Within-city Income Inequality, Residential Sorting, and House Prices *

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Abstract

This paper studies the effects of rising income inequality on residential patterns and house prices within a city. I develop a monocentric city model where household location, house prices, and housing supply respond to changes in income distribution. The household chooses housing quality and a residential location within the city. Commuting costs consist of distance-based financial costs and income-dependent time costs. The model can generate imperfect sorting of income across locations: there is income mixing at non-central locations; households at the top and bottom of the income distribution compete for central locations. When income inequality increases, the changes in equilibrium resemble the gentrification phenomenon. Income growth increases the rich's demand for central locations, and higher housing costs force the incumbent poor residents to move further from the city center. A greenbelt places pressures on house prices and reduces welfare when inequality increases, and the welfare loss is most significant for middle-income households who live near the greenbelt.

Keywords: Residential sorting, Monocentric model, Income inequality, Assignment problem

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1 Introduction

Since the 1990s, cities in North America have experienced a wave of "central city revival", during which the influx of affluent, well-educated residents into central neighborhoods drives up housing costs and displaces existing poor residents.[1] This striking gentrification phenomenon contrasts with a long earlier period of declining central cities and a “flight” of affluents residents in the suburbs. (Baum-Snow 2007, Boustan 2010) In the meanwhile, it is widely discussed in the press that increasing inequality and city-wide housing booms has caused a "housing affordability" crisis. In this paper, I establish the link between these two trends. Income growth for the rich increases their demand for central locations, where jobs and amenities are concentrated but land supply is limited. Facing higher housing costs, many of the relatively poor have to move to non-central locations.

This paper aims to explain how household location and the housing market in a city respond to changes in income inequality, and the implications for household welfare and urban policies. I build a model of residential sorting within the city that has two key features. First, commuting costs consist of a distance-based component and income-dependent time costs. Second, households can choose different housing qualities, and housing supply is endogenous. In the model, households at both ends of the income distribution prefer to live closer to the city center: the rich have high opportunity cost to travel, and the poor are sensitive to financial costs. When income inequality increases, higher time costs for the rich increase their demand for high-quality houses near the city center, where housing supply is limited by availability of land. Higher land rent near the city center increases the cost to supply low-quality houses that the existing poor households demand, who are "priced out" to non-central locations.

[1] Edlund et al. (2015) find that income level, house values, and rent have increases significantly in census tracts in the downtown in the last three decades. Couture et al. (2019) document that the propensity of the rich households to live in downtown neighborhoods has increased since the 1970s.

I begin by documenting some features on income-location relationships in U.S. cities that the model aims to capture. First, sorting of income across locations is not perfect; there are more than one income levels at each location. Second, the income gradients in cities exhibit different patterns. For example, Glaeser et al. (2008) find that in cities like Atlanta and Los Angeles, income tends to increase with distance from the city center. In Chicago and New York, the income gradient exhibits a U-shape pattern. Lastly, the location of the richest and poorest neighborhoods within a city has changed drastically during the last three decades. The richest neighborhoods have become more concentrated near the city center; the poorest ones, which used to locate near the city center, now locate within 5 to 15 kilometers from the city center.

The first goal of this paper is to account for the variations in income-location relationship in cities. I develop a monocentric city model [2] with a continuous income distribution and endogenous housing. Households of different income levels choose their residential locations within the city, as well as their preferred housing quality at the chosen locations. In the city, there is a competitive construction industry that supplies houses of different quality levels. Commuting from households' residence to the city center incurs commuting costs. The key feature of the model is that commuting costs consist of two components: financial costs including car maintenance, insurance, gas, etc, and income-dependent time costs. It is assumed that the time costs to travel increase with income. The relevance of such income-

[2] While the assumption that the city is monocentric has never accurately described urban landscape, recent empirical evidence shows that the monocentric city assumption is still relevant. For example, Rappaport (2014) finds that the share of urban employment in the CBD is 18.6 % in mid-sized metros (with population of 1 to 2 millions) in the U.S., and the share of agglomerative occupations, such as finance, insurance, and real estate, is even higher. Also, agglomerative occupations remain far higher in the CBD than in the remainder in mid-sized metros. On average, agglomerative workers in a CBD experience a density that was 21 times higher. Baum-Snow (2014) finds that in the 1960-2000 period, the share of urban job that shifted to suburb was only one third of the share of residents that shifted there. Firms located in the CBDs tend to larger and more productive than firms located elsewhere in metros. (Black et al. (2014)) The joint centralization of agglomerative occupations and productive firms tends to anchor the geographic distribution of residents within cities.

dependent time costs is supported by recent empirical work, such as Craig (2019) [3], Small et al. (2005) and Koster and Koster (2015).

The model generalizes the prediction on the relative location of income in the standard monocentric model with two income types.[4] In the model, the relative location of households depend on how the effects of financial costs and time costs differ across income levels. When the effects of time cost dominate for all income levels, in equilibrium, income decreases monotonically with distance, as richer households would like to live closer to the city center to save time. When the effects of financial costs dominate for all households, income increases monotonically with distance; the richer prefer to locate in the suburbs where land is cheaper.

With income-dependent commuting costs and housing demand, the model delivers novel predictions on sorting patterns of income within a city. When the income elasticity of time costs is large enough, and the financial cost is moderate, the effects of the two components of commuting costs differ by income groups. For high-income households, the effects of time costs dominate, and households would live closer to the city center as they get richer. Low-income households are more sensitive to financial costs, and households would live closer to the city center as income decreases. In such equilibria, there is income mixing at non-central locations. A location can be occupied by two types of households: the high-income households have higher time costs and live in better houses; the low-income type live in houses of lower quality. There is competition for central locations between households at

[3] By estimating a discrete choice model of residential location and transit mode choice using household level data in Vancouver, Craig (2019) finds that there is significant variation across households on the willingness to pay to reduce commute time. In particular, for the mean-income household, the value of travel time is about $14 dollars per hour. For households with an income of three-hundred thousand dollars per year, the estimated value of travel time is about $32 per hour. For the 25$^{\text{th}}$ income percentile, the value is close to $12 dollars per hour.

[4] In the monocentric city model with two income types, the relative location of the rich and the poor depends on the comparison between the income elasticity of marginal commuting costs and the income elasticity of housing demand. See Duranton et al. (2015) for a more details.

both ends of the income distribution: both the richest and the poorest households would like to live at the city center, where the city is not able to sprawl to accommodate for their housing demand. Depending on which type of households benefits more from living closer to the city centre, the model can generate different equilibrium configurations: there are two income levels at each location in the suburbs; the city centre is either occupied by the richest, or the poorest, or both.

Another contribution of the paper is that I develop an assignment model approach to the monocentric model when there is a continuum of income levels. Households with different income levels and housing of different quality are jointly assigned to residential locations within the city. In equilibrium, the sorting pattern is solely determined by the income elasticity of marginal commuting costs, because prices, housing supply, and the city size adjust to support this proposed sorting pattern as an equilibrium. In this framework, rising inequality affects household's locations and prices through two channels. First, income growth at the top of the distribution increases the rich households' willingness to pay for the their desired locations. Second, pressure on house prices at certain locations spills over to other segments of the city, which induces other households to re-sort within the city.

The theoretical predictions on imperfect sorting links gentrification to higher income inequality. Substantial income growth at the top of the distribution increases the rich's time costs, which makes them prefer to live closer to the city center. As income becomes more concentrated at the top, the demand for central locations increase significantly. In response to such changes in demand, the supply side of the housing market adjusts endogenously by supplying high-quality houses to the rich near the city center. Because there is no room for the city to sprawl to accommodate for increasing housing demand, land becomes more expensive at central locations, making it more costly to supply the low-quality house that the existing poor households demand. As a result, low-income households optimally move to non-central locations.

I use the model to evaluate the distributional effects of higher inequality across locations and over households. The pressures on land rent near the city center spill over to other segments of the city. The effects of higher income inequality on housing quality and house prices differ for different segments of the housing market. At a location where there are two income levels, the quality and price for a house increases for the high-income households, and decrease for the low-income households. Such changes are driven by the endogenous relocating of households within the city when inequality increases. In the model, when income inequality increases, low-income households' ability to move to non-central locations where the housing supply can accommodate their housing needs, thus mitigating the welfare gap from income inequality.

With both income and location heterogeneity, the model is flexible enough to evaluate the distributional effects of urban policies that target certain locations or income groups. I consider the effects of a specific zoning policy, a greenbelt that prohibits construction beyond the city edge. With the greenbelt, housing supply in the suburbs is restricted as the city is not able to sprawl. The city becomes more compact when inequality increases, and land rent increases at all locations. With the greenbelt, household welfare is reduced when inequality increases. The welfare loss is most significant for low-income households who choose to live close to the greenbelt.

This paper contributes to the growing literature that explores the driving forces of neighborhood gentrification.[5] Guerrieri et al. (2013) build a model where the rich households' preference for neighborhood externalites cause low-price neighborhoods adjacent to rich ones to become gentrified during housing booms. Edlund et al. (2015) and Su (2018) both find that high-skilled workers' increasing time value due to longer work hours accounts

[5]Examples include aging cycle of housing stocks (Brueckner and Rosenthal 2009), the role of reduction crime rates in the central city (Ellen et al. 2019)

for neighborhood gentrification. Baum-Snow and Hartley (2019), Couture and Handbury (2017), and Couture et al. (2019) [6] highlight the role of non-homothetic preferences for amenities in central neighborhoods. A commonality of these papers is that the forces that cause gentrification have differential effects over income levels. In my model, the key determinants of household location, commuting costs and house quality, are both income-dependent. The income-dependent time costs in my model can be broadly interpreted as income-variant time values or preferences for central amenities, consistent with the stories in other papers. My model features a continuum of income levels and endogenous housing supply in a monocentric city framework. Gentrification is the equilibrium outcome of different households competing for central locations when inequality increases.

My work also provides additional insights on the welfare implications of gentrification and displacement. There is mixed evidence for displacement in gentrifying neighborhoods. For example, Vigdor et al. (2002), Ellen and O'Regan (2011), Brummet and Reed (2019) find little evidence that incumbents residents are displaced by gentrification. Waights (2014), however, finds that the poor renters in the UK are more likely to exist gentrifying neighborhoods. Couture et al. (2019), Su (2018) both find that welfare gap is widened more than income gap once gentrifying forces are accounted for. In my paper, whether or not displacement happens and the welfare implications of income inequality when it does depend on the income level of the poor, transit cost and the response of housing supply.

My paper contributes to monocentric city model of urban land use (Alonso et al. 1964,

[6] Couture et al.(2019) look at the effects of rising inequality on residential sorting and household welfare. They build and estimate a quantitative spatial model with non-homothetic preference for neighborhood amenities and endogenous neighborhood development. Their model is able to match the empirical regularity that the propensity to live in downtown for different income levels exhibit a U-shape pattern. Income mixing at each neighborhood is achieved by perturbing perfect sorting with idiosyncratic preferences for locations. My paper develops a monocentric model with a continuum of income levels. Income mixing at a location is the equilibrium outcome of the multi-dimensional sorting problem of households across locations. The income-location relationships in the non-monotonic equilibria is consistent the with empirical features of income-location relationship in U.S. cities.

Muth 1969, and Mills 1984).[7] The canonical monocentric model usually features two income levels, and predicts that the relative location of households is determined by the comparison between the income elasticity of commuting costs and the income elasticity of housing demand. Despite its theoretical elegance, the model is too simplified to capture several salient features of residential patterns observed in the data. There are limited works on monocentric model with continuous income distribution, the few examples are Beckmann (1969), Montesano (1972), De Bartolome and Ross (2007).[8] In this paper, I propose an assignment approach to the monocentric model with endogenous housing supply. The model generalize the prediction of the canonical model on the relative location of different income levels, and generates non-monotonic sorting patterns. Moreover, with a continuous income distribution, the model is flexible enough to study the distributional effects of higher income inequality and policy change on prices and household welfare.

The approach to the sorting problem of a continuum of income levels across a continuum of locations follows the large assignment literature. [9] The model is mostly related to Davis and Dingel (2020), which uses the assignment model to study the distribution of skill, occupation and industry across cities. In their model, the skill distributions in cities overlap due to intra-city geographic heterogeneity. This chapter looks at within city distribution of income, and my model is able to generate imperfect sorting equilibria, where a location could be occupied by two types of households with different housing demand.

The model also complements recent works that study the interactions of different seg-

[7]See Duranton et al. (2015) and Fujita (1989) for review.

[8]Beckmann (1969), and Montesano (1972) build a monocentric model with a continuous income distribution, and the marginal commuting cost is a constant. As shown later in the paper, this can be considered as a special case of my model. De Bartolome and Ross (2007) considers a monocentric city with fiscal jurisdictions and a continuous income distribution. In their work, lot size is fixed, so within a jurisdiction, income decreases with distance, and income gradients for two jurisdictions overlap. In my model, households can choose housing quality at each location, and the model can generate an equilibrium where households of different income levels live at the same location with different types of houses.

[9]Sattinger (1993) and Costinot and Vogel (2015) survey the assignment model literature.

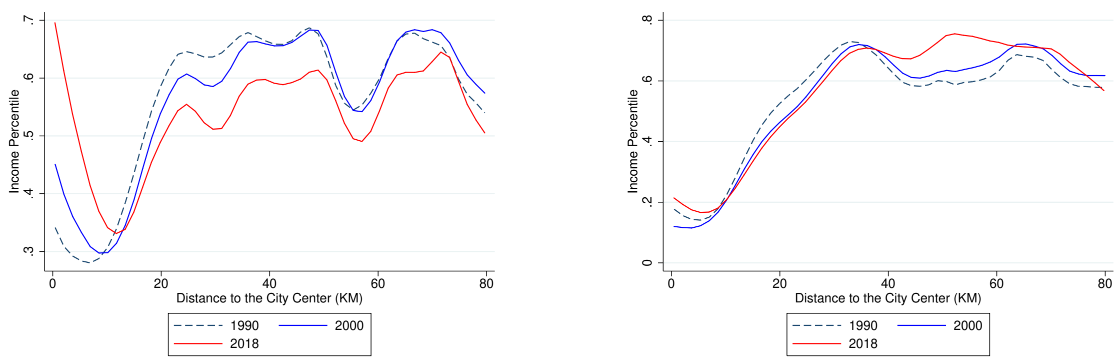
ments of the housing market using assignment models. For example, Landvoigt et al. (2015) study the cross section of house prices in San Diego during the 2000s boom. Nathanson (2019) looks at the effects of construction at the lower end of the housing market on prices and welfare when houses are indivisible. In these works, housing supply is exogenous and houses differ by a single quality index. In my work, housing is modeled as a combination of land and capital at a location, and housing value depends on its proximity to the jobs and amenities. Moreover, the distribution of housing stock is endogenous in my model: housing supply and city size adjusts to changes in environment and household characteristics

The rest of the paper is organized as follows. I start by documenting some facts on within-city residential patterns to motivate the theoretical model. Then I present the setup of the model and characterize the equilibrium income-location relationship. I use the model to study the effects of increasing income inequality on household location, the local housing market, and the implications for household welfare. Lastly, I use the model to evaluate the effects of a specific zoning policy, the greenbelt, when income inequality increases.

