, t) + x - S_0 e^{\mu t}$$

$$b(k, S_0, x, \mu, \sigma, t) = a(k, S_0, x, \mu, \sigma, t) - S_0 e^{\mu t} \frac{\sigma\sqrt{t}}{\sqrt{k!}}, \qquad k > 0.$$

Let $C(S_0, x, \mu, \sigma, t)$ and $P(S_0, x, \mu, \sigma, t)$ denote the market prices for the European call and put option prices with maturity t, exercise x, initial price S_0, and reference measure statistics of μ and σ^2 for the mean and variance rates. It follows from Theorem 4.1 that

$$C(S_0, x, \mu, \sigma, t) = \sum_{k=0}^{\infty} a(k, S_0, x, \mu, \sigma, t)\pi_k$$

and

$$P(S_0, x, \mu, \sigma, t) = \sum_{k=0}^{\infty} b(k, S_0, x, \mu, \sigma, t)\pi_k.$$

From observed prices of European call and put options one may infer the prices of basis elements. We illustrate this extraction of implicit basis prices in two ways, first assuming the validity of the Black–Scholes asset pricing model and, second, using observed option prices.

5.1. Black–Scholes Prices for Basis Elements

Under the Black–Scholes model the true probability measure is a geometric Brownian motion, and we may use this as the reference measure itself. The density of the futures price law with respect to the true or reference measure is well known in this case, and it may be written in terms of z as

$$\lambda(z) = e^{\alpha\sqrt{t}z - \alpha^2 t/2},$$

where $\alpha = (r - \mu)/\sigma$ (see Dothan 1990, p. 210). Using the constant interest rate assumption of the Black–Scholes model we obtain that

$$\pi_k = e^{-rt} \int_{-\infty}^{\infty} e^{\alpha\sqrt{t}z - \alpha^2 t/2} \phi_k(z)n(z)\, dz.$$

For an explicit evaluation of π_k, observe that on rearrangement

$$\sqrt{k!}\, \pi_k = \frac{e^{-rt}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z-\alpha\sqrt{t})^2/2}\, \Phi_k(z)\, dz,$$

so substituting $u = \alpha\sqrt{t}$ in the generator for the Hermite polynomials we obtain

$$\frac{e^{-(z-\alpha\sqrt{t})^2/2}}{\sqrt{2\pi}} = \sum_{j=0}^{\infty} \Phi_j(z)n(z)\frac{\alpha^j t^{j/2}}{j!}.$$

Performing the integration and using the orthogonality of the polynomials Φ_k with respect to the density $n(z)$ we get

$$\pi_k = e^{-rt}\frac{\alpha^k t^{k/2}}{\sqrt{k!}}.$$

Since α is negative, the prices alternate in sign and tend to zero as k tends to infinity. In particular, if the reference measure is taken to be Q, the equivalent martingale measure, then $\pi_0 = e^{-rt}$ and $\pi_k = 0$ for all $k > 0$. It is further interesting to note that, though the European call and put options do not themselves depend on μ, the mean rate of return, both the call coefficients $a(k, S_0, x, \mu, \sigma, t)$ and the put coefficients $b(k, S_0, x, \mu, \sigma, t)$ do depend on μ, as does the price π_k of the basis element $\phi_k(z)$. In fact, market values of cash flows are given by

$$\sum_{k=0}^{\infty} a_k\pi_k = \sum_{k=0}^{\infty} e^{-rt}\left.\frac{\partial^k\Gamma}{\partial u^k}\right|_{u=0}\frac{(\alpha\sqrt{t})^k}{k!} = e^{-rt}\Gamma(\alpha\sqrt{t}, S_0, \mu, \sigma, t),$$

where Γ is the generator of the arbitrary cash flow $g(z, S_0, \mu, \sigma, t)$ defined by

$$\Gamma(u, S_0, \mu, \sigma, t) = \int_{-\infty}^{\infty} g(z, S_0, \mu, \sigma, t)\frac{e^{-(z-u)^2/2}}{\sqrt{2\pi}}\,dz.$$

A change of variable shows that

$$\Gamma(\alpha\sqrt{t}, S_0, \mu, \sigma, t) = \int_{-\infty}^{\infty} g(z + \alpha\sqrt{t}, S_0, \mu, \sigma, t)\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz.$$

For $\alpha = (r - \mu)/\sigma$ the independence of valuations from μ follows for cash flows, where $g(z - (\mu - r)t/\sigma\sqrt{t}, S_0, \mu, \sigma, t)$ does not depend upon μ. This is true in particular for all cash flows that depend on z through their dependence on S, as is the case for call and put options.

