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The Decomposition of a GDP Increase

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Accounting for a GDP Increase

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Abstract

We account for the price and quantity components of a decomposition of the expression for a GDP increase accross periods. Our linear measure is compared with Irving Fisher's multiplicative measure. Where technical progress enters is noted.

- key words: price and quantity components; linear decomposition

1 The Decomposition of a GDP Increase

Suppose we have an annual change in GDP over two years, as in

$$\Gamma = \frac{p_1^1 q_1^1 + p_2^1 q_2^1 + \ ... \ + p_n^1 q_n^1}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}.$$

We have n commodities q_i^t with prices p_i^t recorded in consecutive periods. $\Gamma > 1$ is associated with an increase in GDP. We observe that some of the increase expressed in Γ is due to prices changing between periods and some due to quantities changing. We are interested in factoring Γ precisely into a price effect and a quantity effect. Irving Fisher discovered an elegant approach to this exercise: the product of his Ideal quantity index, I^Q and Ideal price index, I^P came to Γ. See the following:

$$\begin{aligned} I^P \quad = \quad & \{[\frac{p_1^1 q_1^0 + p_2^1 q_2^0 + ... + p_n^1 q_n^0}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}] \\ & \times [\frac{p_1^1 q_1^1 + p_2^1 q_2^1 + \ ... \ + p_n^1 q_n^1}{p_1^0 q_1^1 + p_2^0 q_2^1 + ... + p_n^0 q_n^1}]\}^{(1/2)} \end{aligned}$$

*I am indebted to Bert Balk and Robert Cairns for helpful comments.

and

$$I^Q = \{[\frac{p_1^0 q_1^1 + p_2^0 q_2^1 + \ ... \ + p_n^0 q_n^1}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}] \times [\frac{p_1^1 q_1^1 + p_2^1 q_2^1 + \ ... \ + p_n^1 q_n^1}{p_1^1 q_1^0 + p_2^1 q_2^0 + \ ... \ + p_n^1 q_n^0}]\}^{(1/2)}.$$

Each of I^P and I^Q is a geometric mean. The chained version of I^Q forms the framework[1] for the construction of Canada's GDP at Statistics Canada.[2]

We however are interested in a LINEAR decomposition of Γ into price and quantity components.[3] To this end, we define ΔQ in $\Gamma - L^P$, for L^P a Laspeyres price index. We have then

$$\begin{aligned} \Gamma &= L^P + \Delta Q \\ \text{for } L^P &\equiv \frac{p_1^1 q_1^0 + p_2^1 q_2^0 + \ ... \ + p_n^1 q_n^0}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0} \\ \text{and } \Delta Q &\equiv \frac{p_1^1[q_1^1 - q_1^0] + p_2^1[q_2^1 - q_2^0] + \ ... \ + p_n^1[q_n^1 - q_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}. \end{aligned}$$

This "reads": GDP-increase equals a price change term (a Laspeyres price index) plus a quantity-change term, ΔQ. We can then write a similar identity with first a Laspeyres quantity index. This defines ΔP.

$$\begin{aligned} \Gamma &= L^Q + \Delta P \\ \text{for } L^Q &\equiv \frac{p_1^0 q_1^1 + p_2^0 q_2^1 + \ ... \ + p_n^0 q_n^1}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0} \\ \text{and } \Delta P &\equiv \frac{q_1^1[p_1^1 - p_1^0] + q_2^1[p_2^1 - p_2^0] + \ ... \ + q_n^1[p_n^1 - p_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}. \end{aligned}$$

We have again a linear decomposition of GDP-change, now into a Laspeyres quantity index plus a price-change term.

[1] In the Consumer Price Index, the base quantities (a standardized vector of purchases by a representative consumer) stay the same as each new "time-term" is introduced as time moves forward, period by period. Chaining, in contrast, has the base period quantity updated period by period. The current quantity vector in period t becomes the base quantity vector in period $t+1$. This is chaining of a price index. For chaining a quantity index, the quantity vector above is replaced by the price vector.

[2] See Statistics Canada 13-017, Section 2.3 "Other aspects of the Income and Expenditure Accounts Data: Measures of volume and prices in the Income and Expenditure Accounts".

[3] An early formulation is due to Bennett (1920). See Diewert (2005) and Balk (2012; p. 130).

Since $L^P = 1$ for no prices changing between periods, we focus on $L^P - 1$. $L^P - 1$ might be 0.04. Hence we have

$$\Gamma - 1 = \{L^P - 1\} + \Delta Q.$$

Similar steps we followed above lead to

$$\Gamma - 1 = \{L^Q - 1\} + \Delta P.$$

We proceed to switch ΔP and ΔQ. Our linear decomposition of $\Gamma - 1$ is now

$$\Gamma - 1 = \frac{1}{2}\left[\{L^P - 1\} + \Delta P\right] + \frac{1}{2}\left[\{L^Q - 1\} + \Delta Q\right].$$