2 Motivating Facts

In this section, I provide some facts on residential patterns in U.S. cities to motivate the theoretical model. Specifically, I look at the income gradients and the location of low-income neighborhoods in cites. The data comes from the National Historical Geographic Information System (Manson et al. 2017), which provides summary tables and geographic boundary files of the Decennial Census and the ACS at the census tract level. In particular, I consider the relationship between the income percentile of a census tract and its distance to the center of the Core-Based Statistical Area (CBSA). Census tract income is measured by the median family income in that census tract. City centers are geographically located by the coordinates returned by Google Earth upon entering each CBSA's principle city's

Figure 1: Income Gradients of Chicago and Detroit



(a) Chicago (b) Detroit

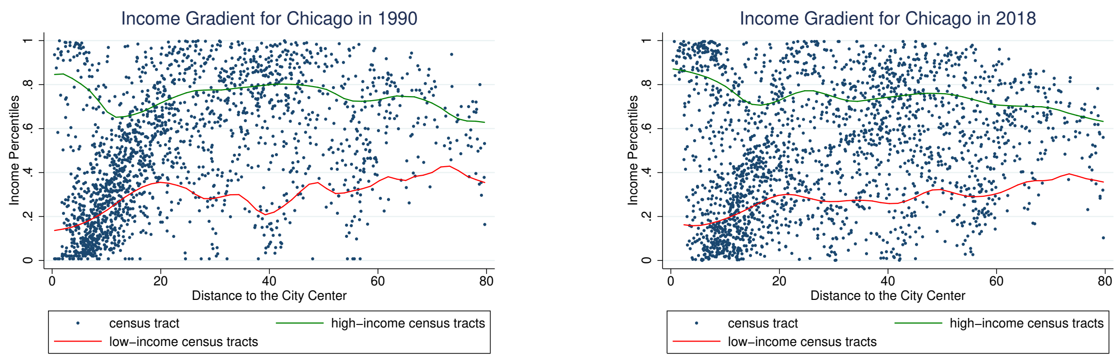
Notes: This figures show the percentile rank of each census trace income against its distance to the city center. Each dot represents a census tract within the CBSA. The lines are the kernel-weighted local polynomial smoothing curve, and Epanechnikov kernel functions.

name, as in Holian and Kahn (2012).[10]

Fact 1: within-city income-location relationships vary across cities and change over time First I revisit the well-documented fact that there is large variations in the income-location relationships across cities and over time. Figure 1 show the income gradients for Chicago and Detroit for the year 1990, 2010, and 2018. The income gradient in Chicago is U-shaped, and the U-shape pattern has become more pronounced over time. While in Detroit, income tends to increase with distance. Figure A.1 is a binscatter points of the percentiles of census tract income to their distances to the city center. In Chicago, high-income neighborhoods have become more concentrate near the city center since the 1990s. In Detroit, high-income census tracts still tend to locate in the suburbs in 2018.

[10] Although how Google Earth selects the centroids of a city is not clear, a close inspection of these points show that the selections usually correspond to subjective judgment of CBD location. The centroid returned is usually close to the main railway station, signature sights, or historical monuments.

Figure 2: Income Gradients High- and Low-income Census Tracts of Chicago in 1990 and 2018



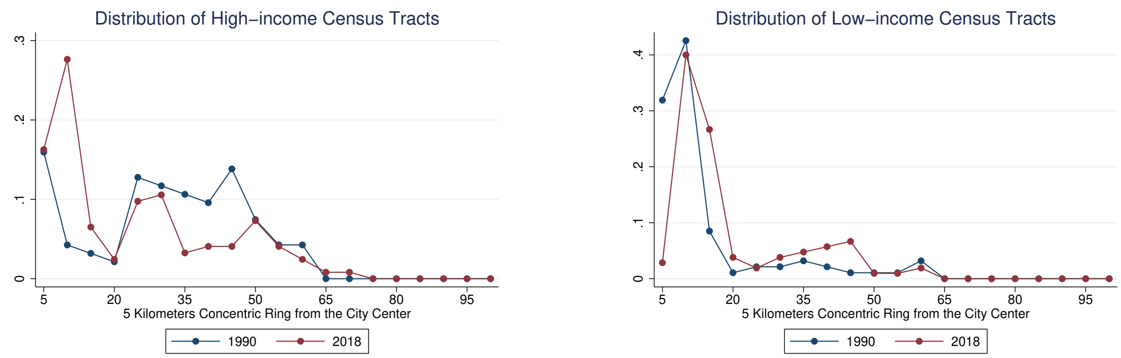
(a) 1990 (b) 2018

Notes: This figures show the income percentile of each census tract against its distance to the city center. Each dot represents a census tract within the Chicago MSA. The lines are income gradients for census tracts above and below the median, respectively, which are the kernel-weighted local polynomial smoothing curve, and Epanechnikov kernel functions.

Fact 2: there is a mixed income levels at a given distance, and both the richest and poorest are over-represented near the city center While the income gradients in Figure 1 describes how income generally changes with distance within the city, it does not reflect the fact that at a certain distance, there are many census tracts with different income levels. Figure 2 and A.2 shows the scatter points of the income percentile for each census tract within Chicago and Detroit in 1990 and 2018. We can see that in both cities, the lowest-income census tracts tend to concentrate near the city center, and relatively richer census tracts tend to locate in the suburbs. What drives the differences of the shapes of income gradients is the location of the richest neighborhoods. If the richest census tracts are near the city center, the income gradient exhibits a U-shape pattern, like in Chicago. If the lowest-income census tracts cluster near the city center, income gradient is upward-sloping.

To reflect the fact that there are more than one income levels at certain distance, and

Figure 3: Distribution of Highest- and Lowest-income Census Tracts in Chicago in 1990 and 2018



Notes: The figures show the distributions of highest- and lowest-income census tracts within the city. Each dot represents the percentage of the richest/poorest census tracts that location within that 5 kilometer concentric ring from the city center. High-income census tracts are those with median income above 95^{th} *percentile within the city; low-income census tracts are those with median income below the* 5^{th} *percentile within the city.*

that both the richest and poorest census tracts tend to be close to the city center, in Figure 2 and A.2, I show the income gradients for census tracts with income above and below the median for Chicago and Detroit in 1990 and 2018. For high-income census tracts, income tends to decrease with distance; for low-income census tracts, income tends to increase with distance. Having separate gradients for high- and low-income neighborhoods is a better description of the income-location relationship within a city.

Fact 3: high-income neighborhoods have become more concentrated near the city center, and low-income ones have become further from the city center. Returning to Figure 1 and Figure 2, from 1990 to 2018, the income percentiles for neighborhoods within 10 kilometers from the city center in Chicago have increased significantly, and the location of the bottom of the U-shape gradients has become further from the city center. Figure 3 show the distributions of census tract with income above/below the top/bottom 5$^{\text{th}}$ percentile in Chicago. The richest census tracts used to cluster within 5 kilometers from the city center, as well as in the suburbs in 1990. In 2018, more than half of the richest census tracts are within 10 kilometers from the city center. In the meanwhile, the share of the lowest-income census tracts that locate within 5 kilometers from the city center falls drastically from over 30% to less than 5% from 1990 to 2018, and the distribution for the lowest-income census tracts have become more dispersed within the city.

3 The Model

In this section, I develop a model of residential sorting within a city. The household chooses a residential within a city, as well as housing quality at that location. There is a competitive construction industry supplying housing at each location. The key feature of the model is that the time costs to travel is income-dependent. High income elasticity of time costs generates imperfect sorting in equilibrium, and causes central locations to gentrify when inequality increases. I do not micro-found the source of variation of time costs by incomes, but the model is flexible enough for this to be interpreted as possible stories, for example, higher time value for high-income workers, or non-homothetic preferences for city-type luxury amenities.

3.1 Environment

The setup of the model follows that of a canonical monocentric model with endogenous housing supply, except that the distribution of household income is continuous. Consider a city populated by N households. Household income is denoted by $y \in \left[\underline{y}, \bar{y}\right]$ and distributed according to a probability distribution function $f(y)$. The city is represented by the real line and each point on the line $x \in (-\infty, +\infty)$ is a different location. Point $x = 0$ is normalized as the center of the city.

Agents are fully mobile and can choose to live at any location within the city. All residents have to travel to the city center $x = 0$ for work and to access amenities.[11] Traveling incurs commuting costs $T(y, x)$, which is

$$T(y, x) = ax + \tau(y)x. \tag{1}$$

where a is the financial cost to travel, which includes gas, maintenance, insurance, etc. $\tau(y)$ is the time cost to travel one unit of distance. $\tau(y)$ is increasing in income level, as the opportunity costs to travel is higher for richer households. To facilitate the analysis, it is assumed that time cost $\tau(y)$ takes the following functional form:

$$\tau(y) = \rho * y^{\gamma},$$

with $\rho > 0$ being the travel time, and γ being the income elasticity of marginal time costs to travel[12]. Time costs increase both in traveling time and income levels.

The specification of income-dependent time costs generally captures the idea that the cost or disutility from living further from the city center increases with income level. Potentially there are several channels through which commuting costs to the city center vary with income. For example, if the time value increases with income level, the opportunity cost from traveling to the city center is higher for the rich. If high-income households consume more city-type amenities, such as theatres, cinemas, fancy restaurants, etc, and there is a greater variety and higher density of such amenities near the city center, then the disutility

[11]Cities have been traditionally viewed as centers for production. Glaeser and Gottlieb (2006) argues that high density in urban areas facilitates the role of cities as centers for consumption.

[12]In monocentric models with two income types, it is assumed that the marginal commuting costs for each income type is a constant, and is higher for the rich than for the poor. In my model, how much time costs vary with income level is governed by the parameter γ.

from living further from such amenities will be higher for richer households.

Each household chooses a residential location within the city, and the housing quality h at each location. Following Moen (1997) [13], I assume that at each location, there is a market maker that can separate the housing market into a continuum of sub-markets, each consisting of houses of quality h. The market maker announces the prices for houses of different qualities $P(h, x)$: in each submarket, houses are of the same price, but the prices differ across submarkets. Households are free to choose housing quality h, taking the prices for houses of different quality levels $P(h, x)$ as given.

3.2 Construction Sector

At each location x, there are $L(x)$ units of land available, with $L'(x) \geq 0$. Construction firms in a perfectly competitive construction industry use land and capital to produce housing services under constant returns to scale technology. Denote the input of land as l, and capital as k. Assume that the production function for housing is Cobb-Douglas:[14] $f(l, k) = l^{\alpha} k^{1-\alpha}$, with $0 < \alpha < 1$. Denote the rental price for land as $R(x)$, and the rent for best alternative land use as $\underline{R}$. A location will be developed whenever $R(x) > \underline{R}$.

Assume that the rental price of capital is constant and exogenously given. The unit cost function $C(R(x), 1)$ is defined as the minimum cost to produce one unit of housing service at x:

$$\begin{aligned} C(R(x), 1) = \min_{l,k} \quad & R(x)l + rk \\ \text{subject to} \quad & l^{\alpha} k^{1-\alpha} \leq 1. \end{aligned}$$

[13] Moen (1997) constructs an competitive equilibrium for labor market with frictions. In the setup, it is assumed that a market maker can separate the labor market into submarkets, each consisting of a subset of unemployed workers and firms with vacancies. The firms in a give submarket search for workers in the same submarket, and vice versa for the workers.

[14] Ahlfeldt and McMillen (2014) estimate the elasticity of substitution of land for capital using data from Berlin, Chicago and Pittsburgh. They find that the elasticity is close to one, which suggests that a Cobb-Douglas production for housing is a reasonable specification.

Solving the minimization problem yields the unit cost function

$$C(R(x), x) = AR(x)^{\alpha}, \tag{2}$$

where $A = \frac{1}{\alpha^{\alpha}} \frac{r}{(1-\alpha)^{1-\alpha}}$. The cost to produce one unit of housing services is higher at a location where land is more expensive. After applying Shepard's lemma, it follows that the conditional demand for land to produce one unit of housing is simply

$$\frac{dc(x)}{dR(x)} = \alpha AR(x)^{\alpha-1}. \tag{3}$$

Given $L(x)$ units of land at each location, the total housing services $Q(x)$ supplied at location x is

$$Q(x) = \frac{R(x)^{1-\alpha}}{A\alpha} L(x). \tag{4}$$

The total quantity of housing services that will be supplied at a location depends on the construction technology, the amount of land available, as well as the land rent at that location. Free entry of firms implies that at each location, construction firms make zero profits, so the price for a house of quality h equals its cost, that is

$$P(h, x) = C(R(x), 1) * h = AR(x)^{\alpha}h. \tag{5}$$

Given the rental price for land $R(x)$ at location x, the zero profit condition establishes a one-to-one relationship between the rental price for land and the price for a house at each sub-market. It can be understood as the market maker offering the households a "menu" at each location, which specifies the prices for houses of different qualities.

From the above zero profit condition, the unit price for housing service, denoted as $p(x)$, equals the unit cost, i.e.,

$$p(x) = AR(x)^{\alpha} \tag{6}$$

3.3 Household Problem

Each household derives utility from consumption good, c, and a house of quality h. Assume that household utility is Cobb-Douglas. Taking the "menus" for different houses at each location as given, a household with income level y_0 simultaneously chooses a location within the city, and the house quality at that location. Conditional on choosing to live at location x, the maximization problem of a household is simply

$$\begin{aligned} \max_{c,h} \quad & (1-\beta)\ln c(x) + \beta \ln h(x) \\ \text{subject to} \quad & c(x) + P(h,x) = y_0 - T(y_0,x), \\ & P(h,x) = AR(x)^{\alpha} h. \end{aligned} \tag{7}$$

For any given location x, the optimal housing quality of household of income level y_0 is

$$h(y_0,x) = \frac{\beta\left(y - T(y_0,x)\right)}{AR(x)^{\alpha}}. \tag{8}$$

At each location x, housing consumption will be higher, if the household's income net of commuting cost is higher, or if rental price for land is lower. Plugging the demand function $h(y_0,x)$ and $c(y_0,x)$ into the utility function yields the indirect utility for the household if he chooses to live at location x

$$v(y_0,x) = K + \ln\left(y_0 - T(y_0,x)\right) - \alpha\beta \ln R(x), \tag{9}$$

where $K = (1-\beta)\ln(1-\beta) + \beta\ln(\frac{\beta}{A})$. The optimal location of household with income y_0, denoted as $x^*(y_0)$, should maximize his utility, i.e.,

$$x^*(y_0) = \arg\max_x v(y_0, x)$$

4 Equilibrium

With both income and location heterogeneity, the allocation of households across locations is a complicated problem: solving the model requires obtaining the distribution of prices and quantities over locations simultaneously. In equilibrium, household location choice depends on prices and commuting costs, which in turn depend on households' locations. Potentially there are many sets of allocations and prices that can be sustained as equilibria. The common approach to determine the equilibrium location of different income types in the monocentric model, the bid-rent approach, is of limit use in this setting.[15] However, I show that under mild restrictions on commuting cost function and utility function, it is possible to characterize the income-location relationship in equilibrium.[16]

The approach in this paper follows the example of the differential rent model in Sattinger (1993).[17] I proceed by first proposing a potential assignment relationship $y^*(x)$, and then I find the conditions under which the proposed assignment is an equilibrium and no other

[15] In the canonical monocentric model, the bid-rent function is defined as the maximum price a household is willing to pay for a unit of housing at a location. In equilibrium, a location is allocated to the income type with the highest bids. Application of the bid-rent approach requires that the bid-rent functions of different income groups intersect only once and the steepness of the bid-rent functions can be ordered. With a continuum of income levels, there is a continuum of equilibrium utility levels. It is hard to define the bid-rent function with a continuous income distribution.