5.2. Basis Prices Implicit in Market Data and the Empirical Futures Price Law

Let w be a vector of observed market prices for European call and put options of a fixed maturity t on an asset with current asset price S_0. Theorem 4.1 implies that we should be able to write

$$w_i = \sum_{k=0}^{\infty} A_{ik}\pi_k,$$

where the coefficients A_{ik} can be obtained from Taylor series expansions in u of the generators $\Phi(u, S_0, x, \mu, \sigma, t)$ and $\Psi(u, S_0, x, \mu, \sigma, t)$ for calls and puts respectively.

For the purposes of empirical implementation on a finite set of assets, consider the truncation of the countably infinite basis representation at some finite polynomial of order N. In this case we may approximate the prices of basis elements by

$$w = A^N \hat{\pi} + \varepsilon,$$

where A^N is a matrix of N columns and rows equal to the number of options. Estimates for $\hat{\pi}$ could then be obtained by an application of least-squares methods.

One may construct from π, using Theorem 4.1, the empirically implied futures price law. It follows from Theorem 4.1, using undiscounted prices, that

$$\lambda(z) = \sum_{k=0}^{\infty} e^{rt} \pi_k \phi_k(z)$$

with an approximation

$$\lambda^N(z) = \sum_{k=0}^{N} e^{rt} \hat{\pi}_k \phi_k(z).$$

Asset values for an asset with the time t cash flow of $g(z)$ equal to $\Sigma_{k=0}^{\infty} a_k \phi_k(z)$ and truncation $g^N(z)$ equal to $\Sigma_{k=0}^{N} a_k \phi_k(z)$ are then approximated by $e^{-rt}\langle \lambda^N, g^N \rangle$ in place of the required actual valuation of $e^{-rt}\langle \lambda, g \rangle$. By the Cauchy–Schwartz inequality the error in this approximation may be bounded by $\|g\| \, \|\lambda - \lambda^N\| + \|\lambda^N\| \, \|g - g^N\|$.[7] At any point of application for any prospective claim, we have information on three of the relevant norms in this bound, namely $\|g\|$, $\|g - g^N\|$, and $\|\lambda^N\|$. Hence the choice of truncation is essentially based on a judgment about $\|\lambda - \lambda^N\|$. For the Black–Scholes case this norm is bounded by $(\alpha^2 t)^{N+1}/(1 - \alpha^2 t)$ and depends on the level of standardized excess returns $(\mu - r)/\sigma$. For a mean rate of return of 30%, an interest rate of 10%, a volatility of 30%, and a time to maturity of 150 days, this bound for quartics is 0.00025. For practical applications, one would need to view the evolution of the norm of λ and set up a stopping criteria based on the changes thereof. Our illustration here will be based on quartics.

In the next subsection we illustrate these calculations of empirically implied basis prices and futures price laws using data on S & P 500 options. The results are compared with the theoretical model for Black–Scholes basis pricing as described in Section 5.1. For another recent approach at estimating the empirical futures price laws from options price data the reader is referred to, for example, Longstaff (1992). Longstaff uses call option prices to construct the futures price law histogram.

5.3. Illustrative Evaluations Using S & P 500 Options

Data for closing prices on S & P 500 call and put options was obtained from the *Wall Street Journal* of October 31, 1990 and November 28, 1990, henceforth dates 1 and 2,

[7] This follows from noting that $|\langle \lambda, g \rangle - \langle \lambda^N, g^N \rangle| = |\langle \lambda, g \rangle - \langle \lambda^N, g \rangle + \langle \lambda^N, g \rangle - \langle \lambda^N, g^N \rangle|$. The relevant norms are the L^2-norms, e.g., $\|g\| = \langle g, g \rangle^{1/2}$.