[16] The bid-rent function is the inverse of household's indirect utility function in the distance space. In this paper, I show that it is easier to work with the indirect utility function with a continuous distribution.

[17] Sattinger (1993) presents a differential rent model with a continuum of worker talents and a continuum of machine sizes. In his setup, when the production function is super-modular in talent and machine size, the equilibrium allocation is a positive assortative matching: more talented workers are assigned to better machines.

equilibrium could arise. Finally, I derive the market clearing conditions under the proposed assignment. Before characterizing the equilibrium income-location relationship, let us first define the equilibrium in this environment.

Definition. Given an income distribution $F(y)$, with $y \in [\underline{y}, \bar{y}]$, a distribution of land available $L(x)$ across locations, and the value for alternative land use $\underline{R}$, a residential sorting equilibrium within a city consists of the city edge, $\bar{x}$, the assignment relationship $y^*(x)$, rental price for land, $R(x)$, housing quality, $h(y^*(x), x)$, and the prices for houses of at different sub-markets, $P(y^*(x), x)$, total quantity of housing services at each location, $Q(x)$; such that

1. $h(y^*(x), x)$ and $y^*(x)$ satisfy the household maximization problems (7) and (9), with the associated unit house prices $p(x)$;
2. taking the rental price for land $R(x)$ as given, construction firms minimize cost at each location x, and make zero profit;
3. in all submarkets between x and $x + dx$, the housing market clears;
4. at the city edge $\bar{x}$, the rental price for land $R(\bar{x})$ equals the value for alternative land use, $\underline{R}$.

Lemma 1. *In equilibrium, the rental price for land $R(x)$ is continuously differentiable in $x \in [0, \bar{x}]$.*

Assume that the commuting cost function $T(y, x)$ is differentiable, with a continuum of income levels and a continuum of locations, the rental price for land $R(x)$ is differentiable,[18]

[18]In the monocentric model with discrete income types, the equilibrium price function is the upper-envelop of the bid-rent functions of different income types. The price function is continuous, with kinks at the points where the bid-rent functions intersect.

otherwise households who locate at the kink point in equilibrium are always better off deviating to adjacent locations.

Lemma 1 allows us to characterize household's location choice within the city. Differentiating the indirect utility (9) with respect to x gives the first order condition that characterizes household's optimal location choice

$$\frac{\partial v(y_0, x)}{\partial x} = 0, \tag{10}$$

which can be organized as

$$\frac{-T_x(y_0, x)}{y - T(y_0, x)} = \alpha\beta \frac{R'(x)}{R(x)} = \beta \frac{p'(x)}{p(x)}. \tag{11}$$

The first order condition (11) says that, when choosing x, a household weights the trade-off between proximity to the city center and unit price for housing service. Around the optimum, the growth rate of income net commuting costs must equal the growth rate of unit price.

In order for the solution of the first order condition to be the optimal location, the households' indirect utility function $v(y_0, x)$ must be concave in x, i.e., the second order condition for the household's maximization problem must hold

$$\frac{\partial^2 v(y_0, x)}{\partial x^2} < 0. \tag{12}$$

4.1 Equilibrium Income-location Relationship

In this section, I characterize the equilibrium income-location relationship. I first show that the equilibrium sorting pattern depends on how the marginal utility $\frac{\partial v(y,x)}{\partial x}$ varies with income levels. With Cobb-Douglas utility, the relative location of households in equilibrium depends on the comparison between the income elasticity of marginal commuting costs and the income elasticity of housing demand. Under the specified commuting cost function, the equilibrium sorting pattern depends on how the relative strength of the financial costs and time costs varies with income.

Proposition 1. *If* $\frac{\partial v(y,x)}{\partial y \partial x} \neq 0$*, the optimal location for each income level is unique.*

Proof. In Appendix B.1. $\square$

This proposition proves that if the marginal utility $\frac{\partial v(y,x)}{\partial x}$ varies with income level, then for each income type, the optimal location is unique. Having a continuum of incomes makes the sorting problem concave. The proof proceeds by considering another optimal location for a household, because there is a continuum of income levels and locations, the household always has incentive to deviate by living closer to the original optimum.[19]

Next we consider the assignment relationship $y^*(x)$ in the model when household's utility is Cobb-Douglas. For any proposed assignment $y^*(x)$ to be an equilibrium, it must satisfy the household's first order condition (10), which is

$$\left. \frac{\partial v(y,x)}{\partial x} \right|_{y=y^*(x)} = 0,$$

Under Cobb-Douglas utility, the above condition becomes

$$\beta \frac{p'(x)}{p(x)} = \left. \frac{-T_x(y,x)}{y - T(y,x)} \right|_{y=y^*(x)}. \tag{13}$$

From this condition, we can see that price $p(x)$ and its differential at $p'(x)$ at location x depend on the income level of the household that is assigned to that location, as well

[19]This result contrasts with the prediction of the monocentric model with discrete income types, where households of the same income level occupy a segment of the city, and price differentials make the utility level equalized across locations within the segment.

as his commuting costs. When income level changes at a location, price at that location adjusts accordingly. Changes in price at location x also spill over to adjacent locations: price differentials must also change for the marginal households to be indifferent over infinitesimal change in distances.[20]

The validity of the proposed assignment $y^*(x)$ can then be checked. For the assignment $y^*(x)$ to be an equilibrium, the second order condition of the household's optimal location choice should be satisfied at $y^*(x)$, that is,

$$\left.\frac{\partial^2 v(y,x)}{\partial x^2}\right|_{y=y^*(x)} < 0.$$

Lemma 2. *If the households' indirect utility $v(y,x)$ is supermodular within an income range $y \in [y_0, y_1]$, then income increases with distance in this range. If $v(y,x)$ is submodular for $y \in [y_o, y_1]$, in equilibrium income decreases with distance in this range.*

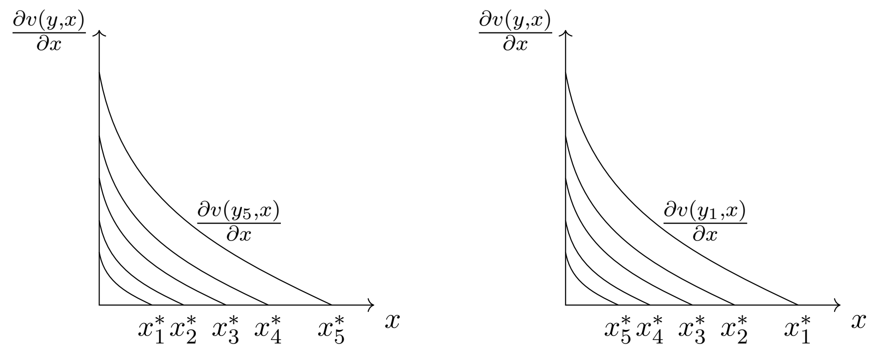
Proof. In Appendix B.2. □

This lemma states that, the optimal locations of households with different income levels follow the same order of their marginal utility over locations, $\frac{\partial v(y,x)}{\partial x}$. This is illustrated in Figure 4.[21] If $\frac{\partial v(y,x)}{\partial y \partial x} > 0$ for all x and $y \in [y_0, y_1]$, then there is complementarity between income and distance: richer households disproportionately benefit more from living further from the city center. If $\frac{\partial^2 v(y,x)}{\partial x^2} < 0$ for all x and $y \in [y_0, y_1]$, richer households benefit from living closer to the city center.

[20] In the canonical monocentric model with two income levels, utility equalization condition has to hold: price differentials across locations must make the utility levels for the same income type equalized across locations. With a continuous distribution, the optimal location for each income type is unique, so there is no "conventional" utility equalization condition. Equation (13), however, plays a similar role: the price differential at a location should make the marginal households indifferent over an infinitesimal change in distance.

[21] This condition for determining the relative location of different income levels is analogous to the one in monocentric model with discrete income types. In the model with discrete income types, the locations of different incomes are ordered by the steepness of their bid-rent functions. Given that the bid rent function is the inverse of the indirect utility function in the distance space, these two conditions are analogous.

Figure 4: Ordering of Households' Optimal Locations



Note: Ordering of marginal utility for income in the range $y \in \{y_1, y_2, ..., y_5\}$, with $y_1 < y_2 < ... < y_5$, and ordering of their optimal locations $\{x^*(y_1), x^*(y_2), ..., x^*(y_5)\}$. In the left panel, marginal utility over locations increases with income; in the right one, marginal utility over locations decreases with income.

This condition for determining the relative location of households holds generally, regardless of preference and the functional form for commuting costs. Next we consider the sorting patterns in the current specification where utility is Cobb-Douglas and commuting costs consist of financial and income-dependent components.

Proposition 2. *Assume that household's utility is Cobb-Douglas over consumption and housing. The relative location of households with income $y \in [y_0, y_1]$ depends on the comparison between the income elasticity of marginal commuting costs and the income elasticity of housing demand. In particular, when the income elasticity of commuting costs dominate, income decreases with distance for $y \in [y_0, y_1]$. When the income elasticity of housing demand dominates, income increases with distance for $y \in [y_0, y_1]$.*

Proof. By Lemma 2, if $\frac{\partial v(y,x)}{\partial y \partial x} > 0$ for all $y \in [y_0, y_1]$, in equilibrium, income increases with distance in this range. From Equation (9), this partial derivative is

$$\frac{\partial v(y,x)}{\partial y \partial x} = \frac{(1 - T_y)T_x}{(y - T(y,x))^2} - \frac{T_{xy}}{y - T((x),x)},$$

where T_x, T_y, and T_{xy} denote the partial derivative of $T(y, x)$ with respect the the variable in the subscripts. After re-arranging, it can be shown that $\frac{\partial v(y,x)}{\partial y \partial x} > 0$ becomes

$$\frac{T_{xy}}{T_x} < \frac{1 - T_y}{y - T(y, x)}.$$

Dividing both sides by $\frac{1}{y}$, the above expression becomes the comparison between two elasticities:

$$\frac{d \ln(T_x)}{d \ln y} < \frac{d \ln(y - T(y, x))}{d \ln y},$$

where the term on the left hand side is the income elasticity of marginal commuting costs. From housing demand (8), the total housing expenditure at a location is always a constant fraction of income net of commuting costs:

$$P(h, x) = \beta \left(y - T(y, x)\right).$$

So the the right hand side of the inequality equals the income elasticity of housing demand.

□

Proposition 2 is a generalization of the Alonso-Muth condition in the monocentric city model when income distribution is continuous.[22] In this setup, this condition holds because of Cobb-Douglas utility. In the indirect utility function, income net of commuting costs is separable from the unit price. When varying around the optimal location, the household will re-adjust consumption and housing bundle, such that total housing expenditure is

[22] In the model with two income types, the relative location of income depends on the comparison between the income elasticity of marginal commuting costs and the income elasticity of housing size. This is actually consistent with the result in this paper. With two income levels, at the location where the two bid-rent functions intersects, unit price $p(x)$ is the same for households of both types, thus the income elasticity of housing size equals the income elasticity of total housing expenditure.

always a constant fraction of income net of commuting costs.

4.2 Equilibrium Sorting Patterns

Proposition 2 establishes the relationship between households' relative location and the income elasticity of marginal commuting costs. Next I consider the equilibrium residential patterns when commuting costs consist of distance-based financial costs and income-dependent time costs. Under this specification, the equilibrium income-location relationship depends on how the relative strength of financial costs and time costs varies with income levels.

Proposition 3. *Assume that the commuting costs function is $T(y,x) = ax + sy^{\gamma}x$. If $\gamma < 1$, income increases with distance in equilibrium. If $\gamma > 1$, there exists cutoff values to the financial costs, $a_1 = \tau(\bar{y})(\gamma - 1)$ and $a_0 = \tau(\underline{y})(\gamma - 1)$, such that when $a > a_1$, income increases with distance; when $a < a_0$, income decreases with distance.*

Proof. In Appendix B.3. □

Proposition 3 proves that if either one of the commuting costs dominates for all households, in equilibrium, income always changes monotonically with distance. In particular, when the income elasticity of time costs γ is less than one, in equilibrium, income always increases with distance.[23] When $\gamma > 1$, if the financial cost, a, is high, then in equilibrium, only the rich can afford suburban locations, and low-income households live in the central city. When $\gamma > 1$ and financial cost is small, rich households concentrate near the city center, and low financial costs enable the low-income households to move to the suburbs.

[23]Beckmann (1969) considers the monocentric model with a continuous distribution when marginal commuting cost is a constant. His specification is a special case of this model when $a = 0$ and $\gamma = 0$. In the paper, he considers an equilibrium where income monotonically increases with distance, which is consistent with the prediction of Proposition 3.

Lemma 3. *When $\gamma > 1$ and $a_0 < a < a_1$, there exists a cutoff income level $\hat{y} = \left[\frac{a}{\rho(\gamma-1)}\right]^{\frac{1}{\gamma}}$, such that for $y \in [\hat{y}, \bar{y}]$, income decreases with distance; for $y \in \left[\underline{y}, \hat{y}\right]$, income increases with distance.*

Proof. The proof is straightforward from Proposition 3. □

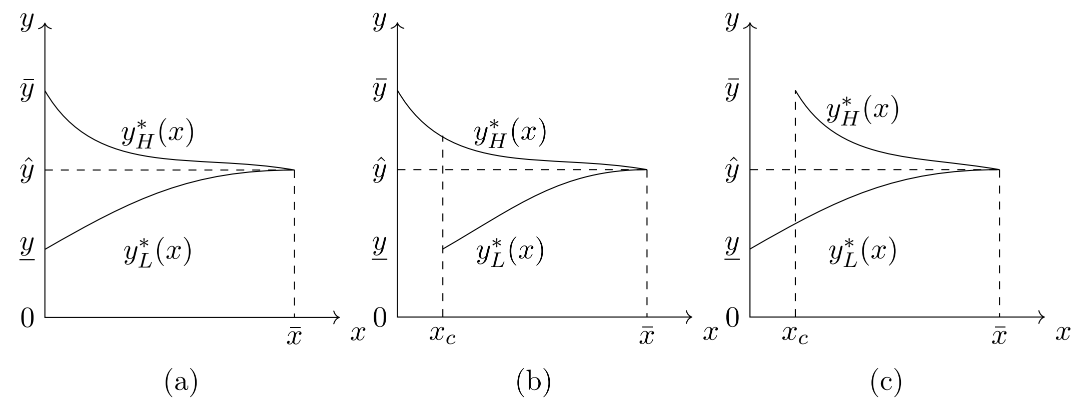
When neither the financial cost or time cost dominates for all incomes, the effects of the two types of commuting costs differ by income groups. This lemma implies that for high-income households with income $y > \hat{y}$, the effect of time costs dominates: households tend to live closer to the city center to save traveling time as they get richer. When the income level is lower than $\hat{y}$, households are more sensitive to financial costs and prefer to live closer to the city center as they get poorer. From Proposition 2, income decreases with distance within the high-income group, and increases with distance within the low-income group.

Denote the assignment relationship for households with income $y \in \left[\underline{y}, \hat{y}\right]$ as $y_L^*(x)$, and that for households with income $y \in [\hat{y}, \bar{y}]$ as $y_H^*(x)$. We consider a potential equilibrium which allows households of different income levels to live at the same location. For this to happen, the following *price match condition* must be satisfied, that is

$$\beta \frac{p'(x)}{p(x)} = -\frac{T_x\left(y_H^*(x), x\right)}{y_H^*(x) - T(y_H^*(x), x)} = -\frac{T_x\left(y_L^*(x), x\right)}{y_L^*(x) - T(y_L^*(x), x)}. \tag{14}$$

This condition is obtained by equating the first order conditions 13 for the two types of households that are both assigned to location x. The equilibrium unit price $p(x)$ and its differential should be such that households with different income levels and different commuting costs both find x optimal. Such an equilibrium is supported by each household's ability to adjust consumption and housing bundle. Intuitively, facing the same unit price for housing services at x, households with income $y_H^*(x)$ live at better houses than households with income $y_L^*(x)$ do, such that the both prefer the same location, even though they have different income levels and pay different commuting costs.

Figure 5: Imperfect Sorting Equilibria



Note: Different equilibrium configurations when $\gamma > 1$ and $a_0 < a < a_1$.

Proposition 4. *When $\gamma > 1$ and $a_0 < a < a_1$, the allocation of households across locations is represented by a cutoff income level $\hat{y} = \left[\frac{a}{\rho(\gamma-1)}\right]^{\frac{1}{\gamma}}$, two assignment functions $y_H^*(x)$ and $y_L^*(x)$, and a cutoff location x_c, such that*

1. *$\hat{y} \leq y_H^*(x) \leq \bar{y}$, with $\frac{dy_H^*(x)}{dx} < 0$, and $\underline{y} \leq y_L^*(y) \leq \hat{y}$, with $\frac{dy_L^*(x)}{dx} > 0$*

2. *households with income $y = \hat{y}$ live at the city edge $\bar{x}$, i.e., $y_H^*(\bar{x}) = y_L^*(\bar{x}) = \hat{y}$*

3. *if $\frac{T_x(\bar{y})}{\bar{y}} = \frac{T_x(\underline{y})}{\underline{y}}$, then $x_c = 0$, and $y_H^*(0) = \bar{y}$, $y_L^*(0) = \underline{y}$; if $\frac{T_x(\bar{y})}{\bar{y}} > \frac{T_x(\underline{y})}{\underline{y}}$, then $x_c > 0$, and $y_H^*(0) = \bar{y}$, $y_L^*(x_c) = \underline{y}$; if $\frac{T_x(\bar{y})}{\bar{y}} < \frac{T_x(\underline{y})}{\underline{y}}$, then then $x_c > 0$, and $y_H^*(x_c) = \bar{y}$, $y_L^*(0) = \underline{y}$.*

Proof. In Appendix B.4. □

This proposition states the sorting patterns when the effects of financial cost and time cost differ by two income groups. For households with income level $\hat{y}$, the effects of the two types of costs are equalized, and they choose to live at the city edge $\bar{x}$ in equilibrium. Beyond or above this income level, households' demand for central locations increase. Households at both the top and bottom of the income distribution have high demand for central locations. High-income households would like to live closer to the city center in order to save time, and households with low income levels live close to the city center to avoid paying commuting costs.[24]

In these equilibrium configurations, there is competition for central locations near $x = 0$: both the richest and poorest households have highest demand for the city center where jobs and amenities are concentrated, but the city cannot spral any further. Depending on which type has comparative advantage in living at the city center $x = 0$, the model can generate three equilibrium configurations which differ with respect to which households locate in central locations. Figure 5 shows the three possible equilibrium configurations. If the saving in marginal commuting costs relative to income level is higher for the richest, i.e., $\frac{T_x(\bar{y})}{\bar{y}} > \frac{T_x(\underline{y})}{\underline{y}}$, central locations will be occupied by the richest households. If the poorest benefit more from living at the city center, central locations will be occupied by the poorest households. If the effects of moving to $x = 0$ are equalized, then the city center is occupied by both the richest and the poorest.

The imperfect sorting equilibria in Figure 5 are qualitatively consistent with the income-location relationships documented in Section 2. In each equilibrium configuration, both the richest and the poorest households prefer to live close to the city center, in line with the fact both the highest- and lowest-income neighborhoods are over-represented in the central

[24]Fujita (1989), DeSalvo (1985) both build a monocentric mode with pecuniary commuting cost and time value for leisure, and their models generate a residential pattern in which both the rich and the poor tend to concentrate near the city center. In these works, such an equilibrium is demonstrated by how the slope of the bid-rent function varies with income levels. My model not only shows both the rich and the poor tend to concentrate near the city center, but there is also competition for central locations. Besides characterizing the equilibrium income-location relationship, I also propose a method to solve the model under such imperfect sorting equilibria.

city. Each equilibrium configuration differs by the income levels at locations near the city center, which is consistent with the fact that the relative location of the poorest and richest neighborhoods differs across cities and changes over time. In particular, Panel (b) of Figure 5 corresponds to the income gradient in Chicago in 2018, where the richest neighborhoods locate in the central city, and the poorest ones concentrate within 5 to 15 kilometers from the city center. The U-shape income gradient can be interpreted as the overlay of the two assignment functions in this case. In Panel (c), central locations are occupied by the poorest households, resembling the income-location relationship in Detroit, where the overall income gradient is upward-sloping.

4.3 Income Distribution, Commuting Costs, and Sorting Patterns

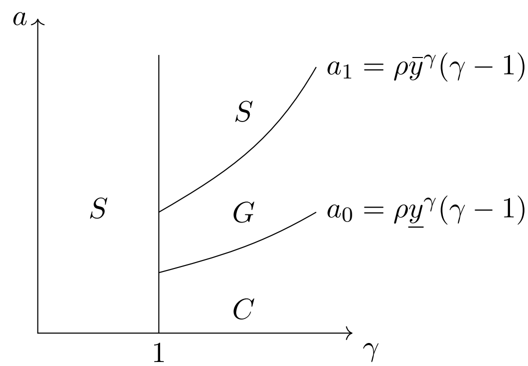
Previous results show how the equilibrium income-location relationship varies in response to commuting costs and the support of the income distribution. These results are summarized in Figure 6, which displays the equilibrium configuration under different values of transit cost, a, and income elasticity of time costs, γ.

When the income elasticity, γ, is large and transit cost, a, is moderate, the model can generate a "gentrified equilibrium" where both the rich and the poor tend to live close to the city center. Whether such equilibrium configurations will arise depends on the relative strength of two commuting costs. It is easy to show that

$$\frac{da_0(\gamma)}{d\gamma} = s\underline{y}^{\gamma} \left[(\gamma - 1)\ln \underline{y} + 1\right] > 0,$$

and similarly $\frac{da_1(\gamma)}{d\gamma} > 0$. From Figure 6, the cutoff values for financial cost, a_0 and a_1, below and above which the "gentrified equilibrium" arises increase in γ. The intuition is that a larger γ reinforces the high-income households' tendency to live near the city center. To sustain the equilibrium where both the rich and the poor concentrate near the city

Figure 6: Commuting Costs, Income Distribution, and Equilibrium Income-location Relationship



Note: Equilibrium residential patterns for different values of a and γ. The "S" region is for the "suburbanized" equilibrium where income increases with distance. The "C" region is for the equilibrium where income decreases with distance. The "G" region is for the "gentrified" equilibrium where households at the both end of the distribution live at central locations.

center, the financial cost has to be larger so that low-income households would like to live near the city center. Moreover, the effects of increasing time value are not symmetric for a_0 and a_1, as $\frac{da_1(\gamma)}{d\gamma} > \frac{da_0(\gamma)}{d\gamma}$. Because when $\gamma > 1$, time costs increase more than income growth, at the upper bound of the "gentrified" region, the financial cost has to increase more aggressively to sustain the "gentrified equilibrium" than at the lower bound.

We can also see the effects of reducing transit time, or increasing travel speed on residential patterns from Figure 6. When travel time ρ decreases, it shifts down both a_0 and a_1. For a financial cost that is large enough, the "suburbanized equilibrium" is more likely to happen when transit time is reduced.[25] For a low financial cost per kilometer, reducing traveling time makes the effect of transit cost relatively large. As the effect of financial cost dominates, low-income households tend to live near the city center.

The income distribution affects equilibrium residential patterns only through the high-

[25]This prediction is consistent with Baum-Snow (2007), in which high speed radial highways causes population decline in the central city.

est and lowest income levels. Income growth at the top of the distribution increases the tendency of rich households to concentrate near the city center, as their time costs increase more proportionately than income. The model also predicts that income growth for the low-income households and low financial cost enable them to move to non-central locations. If the income level for the poorest households remain low, they cannot afford non-central locations and are effectively "stuck" in the central neighborhoods.[26]

5 Model Solution

In this section, I derive the equilibrium conditions to solve the model under different equilibrium configurations. Under the proposed assignment, the equilibrium can be represented by a system of ordinary differential equations, which can be solved numerically.

5.1 Perfect Sorting

Let us first consider the equilibrium configuration where income increases with distance from the city center. Assume that in equilibrium, the market maker closes sub-markets where trade does not occur. In this equilibrium configuration, only one sub-market opens up at each location. From location x to $x + dx$, the change in the income level is $\frac{dy^*(x)}{dx}$. Hence, the total housing demand over this range is given by

$$h(y^*(x), x)Nf(y^*(x))\frac{dy^*(x)}{dx}dx.$$

which is the housing demand of residents that choose to live at locations between x and $x + dx$, times the total measure of such households, $Nf(y^*(x))$. The supply of housing

[26] **?** documents that urban poverty tends to concentrates in the central city, and accessibility to public transit explains why the poor live in central neighborhoods. Couture et al. (2019) documents that both the richest and the poorest households are over-represented in the central city. The "gentrified equilibrium" in my model is consistent with these findings.

services over this interval is $Q(x)dx$. Equating the total housing demand to supply gives the market clearing condition in this submarket gives

$$Q(x)dx = h(y^*(x), x)Nf(y^*(x))\frac{dy^*(x)}{dx}dx.$$

After substituting for housing supply using Equation (4) and housing demand using Equation (8) into the above expression, we obtain

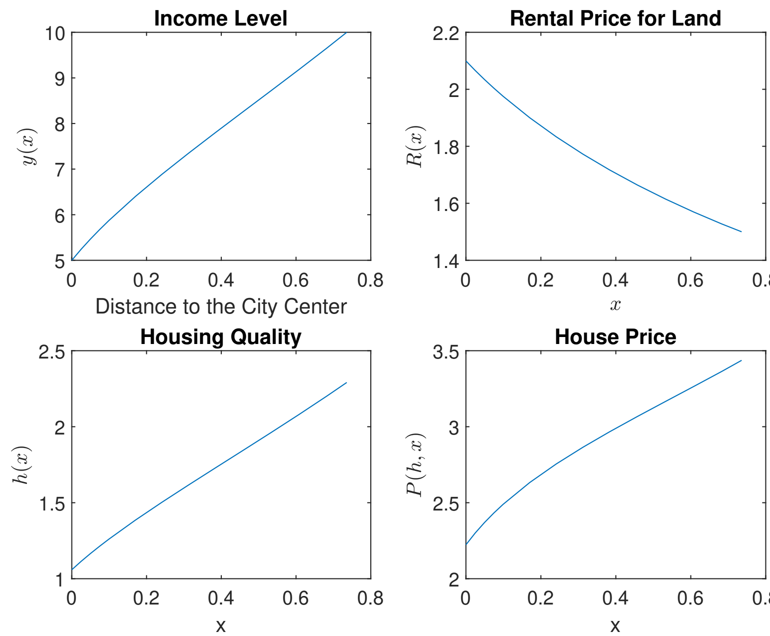
$$\frac{dy^*(x)}{dx} = \frac{R(x)L(x)}{Nf(y^*(x))\alpha\beta\left[y^*(x) - T(y^*(x), x)\right]}. \tag{15}$$

In a competitive equilibrium, the slope of the assignment function is set such that housing supply equals demand. The higher the supply of housing stock at a location, $Q(x)$, relative to the demand, the faster households should be assigned to locations for market to clear. Intuitively, if there many houses concentrated around x, but there are few households with income close to $y^*(x)$, then households a lot richer or poorer than $y^*(x)$ will be assigned to locations close to x. Substituting the zero profit condition, Equation (5), into the household first order condition (11) gives

$$\frac{dR(x)}{dx} = \frac{-T_x(y^*(x), x)R(x)}{\alpha\beta\left[y^*(x) - T(y^*(x), x)\right]}. \tag{16}$$

Under the proposed equilibrium in which income increases with distance, the richest households $y = \bar{y}$ live at $x = 0$, and the poorest households $y = \underline{y}$ live at the city edge $\bar{x}$. So the solution to the model is represented by the a system of ordinary differential equations, consisting of Equation (15) and Equation (16), with the boundary conditions $y^*(0) = \underline{y}$, $y^*(\bar{x}) = \bar{y}$, and $R(\bar{x}) = \underline{R}$. The existence and uniqueness of the equilibrium, and the solution algorithm to solve the model is illustrated in the Appendix C. With the

Figure 7: Model Solution: Income Increases with Distance



Note: This figure shows the numerical solution to the model when the equilibrium is such that income increases with distance. The model is solved under a truncated lognormal distribution for income.

solutions represented by a system of differential equations, the model is straightforward to solve numerically.[27]

Figure 7 shows the solution to this equilibrium. In this equilibrium configuration, the rental price for land decreases with distance, and so does the unit price for housing services. Richer households live in the suburbs with houses of higher qualities, and the price for a house increases with distance.