TABLE 5.1
Minimized Percentage Errors and Volatilities

		Model			
		Black–Scholes		Basis Pricing	
Date	Maturity in Days	Average Percentage Error	Daily Percentage Volatility	Average Percentage Error	Daily Percentage Volatility
October 31, 1990	17	43.80	1.23	6.97	2.0
	52	37.78	1.06	11.29	2.0
	136	45.80	1.07	3.72	2.0
November 28, 1990	24	50.26	0.83	15.24	2.0
	52	46.16	0.83	10.41	2.0
	108	50.59	1.03	4.11	2.0

respectively. Basis prices are like the Treasury yield curve and can be calculated and reported on a daily basis. We examine the one-dimensional model outlined in Sections 5.1 and 5.2 for three maturities. The time-1 maturities in days are 17, 52, and 136 and the time-2 maturities are 24, 52, and 108. The interest rate for both dates is 10%, and the S&P 500 index was 304.06 and 318.10 at times 1 and 2, respectively. The interest rate is supposed constant for the maturities being considered. Exercise prices were taken at $10 intervals, and this gave 12, 19, and 13 options at time 1 for the maturities of 17, 52, and 136 days respectively. At time 2 we have 16, 10, and 14 options for the maturities 24, 52, and 108. Complications due to dividend distributions and early exercise on the American feature of the options are ignored in these illustrative calculations. All computations are made using Mathematica (Wolfram 1988), as this software provides internal routines for analytical Taylor series evaluations.

For each date and maturity we project the observed closing option prices onto the space generated by the Taylor series coefficients of these options with respect to the first five Hermite polynomials. This determines estimated prices for polynomial risks to the quartic order. In constructing the projection we minimized squared pricing errors relative to the option price itself.[8] We used a reference measure that is close to the Black–Scholes futures price law by conducting the Taylor series expansions at $\mu = r = 10\%$. The value of σ for the reference measure was estimated by grid search along with the polynomial risk prices.

Table 5.1 reports the average percentage pricing error for each date and maturity using basis pricing and, for comparison, the Black–Scholes formula with an implied volatility that was allowed to vary with maturity. The Black–Scholes implied volatility estimates are around 1% for both dates and maturities. The basis pricing reference measure volatilities are estimated at 2%. The percentage errors for the Black–Scholes model are quite high, and these are significantly reduced by employing polynomials to the fourth order or the quartic basis pricing model.

[8]One could proceed by minimizing absolute errors, and we grant that the choice of minimizing percentage errors is biased toward exaggerating the differences from Black–Scholes, as Black–Scholes valuations are known to perform relatively poorly for out-of-the-money options. However, the choice of absolute error minimization in a sample cutting across a range of exercise prices essentially ignores the mispricing of out-of-the-money options. Furthermore, from a financial perspective the percentage error represents the expected returns to be extracted from unit investment in the mispricing.

TABLE 5.2
Basis Prices for Polynomial Risks to Quartic Order

		Basis Prices (t values)				
		Hermite Polynomial Order				
Date	Maturity	0	1	2	3	4
October 31, 1990	17	0.9486	−0.0746	−0.2173	−0.1881	0.6406
		(−1.08)	(−4.29)	(−7.37)	(−5.22)	(9.19)
	52	1.003	−0.1028	−0.2498	−0.2112	0.6474
		(0.38)	(−4.55)	(−6.78)	(−4.80)	(7.16)
	136	0.9283	−0.0960	−0.1993	−0.3518	0.2434
		(−1.06)	(−1.06)	(−10.90)	(−9.21)	(1.44)
November 28, 1990	24	0.8286	−0.0888	−0.3139	−0.0130	0.5760
		(−2.90)	(−5.23)	(−9.37)	(−0.045)	(8.41)
	52	0.9626	−0.0652	−0.4177	−0.1182	0.4217
		(−0.22)	(−2.49)	(−6.25)	(−1.28)	(2.00)
	108	0.9051	−0.0454	−0.3644	−0.4066	−0.0867
		(−2.01)	(−4.34)	(−18.6)	(−10.8)	(−0.73)

The prices of the Hermite polynomial risks are reported in Table 5.2 along with t-values for the Black–Scholes hypothesis that $\pi_0 = e^{-rt}$ and $\pi_k = 0$ for $k > 0$. The hypothesis that the risk-free security is priced at e^{-rt} is not rejected at the 5% level in five of the six cases. The hypothesis that the higher-order prices are zero is, however, rejected in most cases. The prices also display a pattern with negative values for the first three orders and positive values for the fourth order.

Graphs of the change of measure densities $\hat{\lambda}(z)$ are presented in Figures 5.1 and 5.2, for days 1 and 2 respectively, using the estimated Hermite polynomial risk prices for the quartic basis pricing model. Under the validity of the Black–Scholes model this function should be identically equal to unity. It can be observed from the graphs that large deviations from current index values on both sides are discounted relative to the lognormal hypothesis for the Black–Scholes futures price law. The adjustment to the lognormal is also seen to be greater for the shorter maturities. The tendency for the tails to dip below zero is a consequence of the quartic polynomial approximation. This is a typical phenomenon in Fourier-type approximations of densities and can be considerably mitigated by taking more terms on a larger data set.[9]

6. CONCLUSION

The space of contingent claims written as functions of a finite set of asset prices at a finite set of dates is viewed as a separable Hilbert space. As such it has a countable orthonormal basis that may be used to price and hedge statically all claims in the space. In general, this basis is difficult to construct as it requires an intimate knowledge of the stochastic process of asset prices. Under fairly general conditions, including market completeness, it is shown

[9]To ensure against any valuation problems arising from such an approximation, one may normalize to a unit integral the positive part of the approximation.

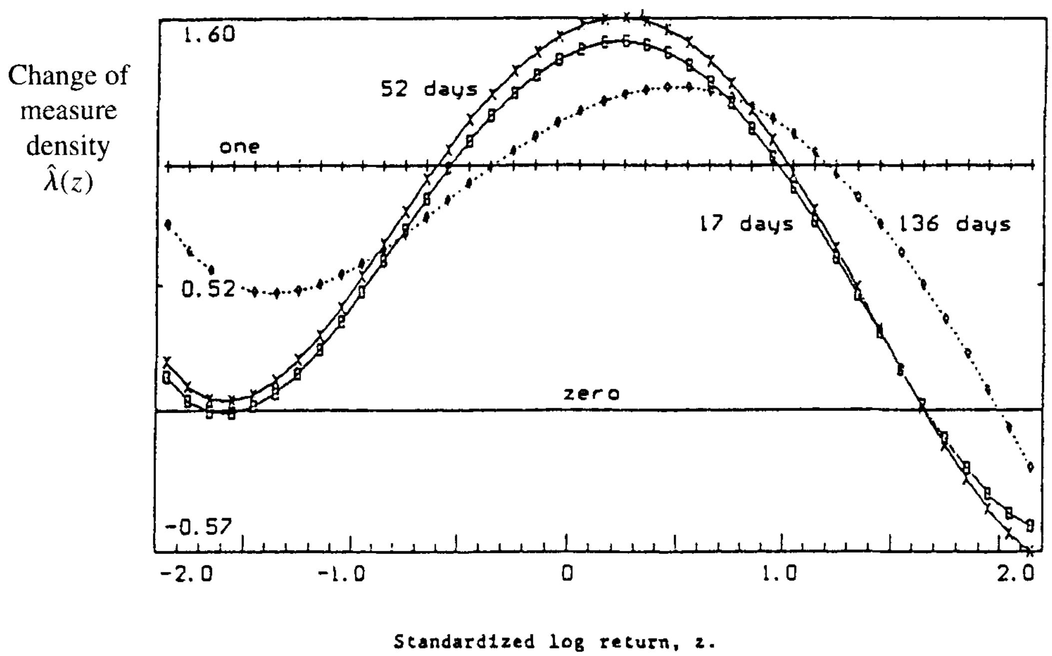

FIGURE 5.1. Equivalent martingale measures for October 31, 1990, maturities of 17, 52, and 136 days.