When the equilibrium is such that income decreases with distance, from location x to $x + dx$, income decreases by $\frac{dy^*(x)}{dx}$, and change in total measure of such households is

[27] In monocentric model with discrete income types, solving the model involves obtaining a sequence of equilibrium utility levels, boundaries for different income levels, and equilibrium prices. Solving the model is very complicated, if possible. See Fujita (1989) for details.

$-Nf(y^*(x))$. The total housing demand in the submarket between x and $x + dx$ is

$$-h(y^*(x), x)Nf(y^*(x))\frac{dy^*(x)}{dx}dx,$$

Equating this with the total housing supply $Q(x)dx$ gives the market clearing condition:

$$Q(x)dx = -h(y^*(x), x)Nf(y^*(x))\frac{dy^*(x)}{dx}dx. \tag{17}$$

The solution to the model when equilibrium is such that income decreases with distance can be represented by the following system of differential equations:

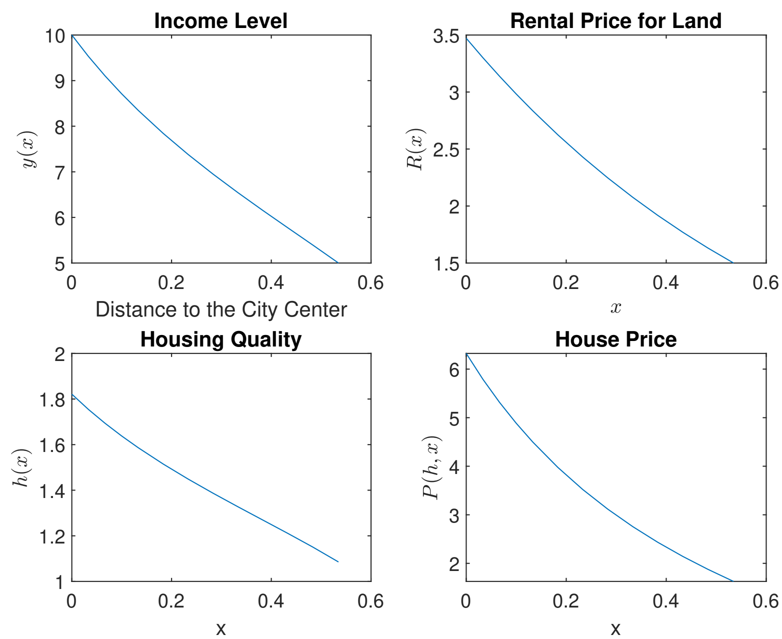
$$\frac{dR(x)}{dx} = \frac{-T_x(y^*(x), x)R(x)}{\alpha\beta\left[y^*(x) - T(y^*(x), x)\right]}, \tag{18}$$

$$\frac{dy^*(x)}{dx} = -\frac{R(x)L(x)}{Nf(y^*(x))\alpha\beta\left[y^*(x) - T(y^*(x), x)\right]}, \tag{19}$$

with the the boundary conditions $y^*(0) = \bar{y}$, $y^*(\bar{x}) = \underline{y}$, and $R(\bar{x}) = \underline{R}$.

Figure 8 shows the solution to this equilibrium configuration where income decreases with distance. Richer households live at better and more expensive houses in the central city. The income gradient is steeper near the city center, reflecting the scarcity of land near the city center. Compared with Figure 7 where income increases with distance, the gradient for rental price is steeper. This is because in this equilibrium configuration, richer households who demand houses of higher quality choose to live closer to the city center. As a result, land rent has to decrease faster with distance in order for the marginal household to be indifferent over small difference in distance.

Figure 8: Model Solution: Income Decreases with Distance



Note: This figure shows the solution to the model when the equilibrium is such that income decreases with distance. The model is solved under a lognormal distribution for income.

5.2 Imperfect Sorting

Next we derive the conditions to solve the "gentrified" equilibrium where both high- and low-income households competing for housing near the city center. Consider the equilibrium where both the richest and poorest live at $x = 0$, and the two assignment functions, $y_H^*(x)$ and $y_L^*(x)$ overlap within the whole segment of the city. From x to $x + dx$, the changes in income levels are $\frac{dy_H^*(x)}{dx}$ and $\frac{dy_L^*(x)}{dx}$, respectively. The total measures of households are $-Nf\left(y_H^*(x)\right)$ and $Nf\left(y_L^*(x)\right)$, respectively. Denote the quality at the two sub-markets that open up at x as $h_H(x)$ and $h_L(x)$, respectively, the market clearing condition between x and $x + dx$ can be written as

$$Q(x)dx = h_L(x)Nf\left(y_L^*(x)\right)\frac{dy_L^*(x)}{dx}dx + h_H^*(x)Nf\left(y_H^*(x)\right)\left(-\frac{dy_H^*(x)}{dx}\right)dx. \tag{20}$$

To obtain the relationship between $y_H^*(x)$ and $y_L^*(x)$, we differentiate the price match condition (14) with respect to x. After some manipulations, we obtain

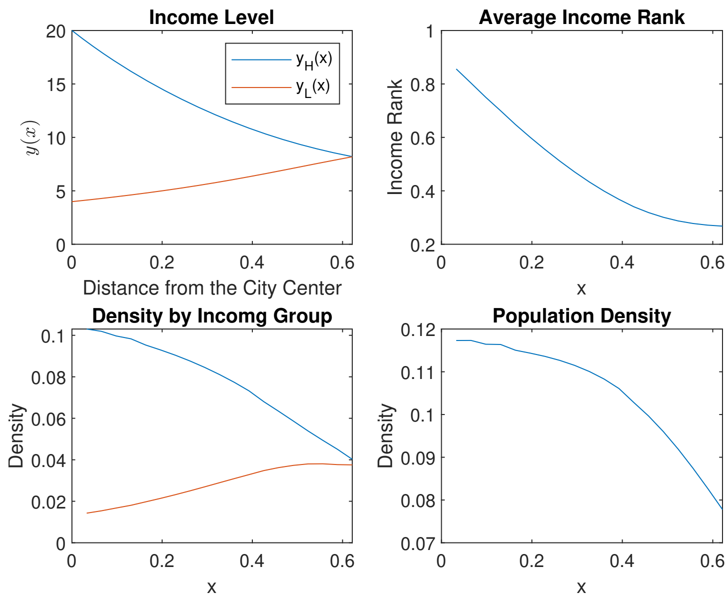
$$\frac{dy_H^*(x)}{dx} = \underbrace{\left[\frac{a + \rho y_H^*(x) - \gamma\rho y_H^*(x)\left(y_L^*(x)\right)^{\gamma-1}}{a + \rho y_L^*(x) - \gamma\rho y_L^*(x)\left(y_H^*(x)\right)^{\gamma-1}}\right]}_{G(y_H^*(x),y_L^*(x))} * \frac{dy_L^*(x)}{dx}. \tag{21}$$

Differentiating the price match condition with respect to x requires that the unit price function $p(x)$ is continuously differentiable: for any infinitesmal change in x, the associated price differential $p'(x)$ cannot "jump".[28]

Substituting this expression into the market clearing condition (20) and organizing, we

[28] Equation (21) is similar to the "smooth pasting condition" in the optimal stopping problem with continuous time. At the optimal stopping point, not only the value for waiting and stopping should be equalized, the marginal value at the optimal stopping point should also be the same.

Figure 9: Model Solution: Income and Population Density When Both the Poorest and Richest Households at the City Center



Note: Income level and average income across locations when the equilibrium is such that the richest and the poorest both live at $x = 0$. The model is solved under a lognormal distribution for income.

can get

$$\frac{dy_L^*(x)}{dx} = \frac{R(x)L(x)}{\alpha\beta}\Big\{\Big(y_L^*(x) - T\big(y_L^*(x), x\big)\Big)Nf\big(y_L^*(x)\big) \\ - \Big(y_H^*(x) - T\big(y_H^*(x), x\big)\Big)Nf\big(y_H^*(x)\big)G(y_H^*(x), y_L^*(x))\Big\}. \tag{22}$$

To sum up, the solution to the equilibrium where both the richest and the poorest live at $x = 0$ is represented by a system of three differential equations: Equation (14), Equation (21), and Equation (22). And the initial conditions are $y_H^*(0) = \bar{y}$, and $y_L^*(0) = \underline{y}$. At the city boundary $x = \bar{x}$, we have $y_H^*(\bar{x}) = y_L^*(\bar{x}) = \hat{y}$, and $R(\bar{x}) = \underline{R}$.

Figure 9 and Figure 10 display solution to the model under the proposed equilibrium configuration. The model is solved under a truncated lognormal distribution for income as

Figure 10: Model Solution: Both the Poorest and Richest Households at the City Center

Note: Other equilibrium objects when the equilibrium is such that the richest and the poorest both live at $x = 0$. The model is solved under a lognormal distribution for income.

in Figure A.4.1. At each location, there are two income levels; from the city center, income decreases with distance from the highest income level, and increases with distance from the lowest level.

The distribution of income and population across locations depends on the shape of the income distribution, as well as the financial cost to travel, a. In this example, the financial cost is relatively low, which enables the low-income households to move further from the city city. In the suburbs, the distribution of housing stock is relatively more dispersed than the distribution of income, as a result, the income gradient for the low-income households is relatively flat. Due to low density of low-income households in the central city, the average income decreases with distance. The overall population density decreases with distance from the city center.

Figure 10 shows other equilibrium objects in this example. The gradients for land rent and the unit price for housing service both decline with distance from the city center. The price gradients near the city center, resulting from that rich households demand high quality houses near the city center, where land is more scarce. At each location, two sub-markets open up; houses of two quality levels are supplied to the high- and low-income households at each location.

In the other two equilibrium configurations where the city center is either occupied by only the richest or the poorest households, the city consists of two segments. Near the city center, income changes monotonically with distance, depending on which income types live at the city center. Beyond x_c, the two assignment functions $y_H^*(x)$ and $y_L^*(x)$ overlap. The solution of the model under these configurations are represented by two sets of differential equations, which can be solved numerically with convenience.

6 Inequality and Neighborhood Gentrification

In this section, I study the effects of income inequality on equilibrium prices, household location, and welfare. I first show that when inequality changes, the equilibrium residential pattern depends only on the support of the income distribution and commuting costs. Then I evaluate the effects of rising inequality by comparing the equilibrium outcomes under two different income distributions. Lastly, I use the model to evaluate the effects of a specific zoning policy, the greenbelt, on prices and welfare when inequality increases.

Proposition 5. *Consider a city starting with the equilibrium configuration where both the richest and poorest households locate at $x = 0$. Suppose the city experiences a change in income distribution from $F(y)$, with support $y \in \{\underline{y}, \bar{y}\}$, to $F'(y)$, with support $y \in \{\underline{y}', \bar{y}'\}$. For a given $\bar{y}'$, there exists an income level $\tilde{y}$, such that*

1. if $\underline{y}' = \tilde{y}$, then the city center $x = 0$ is still occupied by both the richest and poorest households

2. if $\underline{y}' > \tilde{y}$, then the poor in the city center $x = 0$ is fully displaced; only the richest households live at the city center

3. if $\underline{y}' < \tilde{y}$, then the city center $x = 0$ is de-gentrified: only the poorest households live at the city center

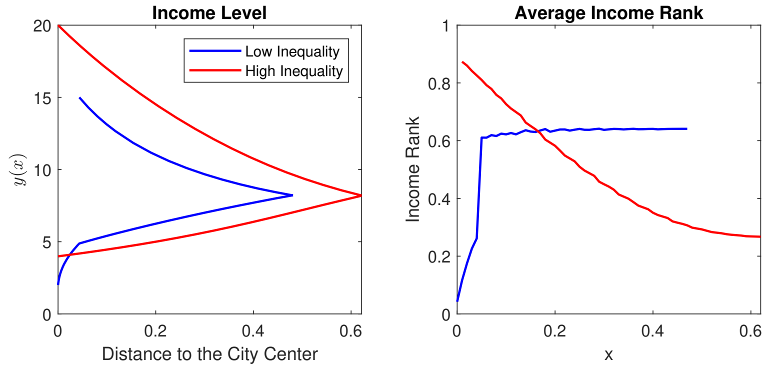
Moreover, $\tilde{y}$ increases with transit cost, a.

Proof. The proof is straightforward from Proposition B.4. □

When income distributions changes, only the income levels of the richest and the poorest, and commuting costs affect the residential pattern. The model predicts that if the income level of the poorest households remains low, in equilibrium, there will always be poor households in the city center. When income growth at the end of the distribution is substantial and financial cost, a, is relatively low, low-income households can re-locate to other parts of the city. This theoretical prediction makes it possible to evaluate the effects of higher inequality on the distribution of income and the local housing market in the long-run equilibrium. Specifically, I solve the model under two different income distributions and compare the equilibrium outcomes.

To highlight the effects of income inequality on neighborhood gentrification and displacement, I consider two initially identical cities that experience different changes in income distributions. In the first city, income grows more substantially for households at the top of the distribution. In the second one, the income level for the poorest households also increase. For illustrative purpose, I assume that the both cities start with the equilibrium configuration where central locations are occupied by the poor, and the rich households

Figure 11: Gentrification without Full Displacement: Income Gradient



Note: This figure shows the income levels and average income rank across locations. The model is solved under two lognormal distributions for income. The blue lines represent the gradients for the city before inequality increases, the red ones are for the distribution with larger variance.