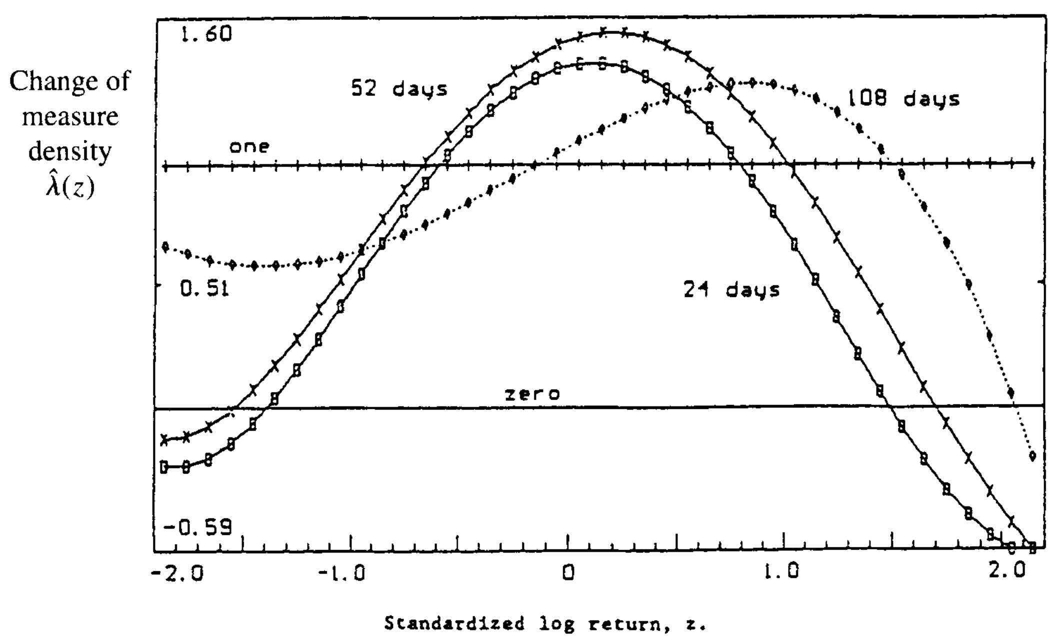

FIGURE 5.2. Equivalent martingale measures for November 28, 1990, maturities of 24, 52, and 108 days.

that one may apply the reference measure approach of filtering theory and first change measure to a reference measure under which asset prices are Gaussian. For the Hilbert space constructed with respect to this reference measure, an explicit basis is provided by the family of Hermite polynomials in the asset prices or returns.

Such a basis for contingent claims plays the role of pure discount bonds in the analysis of fixed income securities or the role of the paths of a tree in the analysis of claims on binomial or multinomial trees. It synthesizes the valuation problem by representing all valuation as derivative to basis pricing. Claim hedges may easily be derived from basis hedges. Finally basis prices completely specify the equivalent martingale measure or futures price law.

Relying on the completeness of call and put options (see Ross 1976, Green and Jarrow 1987, Nachman 1988) the subspace of claims written as functions of a single asset price at a single time point is analyzed further. Specifically we infer from observed option prices the implicit prices of the basis elements and use these prices to construct the empirically implied equivalent martingale measure or futures price law density with respect to the Black–Scholes equivalent martingale measure as the reference measure.

Using data on S&P 500 options for two dates we illustratively calculate the implied prices and densities. It is found that on these dates the futures price law deviated from the Black–Scholes model by relatively discounting large price movements with a compensating premia placed on smaller movements.

Future research could usefully identify bases relevant for various subclasses of claims and estimate and study the time series properties of basis prices. This does for contingent claims what the study of the yield curve does for fixed income securities. A promising area of application would be in the space of interest rate contingent claims.

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