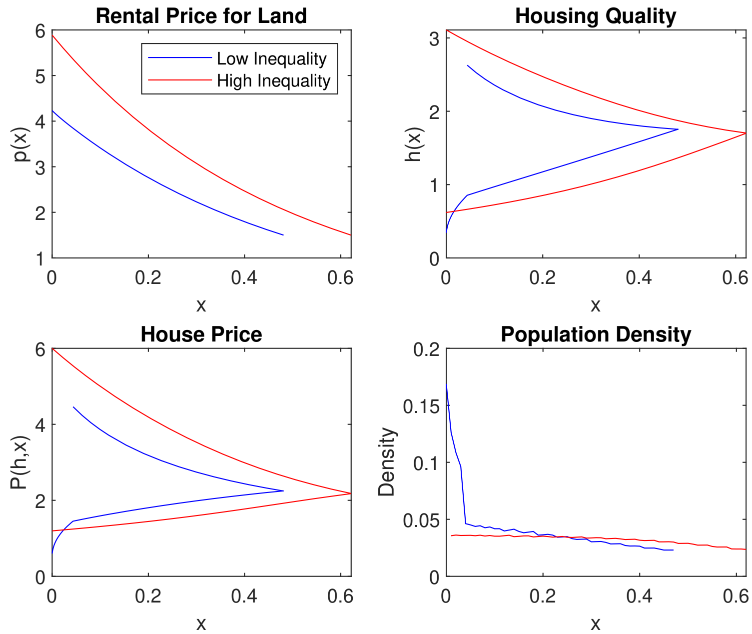
live at locations adjacent to the central city. The changes in income distributions in each example are illustrated in Figure D.4 and Figure D.5.

6.1 Gentrification without Full Displacement

Figure 11 shows the income levels across locations, and the income gradient when the income level of the poorest households remains low after inequality increases. As predicted by Proposition 5, central locations where the poorest households used to live at, are occupied by both the richest and the poorest households after income inequality increases. The city size becomes larger after inequality rises. Income gradient for the high/low-income group shifts up/down. Income growth for the rich causes them to live closer to the city center due to disproportionate increase in time costs; at the same time, relatively richer households in the low-income group move further from the city center.

The shape of the income gradient, which is the average income percentiles at each location, depends on income levels across locations, as well as the shape of the income distribution. Before inequality increases, income gradient is steeper near the city center,

Figure 12: Gentrification without Full Displacement: Distributional Effects Across Locations

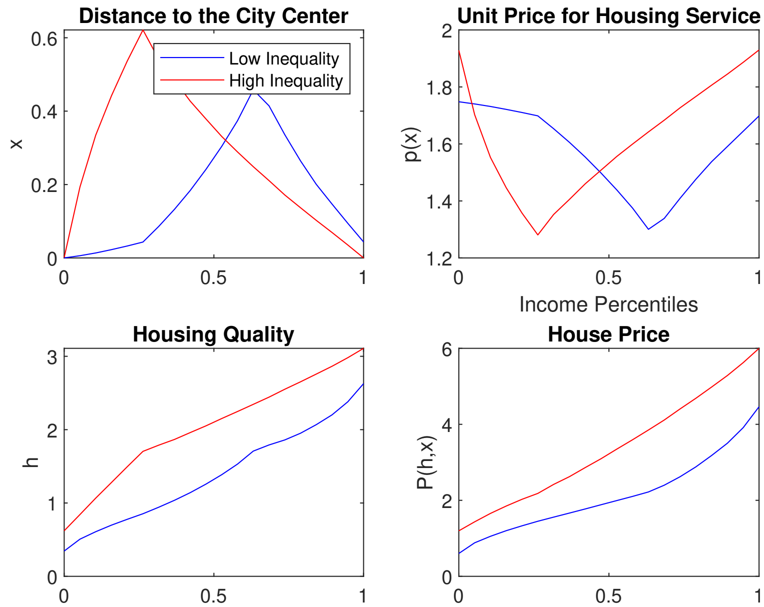


and gradually increases with distance. The shape of the income gradient resembles that in Detroit, where the poorest neighborhoods concentrate near the city center, and the richest ones locate at suburban locations adjacent to the central part of the city. In this example, after inequality increases, the overall gradient declines with distance from the city center.

Figure 12 shows the effects of higher inequality on the local housing market in this example. Higher inequality causes land rent and unit price for housing services to increase at all locations, even though more land gets developed near the city edge. The gradient for land rent becomes steeper after inequality increases, reflecting that the housing market becomes "tighter" closer to the city center: as the rich households get richer, they demand higher-quality houses at locations closer to the city center. Housing supply for high-quality sub-markets, however, is restricted, because there is no new land available in the city center.

Rising inequality induces households of different income groups sort to different segments

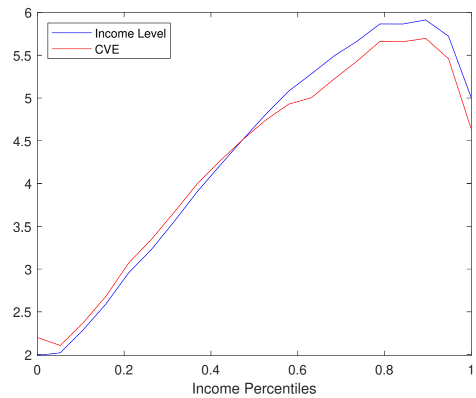
Figure 13: Gentrification without Full Displacement: Distributional Effects Over Households



of the housing market within the city. Before inequality increases, at locations near the city center, only the sub-markets for low-quality houses open up. As the richest households move closer to the city center, high-quality and expensive houses are built. At other parts of the city, the effects of higher inequality differ for the high- and low-quality sub-markets. At each location, the quality and prices for houses increase for the high-quality sub-markets, but decrease for the low-quality sub-markets.

The gradient for population density declines with distance from the city center, and the shape depends on the shape of the income distribution. Before inequality increases, poverty is highly concentrated near the city center. Rising inequality makes the central city more expensive, which leads to a significant drop in population density at locations near the city center. Land beyond the original city edge gets developed; as a result, the city becomes more spread.

Figure 14: Income Growth and Welfare Change

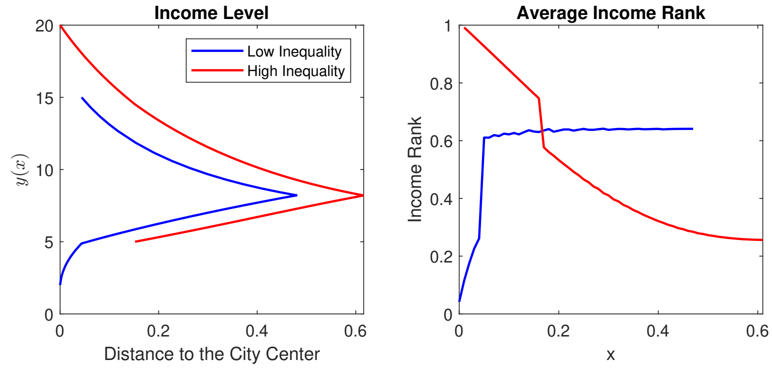


Note: This figure shows changes in income level and household welfare at different income percentile when inequality increases. Welfare change is measured as changes in compensating equivalent variation in consumption units from income growth.

Figure 13 shows the effects of higher inequality on equilibrium outcomes over income percentiles. As inequality increases, households at the bottom of the income distribution, except the poorest ones, move further from the city center. Households with income below the 5$^{\text{th}}$ percentile faces higher unit prices for housing services, as they are "forced" to remain at central locations, where unit prices are bid up by the richest households. Other low-income households are able to move to houses of higher quality at non-central locations, where land rent is lower.

Figure 14 shows the changes in welfare, measured by compensating equivalent variation in consumption units, for households at different income percentiles when inequality rises. In the long run, the effects of higher inequality on welfare gap is mitigated, because households in different income groups re-locate to different types of houses at different locations, and housing supply fully adjust to accommodate for their housing needs. For the richer households, because of high income elasticity of time costs, income growth increases their tendency to concentrate near the city center, where limited land supply causes pressures on land rent and house prices. In the meanwhile, low-income households who can afford

Figure 15: Gentrification with Full Displacement: Income Gradient



the financial costs move to houses of better qualities at non-central locations. As a result, welfare gap is smaller than income inequality.

6.2 Gentrification with Full Displacement

Figure 15 shows the income levels across locations and the income gradient in the second city where inequality increases, but the income level of the poorest household grows more than those in the first city. In this example, the poor residents near the city center are replaced by the richest ones when inequality increases. The overall income gradient declines with distance, with a discrete drop at the location where the poorest households live.

The effects of higher inequality in this city on the housing market are displayed in Figure 16. Same as in the previous example, the gradient for land rent shifts upwards and becomes steeper. The change in urban landscape near the city center is more drastic than in the previous example: low-quality, cheap houses are fully replaced by the best quality, most expensive ones. At locations where both the sub-markets for the high- and low-quality houses open up, quality and prices for houses increase in high-quality sub-markets, and decrease for the low-quality sub-markets. In this equilibrium, income growth at the bottom of the distribution, together with relatively low financial cost to travel, enable the poor

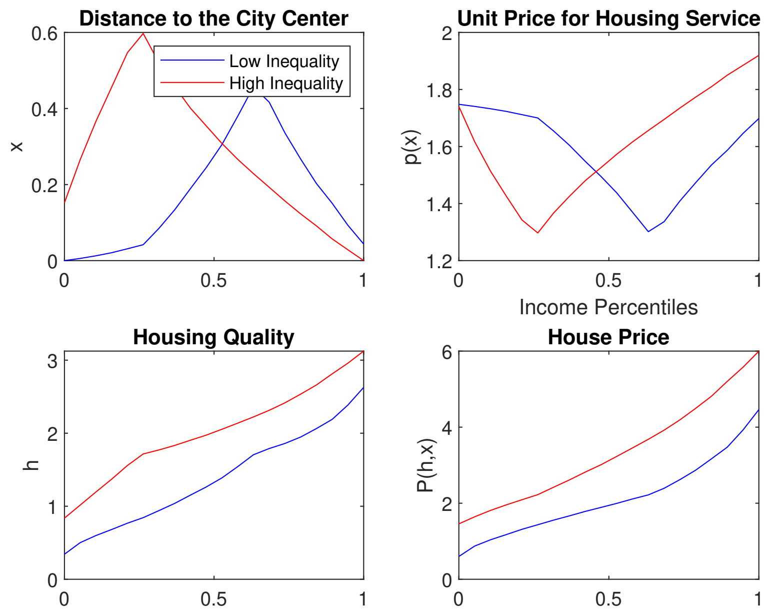
Figure 16: Gentrification with Full Displacement: Distributional Effects Across Locations

households to move to the suburbs, where lower land rent enables them in live at houses of higher quality. As illustrated in Figure 18, the effect of income growth on welfare is more significant for low-income households.

6.3 Policy Analysis: the Greenbelt

I consider the distributional effects of a specific zoning policy, a greenbelt that prohibits housing construction beyond the existing city edge. For illustrative purpose, suppose the city starts from the equilibrium configuration where the central city is occupied by the poor, and changes in inequality cause the poor to be fully replaced by the rich. Figure 19 compares the income gradients for the equilibrium where inequality is higher, with and without the greenbelt. From the left panel, if there is no greenbelt, the city size would be larger in response to rising inequality. The income gradient becomes steeper with the greenbelt: households of distinct income levels have to be assigned to locations with small

Figure 17: Gentrification with Full Displacement: Distributional Effects Over Households



Note: This figure shows the equilibrium outcomes over different income percentiles when the poorest households remain in the city center when inequality increases.

Figure 18: Gentrification with Full Displacement: Income Growth and Welfare Change

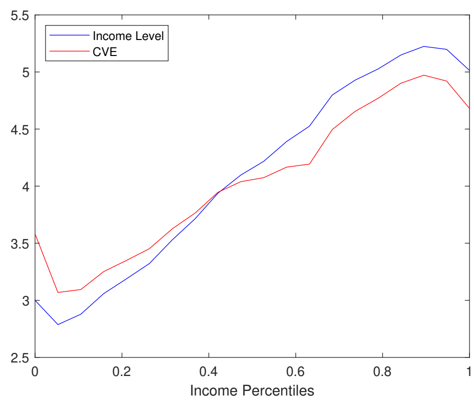
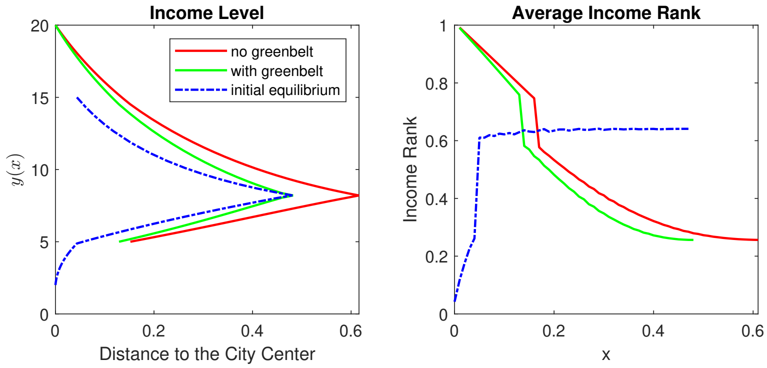


Figure 19: The Effects of the Greenbelt: Income Gradient



Note: Income level and average income rank across locations with and without the greenbelt.

differences in distances from the city center. In this example, the greenbelt causes the overall income gradient to shift downwards, and the city becomes denser because of the greenbelt.

The greenbelt restricts housing supply from accommodating the demand of households with income levels close to $y = \hat{y}$, who would live beyond the greenbelt if there were no such zoning policy. The greenbelt affects the local housing market through re-allocations of households towards the city center, and the spillover effects of price pressures from the submarkets near the city edge. Rental prices for land increase at all locations, even at central locations where income levels would be higher if without the greenbelt. The effects of the greenbelt differ on the high- and low-quality submarkets. At each location, the quality and prices decrease in the high-quality submarket, and increase in the low-quality submarket.

Figure 22 compares the effects of rising inequality on equilibrium outcomes across households with and without the greenbelt. The greenbelt causes all households, except the richest ones, to live closer to the city center. The unit prices increase, and housing qualities decrease for all households because of the greenbelt, and the effects are most significant for households with income $\hat{y}$, who lives at the city edge. Because of Cobb-Douglas utility,

Figure 20: The Effects of the Greenbelt Across Locations

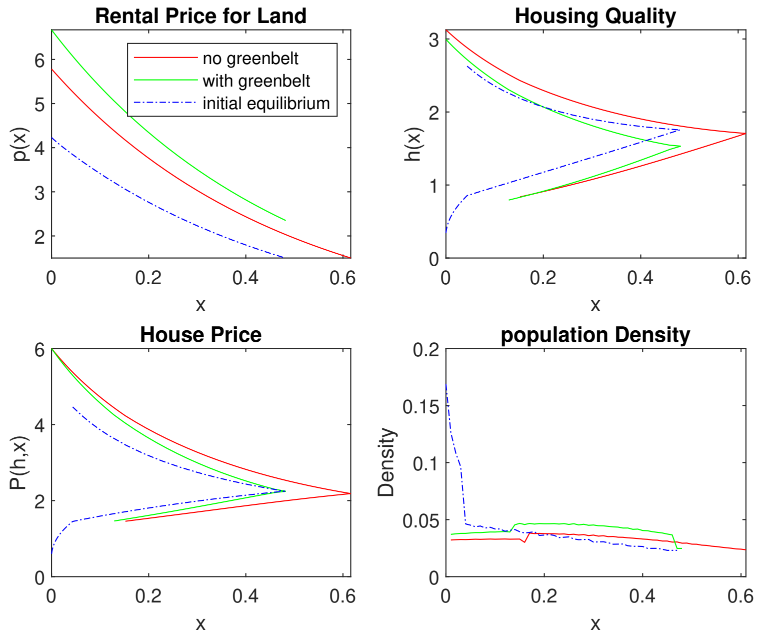
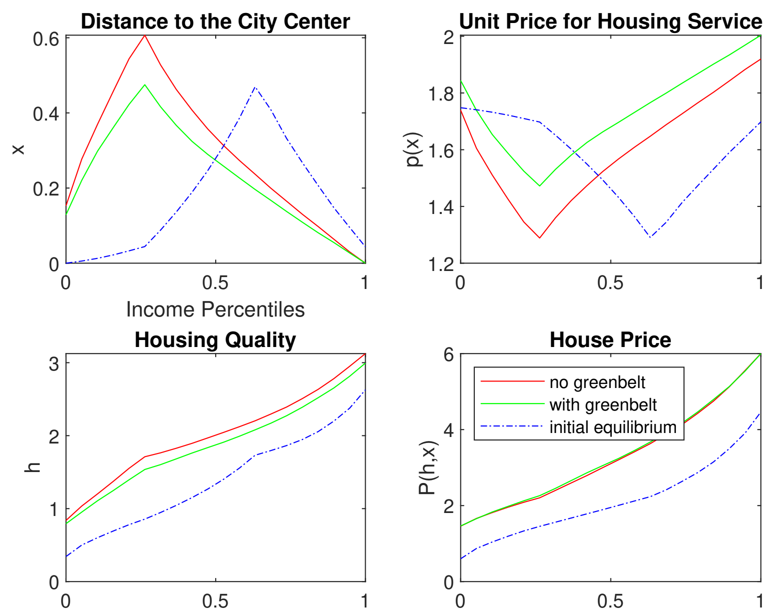
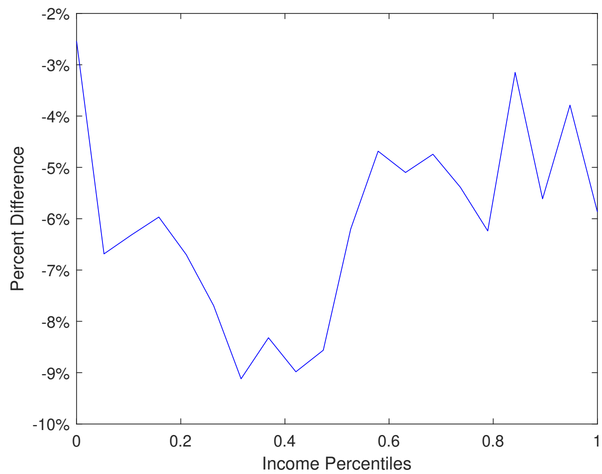


Figure 21: The Effects of the Greenbelt Over Households



Note: Equilibrium outcome over households with and without the greenbelt under the income distribution with higher variance.

Figure 22: The Effects of the Greenbelt: Welfare Loss



Note: This figure shows the percentage change in compensating variation equivalents in consumption units induced by the greenbelt when inequality increases.

house prices, or total housing expenditure, remains the same.

Figure 22 shows the utility loss induced by the greenbelt when inequality increases. The greenbelt reduces the welfare gain from income growth for all households, and the welfare loss is most significant for households who choose to live close to the greenbelt in equilibrium. In particular, for households at the the 25^{th} income percentile, with the greenbelt, welfare gain from income growth is reduced by 9%.

7 Conclusion

This paper presents a theoretical framework which links changes in income distribution to household locations, house prices, and welfare. I develop an assignment approach to the monocentric city model with income-dependent commuting costs and endogenous housing. The model not only generalizes the predictions of the canonical monocentric model with two income types, it is also able to generate imperfect sorting of income across locations that are qualitatively consistent with real-world observations. Specifically, the U-shape income gradients in cities can be interpreted as the overlay of the assignment functions for high- and low-income groups. The model also addresses causes of recent gentrification phenomena in cities: when inequality increases, the changes in equilibrium configurations resemble the gentrification process: income levels near the city center increase, and incumbent poor residents are pushed further from the city center due to higher housing costs.

This paper provides a flexible framework to evaluate the distributional effects of urban policies on city-wide prices and household welfare. According to the model, it is crucial to account for household location choice when designing urban policies. For example, subsidizing the poor households would induce them to move to non-central locations where the supply side is more flexible to adjust. The effects of a zoning policy depend on where the policy is implemented, and which households choose to live close to the location that the policy targets.

The model studies the long-run effects of higher income inequality on the local housing market in a competitive equilibrium framework. There are no market frictions, and housing supply fully adjusts to changes in the economic environment and household characteristics. Under Cobb-Douglas preference over housing and consumption, adjustment in the quality of houses mitigates the effects of higher income inequality on the welfare gap. It would be interesting to explore the welfare implications under different utility functions. For example, if households have to consume a minimum amount of housing, low-income households would be hurt more when higher inequality causes prices to increase.

This paper addresses the distribution of income and housing within cities. There are still lots of important questions in this topic that worth further explorations. With micro-level data on household and neighborhood characteristics available, it is feasible to explore more empirical regularities to guide future work. For example, it would be interesting to look at the various aspects of internal city structure, such as the spatial patterns of housing characteristics and housing costs, and the density and type of neighborhood amenities within cities. Some predictions of the model, for instance, that higher inequality has disproportionate effects on central neighborhoods, can also be empirically tested with data at the census tract level. From the theoretical perspective, the method developed in this thesis is amenable to further extensions, such as how rising inequality and housing costs affects sorting across cities.

References

Gabriel Ahlfeldt and Daniel P McMillen. New estimates of the elasticity of substitution of land for capital. 2014.

William Alonso et al. Location and land use. toward a general theory of land rent. *Location and land use. Toward a general theory of land rent.*, 1964.

Nathaniel Baum-Snow. Suburbanization and transportation in the monocentric model. *Journal of Urban Economics*, 62(3):405–423, 2007.

Nathaniel Baum-Snow. Urban transport expansions, employment decentralization, and the spatial scope of agglomeration economies. *Unpublished Manuscript, Brown University, Providence*, 2014.

Nathaniel Baum-Snow and Daniel Hartley. Accounting for central neighborhood change, 1980-2010. *Journal of Urban Economics*, page 103228, 2019.

Martin J Beckmann. On the distribution of urban rent and residential density. *Journal of Economic Theory*, 1(1):60–67, 1969.

Dan A Black, Natalia Kolesnikova, and Lowell J Taylor. Why do so few women work in new york (and so many in minneapolis)? labor supply of married women across us cities. *Journal of Urban Economics*, 79:59–71, 2014.

Leah Platt Boustan. Was postwar suburbanization “white flight”? evidence from the black migration. *The Quarterly Journal of Economics*, 125(1):417–443, 2010.

Jan K Brueckner and Stuart S Rosenthal. Gentrification and neighborhood housing cycles: will america’s future downtowns be rich? *The Review of Economics and Statistics*, 91 (4):725–743, 2009.

Quentin Brummet and Davin Reed. The effects of gentrification on the well-being and opportunity of original resident adults and children. 2019.

Arnaud Costinot and Jonathan Vogel. Beyond ricardo: Assignment models in international trade. *economics*, 7(1):31–62, 2015.

Victor Couture and Jessie Handbury. Urban revival in america, 2000 to 2010. Technical report, National Bureau of Economic Research, 2017.

Victor Couture, Cecile Gaubert, Jessie Handbury, and Erik Hurst. Income growth and the distributional effects of urban spatial sorting. Technical report, National Bureau of Economic Research, 2019.

Andrea Craig. Commute mode and residential location choice. Technical report, 2019.

Donald R Davis and Jonathan I Dingel. The comparative advantage of cities. *Journal of International Economics*, 123:103291, 2020.

Charles AM De Bartolome and Stephen L Ross. Community income distributions in a metropolitan area. *Journal of Urban Economics*, 61(3):496–518, 2007.

Joseph S DeSalvo. A model of urban household behavior with leisure choice. *Journal of Regional Science*, 25(2):159–174, 1985.

Gilles Duranton, Vernon Henderson, and William Strange. *Handbook of regional and urban economics*. Elsevier, 2015.

Lena Edlund, Cecilia Machado, and Maria Micaela Sviatschi. Gentrification and the rising returns to skill. Technical report, National Bureau of Economic Research, 2015.

Ingrid Gould Ellen and Katherine M O'Regan. How low income neighborhoods change: Entry, exit, and enhancement. *Regional Science and Urban Economics*, 41(2):89–97, 2011.

Ingrid Gould Ellen, Keren Mertens Horn, and Davin Reed. Has falling crime invited gentrification? *Journal of Housing Economics*, 46:101636, 2019.

Masahisa Fujita. *Urban economic theory: land use and city size*. Cambridge university press, 1989.

Edward L Glaeser and Joshua D Gottlieb. Urban resurgence and the consumer city. *Urban studies*, 43(8):1275–1299, 2006.

Edward L Glaeser, Matthew E Kahn, and Jordan Rappaport. Why do the poor live in cities? the role of public transportation. *Journal of urban Economics*, 63(1):1–24, 2008.

Veronica Guerrieri, Daniel Hartley, and Erik Hurst. Endogenous gentrification and housing price dynamics. *Journal of Public Economics*, 100:45–60, 2013.

Paul R Koster and Hans RA Koster. Commuters' preferences for fast and reliable travel: a semi-parametric estimation approach. *Transportation Research Part B: Methodological*, 81:289–301, 2015.

Tim Landvoigt, Monika Piazzesi, and Martin Schneider. The housing market (s) of san diego. *American Economic Review*, 105(4):1371–1407, 2015.

Steven Manson, Jonathan Schroeder, David Van Riper, Steven Ruggles, et al. Ipums national historical geographic information system: Version 12.0 [database]. *Minneapolis: University of Minnesota*, 39, 2017.

Edwin S Mills and Richard Price. Metropolitan suburbanization and central city problems. *Journal of Urban Economics*, 15(1):1–17, 1984.

Espen R Moen. Competitive search equilibrium. *Journal of political Economy*, 105(2): 385–411, 1997.

Aldo Montesano. A restatement of beckmann's model on the distribution of urban rent and residential density. *Journal of Economic Theory*, 4(2):329–354, 1972.

Richard F Muth. Cities and housing; the spatial pattern of urban residential land use. 1969.

Charles Nathanson. Trickle-down housing economics. In *2019 Meeting Papers*, number 537. Society for Economic Dynamics, 2019.

Jordan Rappaport. Monocentric city redux. *Federal Reserve Bank of Kansas City Working Paper*, (14-09), 2014.

Michael Sattinger. Assignment models of the distribution of earnings. *Journal of economic literature*, 31(2):831–880, 1993.

Kenneth A Small, Clifford Winston, and Jia Yan. Uncovering the distribution of motorists' preferences for travel time and reliability. *Econometrica*, 73(4):1367–1382, 2005.

Yichen Su. The rising value of time and the origin of urban gentrification. 2018.

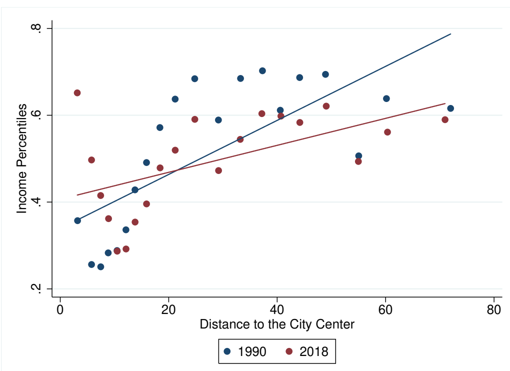
Jacob L Vigdor, Douglas S Massey, and Alice M Rivlin. Does gentrification harm the poor?[with comments]. *Brookings-Wharton papers on urban affairs*, pages 133–182, 2002.

Sevrin Waights. Gentrification and displacement in english cities. 2014.

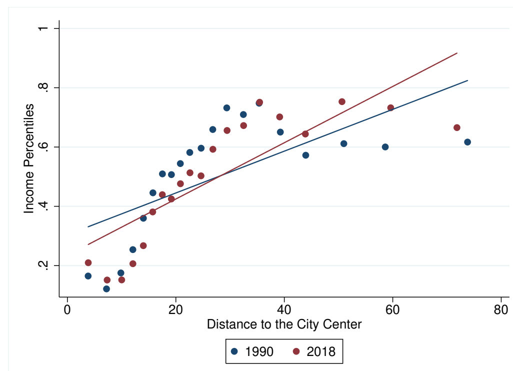
Appendices

A Motivating Facts

Figure A.1: Income-location Relationship in Chicago and Detroit in 1990 and 2018



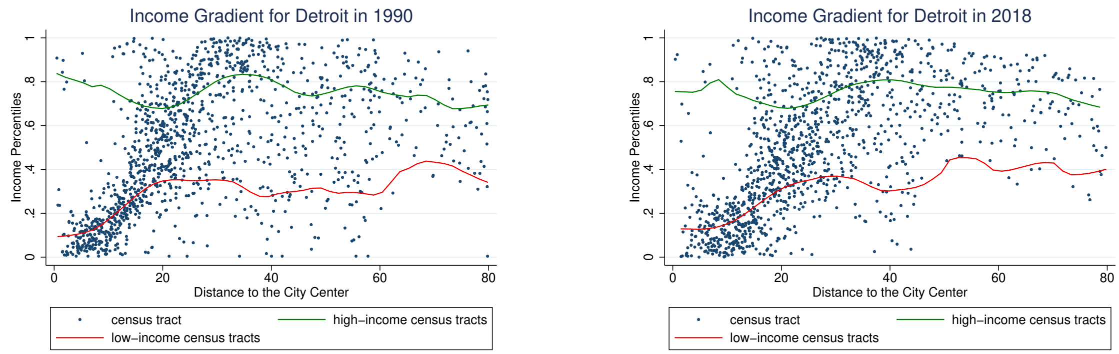
(a) Chicago



(b) Detroit

Notes: This figures are the binscatter plots between the income ranks for census tracts and the distances to the city center for Chicago and Detroit in 1990 and 2018.

Figure A.2: Income Gradients of Detroit in 1990 and 2018



(a) Chicago (b) Detroit

Notes: This figures show the percentile rank of each census trace income against its distance to the city center. Each dot represents a census tract within the Chicago MSA. The lines are income gradients for census tracts above and below the median, respective, which are the kernel-weighted local polynomial smoothing curve, and Epanechnikov kernel functions.

B Mathmatical Proofs

B.1 Proof for Proposition 1

Proof. First let us consider the case when $\frac{\partial v(y,x)}{\partial y \partial x} > 0$. Denote the optimal location of households with income level y_0 as $x^*(y_0)$. From the first order condition of the households' optimal location choice 10, we have

$$\left.\frac{\partial v(y_0, x)}{\partial x}\right|_{x=x^*(y_0)} = 0.$$

Suppose within the city, there is another location $x'(y)$ that yields the same utility as $x^*(y)$: $v(y_0, x'(y)) = \bar{u}$. Without loss of generality, let's assume that $x'(y) > x^*(y)$. Because there is a continuum of incomes, there exists some income level $\tilde{y} = y_0 - \Delta$, with $\Delta > 0$, such that when $\Delta \to 0$, $\tilde{y} \to y_0$. Assume that $x^*(y)$ is a continuous function, we have

$$\lim_{\tilde{y} \to y_0} x^*(\tilde{y}) = x^*(y_0) = x'(y_0).$$

If $\tilde{y}$ is arbitrarily close to y_0, then the optimal location of $\tilde{y}$ should also be $x^*(y_0)$ and $x'(y_0)$. Because $\frac{\partial v(y,x)}{\partial y \partial x} > 0$, $\frac{\partial v(y,x)}{\partial x}$ is increasing in y. As $\tilde{y} < y_0$, we have

$$\lim_{\tilde{y} \to y^-} \left.\frac{\partial v(\tilde{y}, x)}{\partial x}\right|_{x=x'(y_0)} < \left.\frac{\partial v(y, x)}{\partial x}\right|_{x=x'(y_0)} = 0,$$

which means that $\tilde{y}$ would be better off moving to a location $x < x'(y_0)$, thus $x'(y_0)$ cannot be an optimal location for income level y_0.

Similarly, it can be shown that when $\frac{\partial v(y,x)}{\partial y \partial x} < 0$, the optimal location for each income is unique. □

B.2 Proof for Lemma 2

Proof. For the assignment function $y^*(x)$ and the corresponding price $p(x)$ to be an equilibrium, both the first order condition (11) and the second order condition (12) of the household's maximization problem must hold. After plugging in the assignment function $y(x)$, the first order condition becomes

$$\frac{\partial v(y, y^*(x))}{\partial x} = 0. \tag{B.1}$$

Differentiating the above expression with respect to x yields

$$\frac{\partial^2 v(y^*(x), x)}{\partial x \partial y} \left(\frac{dy^*(x)}{dx} \right) + \frac{\partial^2 v(y^*(x), x)}{\partial x^2} = 0,$$

after re-organizing,

$$\frac{\partial^2 v(y^*(x),x)}{\partial x^2} = -\frac{\partial^2 v(y^*(x),x)}{\partial x \partial y}\left(\frac{dy^*(x)}{dx}\right) < 0.$$

If $\frac{\partial v(y,x)}{\partial x \partial y} > 0$ for all x and all y in the income range, then $\frac{\partial v(y^*(x),x)}{\partial x \partial y} > 0$. If $\frac{\partial v(y,x)}{\partial x \partial y} < 0$ for all x and all y in the income range, then $\frac{\partial v(y^*(x),x)}{\partial x \partial y} < 0$. So the derivative of the assignment $y^*(x)$ has the same sign with the partial derivative of the indirect utility function. □

B.3 Proof for Proposition 3

Proof. It is easy to show that $\frac{\partial v(y,x)}{\partial y \partial x} > 0$ is equivalent to

$$g(a,\gamma,y) = \frac{a}{\tau(y)} + 1 - \gamma > 0.$$

If $g(a,\gamma,y) > 0$ for all $y \in \left[\underline{y}, \bar{y}\right]$, then in equilibrium, income increases with distance. Notice that when $\gamma < 1$, $g(a,\gamma,y) = \frac{a}{\tau(y)} > 0$ for all y. When $\gamma > 1$, $g(a,\gamma,y)$ is a decreasing function in y. If $f(\bar{y}) > 0$, then $f(y) > 0$ for all y. This is equivalent to $a > \tau(\bar{y})(\gamma - 1)$.

Similarly, we can show that when $\gamma > 1$ and $a < a_0 = \tau(\underline{y})(\gamma - 1)$, $g(a,\gamma,y) < 0$, which implies that income decreases with distance in equilibrium. □

B.4 Proof for Proposition 4

Proof. The first part of the proposition is straightforward from the proof for Proposition 3. At $y = \hat{y}$, it must be that

$$g(y) = \frac{a}{\tau(y)} + 1 - \gamma = 0,$$

from which we can obtain that $\hat{y} = \left[\frac{a}{\rho(\gamma-1)}\right]^{\frac{1}{\gamma}}$. When $\underline{y} \leq y \leq \hat{y}$, we have g(y) $>$ 0. By the Lemma 2, $\frac{dy_L^*(x)}{dx} > 0$. When $\hat{y} \leq y \leq \bar{y}$, we have $\frac{dy_H^*(x)}{dx} < 0$.

Let us proceed with the second part of the proposition. Denote the assignment function as $y_L^*(x)$ when $\underline{y} \leq y \leq \hat{y}$, and as $y_H^*(x)$ when $\hat{y} < y \leq \bar{y}$. If both $y_H^*(x)$ and $y_L^*(x)$ choose to live at x in equilibrium, $y_H^*(x)$ and $y_L^*(x)$ must satisfy the price match condition (14), which can be re-organized as

$$y_H^*(x)\left[a + \rho\left(y_L^*(x)\right)^{\gamma}\right] = y_L^*(x)\left[a + \rho\left(y_H^*(x)\right)^{\gamma}\right].$$

It is convenient to treat one of the $y^*(x)$'s as a parameter b, with $b \in \left[\underline{y}, \bar{y}\right]$. The price match condition can be re-written as follows:

$$h(y; b) = -\rho b y^{\gamma} + (a + \rho b^{\gamma})y - ab = 0. \tag{B.2}$$

For any b, the solution to $h(y; b) = 0$ gives the income level of the households who is willing to live at the same location with households $y = b$. Characterizing the relationship between $y_H^*(x)$ and $y_L^*(x)$ is equivalent to characterizing the solution to (B.2).

First, it is easy to see that $y = b$ is always a solution to (B.2). Let us denote the other solution to (B.2) as $y_c(b)$: for any b, $y_c(b)$ is the income level of households that live at the same location.

It is easy to show that $h(y; b)$ is concave function with two positive roots when $\gamma > 1$, as

$$h'(y; b) = -\gamma\rho b y^{\gamma-1} + (a + \rho b^{\gamma}),$$

$$h''(y; b) = -\gamma(\gamma - 1)\rho b y^{\gamma-2} < 0.$$

Plugging in $b = \hat{y} = \left[\frac{a}{\rho(\gamma-1)}\right]^{\frac{1}{\gamma}}$ into $h(y; b) = 0$, we get

$$h'(\hat{y}; b = \hat{y}) = -\gamma b \hat{y}^{\gamma} + (a + \rho \hat{y}^{\gamma}) = 0,$$

which means that when $y_c(\hat{y}) = \hat{y}$: $\hat{y}$ is the only solution to $h(y; b) = 0$ when $b = \hat{y}$. Because $y_H^*(x)$ is decreasing in x, and $y_L^*(x)$ is increasing in x, the two assignment functions can only intersect at the city edge $\bar{x}$, with the income level being $\hat{y}$.

Finally, let's determine the income level of households that reside at the same location with the poorest households. When $b = \underline{y}$, we have

$$h'(\underline{y}; b = \underline{y}) = a + (1 - \gamma)\rho \underline{y}^{\gamma}.$$

Recall that $a > a_0 = \rho \underline{y}^{\gamma}(\gamma - 1)$, we have $h'(\underline{y}; b = \underline{y}) > 0$. Because $h(y; b)$ is a concave function, This implies that $y = \underline{y}$ is the smaller root for $h(y; b = \underline{y}) = 0$, so $\underline{y} < y_c(\underline{y})$.

We then need to determine the relationship between $y_c(\underline{y})$, and the highest income level $\bar{y}$. We know that $y_c(\underline{y}) = \bar{y}$, if and only if $h(\bar{y}; b = \underline{y}) = 0$, that is

$$h(\bar{y}; b) = -\rho \underline{y} \bar{y}^{\gamma} + (a + \rho \underline{y}^{\gamma} \bar{y}) - a\underline{y} = 0.$$

This can be re-arranged as

$$\frac{T_x(\bar{y})}{\bar{y}} = \frac{T_x(\underline{y})}{\underline{y}}.$$

This condition says that, when the marginal commuting costs relative to income level are equalized for $y = \underline{y}$ and $y = \bar{y}$, in equilibrium, there are two income levels at each location $x \in [0, \bar{x})$, and $y_H^*(0) = \bar{y}$, $y_L^*(0) = \underline{y}$. Similarly, $y_c(\underline{y}) < \bar{y}$, if and only if $h(\bar{y}; b = \underline{y}) < 0$;

this is equivalent to

$$\frac{T_x(\bar{y})}{\bar{y}} > \frac{T_x(\underline{y})}{\underline{y}}.$$

If the marginal commuting costs relative to income is greater for the richest households, in equilibrium, the poorest households would not match the unit prices at locations near the city center, and they live at $x = x_c > 0$. Lastly, $y_c(\underline{y}) > \bar{y}$, if and only if $h(\bar{y}; b = \underline{y}) > 0$; this is equivalent to

$$\frac{T_x(\bar{y})}{\bar{y}} < \frac{T_x(\underline{y})}{\underline{y}}.$$

If the marginal commuting costs relative to income is greater for the richest households, in equilibrium, the richest households live at $x = x_c$. □

C Existence and Uniqueness of Equilibrium

The existence and uniqueness of equilibrium can be illustrated by the boundary rent curve approach as in Fujita (1989). As an illustrative example, let us consider the equilibrium configuration where income increases with distance. The solution to the model under this equilibrium configuration is represented by

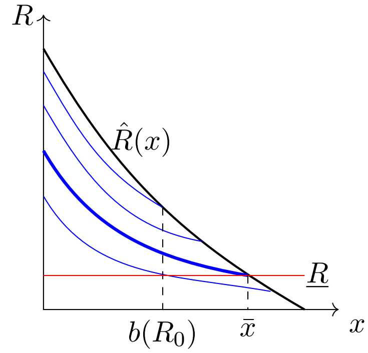
$$\frac{dR(x)}{dx} = \frac{-T_x(y^*(x), x)R(x)}{\alpha\beta\left[y^*(x) - T(y^*(x), x)\right]}, \tag{C.1}$$

$$\frac{dy^*(x)}{dx} = \frac{R(x)L(x)}{Nf(y^*(x))\alpha\beta\left[y^*(x) - T(y^*(x), x)\right]}, \tag{C.2}$$

with the initial condition $y^*(0) = \bar{y}$, and the terminal condition $y^*(\bar{x}) = \underline{y}$, and $R(\bar{x}) = \underline{R}$.

First of all, for a given value for the rental price at the city center $R(0) = R_0$, the

Figure C.3: The Boundary Rent Curve



Note: The boundary rent curve $\hat{R}(x)$ is represented by the dark line. Blue lines are the simulated rental prices for land across locations under different values of R_0, The highlighted one that intersects with $R = \underline{R}$ represents the equilibrium rental price for land.

solution to the system of ordinary differential equations is unique. Under different value of R_0, we simulate the above differential equations and obtain a sequence of income levels and rental prices. At the point where the income level equals $\bar{y}$, we obtain the city boundary $b(R_0)$ under R_0, such that all population are accommodated within this boundary point. By varying R_0, we obtain different values for the boundary points b. The boundary rent curve $\hat{R}(x)$ is the rental price at these boundary points.

Figure C.3 shows the rental price for different values of R_0, and the boundary rent curve $\hat{R}(x)$. In equilibrium, the market land rent must equal $\bar{R}$ at the city boundary. The equilibrium city boundary $\bar{x}$, and the equilibrium rental price $R(x)$ is given by the point where the boundary rent curve $\hat{R}(x)$ meets $R = \underline{R}$. The uniqueness and existence equilibrium for other equilibrium configurations can be illustrated in the same way.

The procedure to construct the boundary rent curve naturally lends to a numerical algorithm to solve the model. Consider the equilibrium where income increases with distance, the model can be solved by the following steps:

1. Guess some R_0 for rental price at $x = 0$ and simulate the differential equations to

obtain $y(x)$ and $R(x)$

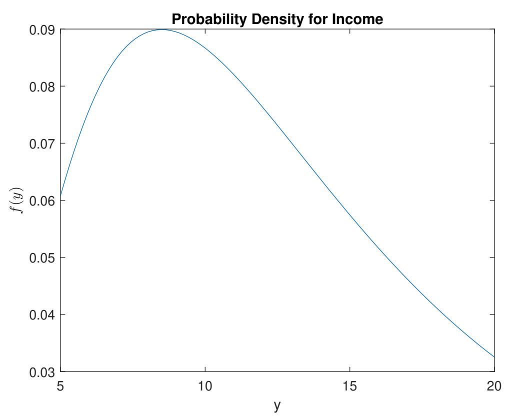
2. Obtain the boundary point $b = b(R_0)$ such that $y(b) = \bar{y}$

3. If $R(b) > \underline{R}$, decrease the value for R_0 and repeat the procedure starting from the first step; if $R(b) < \underline{R}$, increase the value for R_0 and repeat the procedure from the first step; if $R(b) = \underline{R}$, we have found the equilibrium.

D Numerical Solutions

D.1 Income Distribution: Both the Richest and Poorest at the City Center

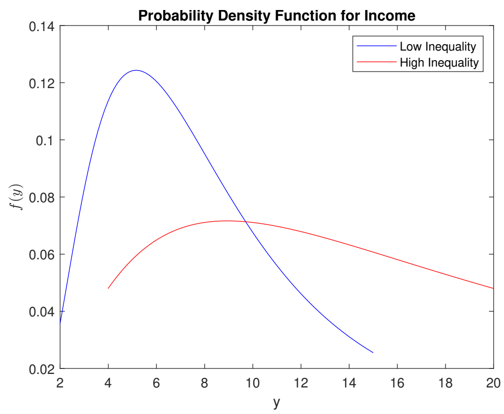
Figure D.4: Probability Density for Income Distribution



Note: This figure shows the probability density for the truncated lognormal distribution when $\bar{y} = 20$, $\underline{y} = 3.99$, $\mu = 3$, *and* $\sigma = 0.9$.

D.2 Gentrification without Displacement of the Poorest Households

Figure D.5: Income Distributions: Gentrification without Displacement of the Poorest Households



D.3 Gentrification with Displacement of the Poorest Households

Figure D.6: Income Distributions: Gentrification without Displacement of the Poorest Households