The identify our price term in $\Gamma - 1$ as

$$\begin{aligned} & \frac{1}{2}\left[\{L^P - 1\} + \Delta P\right] \\ = \quad & \frac{1}{2}[\{\frac{q_1^0[p_1^1 - p_1^0] + q_2^0[p_2^1 - p_2^0] + \ ... \ + q_n^0[p_n^1 - p_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}\} \\ & + \frac{q_1^1[p_1^1 - p_1^0] + q_2^1[p_2^1 - p_2^0] + \ ... \ + q_n^1[p_n^1 - p_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}], \end{aligned}$$

and the quantity term as

$$\begin{aligned} & \frac{1}{2}\left[\{L^Q - 1\} + \Delta Q\right] \\ = \quad & \frac{1}{2}[\{\frac{p_1^0[q_1^1 - q_1^0] + p_2^0[q_2^1 - q_2^0] + \ ... \ + p_n^0[q_n^1 - q_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}\} \\ & + \frac{p_1^1[q_1^1 - q_1^0] + p_2^1[q_2^1 - q_2^0] + \ ... \ + p_n^1[q_n^1 - q_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}]. \end{aligned}$$

We have then a decomposition of $\Gamma - 1$ into price and quantity terms. We can simplify a little. We take each 1/2 inside the appropriate square brackets. We then have an average of the two prices, $\frac{p_i^0 + p_i^1}{2} = \overline{p}_i$ in one case, and of the two quantities, $\frac{q_i^0 + q_i^1}{2} = \overline{q}_i$ in the other case. This leads to

$$\begin{aligned} \frac{1}{2}\left[\{L^P - 1\} + \Delta P\right] &= \frac{\overline{q}_1[p_1^1 - p_1^0] + \overline{q}_2[p_2^1 - p_2^0] + \ ... \ + \overline{q}_n[p_n^1 - p_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0} \\ \text{and } \frac{1}{2}\left[\{L^Q - 1\} + \Delta Q\right] &= \frac{\overline{p}_1[q_1^1 - q_1^0] + \overline{p}_2[q_2^1 - q_2^0] + \ ... \ + \overline{p}_n[q_n^1 - q_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}. \end{aligned}$$

Our basic linear decomposition is now the sum of a price-change term and a quantity-change term:

$$\Gamma - 1 = \frac{\overline{q}_1[p_1^1 - p_1^0] + \overline{q}_2[p_2^1 - p_2^0] + \ ... \ + \overline{q}_n[p_n^1 - p_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0} + \frac{\overline{p}_1[q_1^1 - q_1^0] + \overline{p}_2[q_2^1 - q_2^0] + \ ... \ + \overline{p}_n[q_n^1 - q_n^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + \ ... \ + p_n^0 q_n^0}.$$

We have linear averages of quantities and prices in our expression whereas Fisher's multiplicative decomposition involved the product of two geometric averages.

EXAMPLE : We take up a three commodity numerical example.

$$p_1^0 = 1,\ p_2^0 = 2,\ p_3^0 = 4,\ q_1^0 = 9,\ q_2^0 = 7,\ q_3^0 = 8$$

$$\text{and } p_1^1 = 2,\ p_2^1 = 3,\ p_3^1 = 6,\ q_1^1 = 10,\ q_2^1 = 9,\ q_3^1 = 7.$$

These data give us $\Gamma = 89/55$. The Laspeyres indices are $L^P = 87/55$ and $L^Q = 56/55$. The Δ terms are $\Delta P = 33/55$ and $\Delta Q = 2/55$. This implies $\left[L^P + \Delta P\right] = 120/55$ and $\left[L^Q + \Delta Q\right] = 58/55$ and these figures yield $\frac{1}{2}\left[L^P + \Delta P\right] + \frac{1}{2}\left[L^Q + \Delta Q\right] = 89/55$, which is the value of Γ which we calculated directly at the outset. The quantity part, $\frac{\overline{p}_1[q_1^1 - q_1^0] + \overline{p}_2[q_2^1 - q_2^0] + \overline{p}_3[q_3^1 - q_3^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + p_3^0 q_3^0}$ of the decomposition of $\Gamma - 1$ works out to be 0.02727273. The corresponding Fisher term is:

$$I^Q = \{[\frac{p_1^0 q_1^1 + p_2^0 q_2^1 + p_3^0 q_3^1}{p_1^0 q_1^0 + p_2^0 q_2^0 + p_3^0 q_3^0}] \times [\frac{p_1^1 q_1^1 + p_2^1 q_2^1 + p_3^1 q_3^1}{p_1^1 q_1^0 + p_2^1 q_2^0 + p_3^1 q_3^0}]\}^{(1/2)}$$

which works out to be 1.020582332. This compares with 1.02727273 for the comparable term for the linear decomposition above. The difference between measures appears in the third decimal place.

The price component of our linear decomposition namely $\frac{\overline{q}_1[p_1^1 - p_1^0] + \overline{q}_2[p_2^1 - p_2^0] + \overline{q}_3[p_3^1 - p_3^0]}{p_1^0 q_1^0 + p_2^0 q_2^0 + p_3^0 q_3^0}$, works out to be 0.5909091. The sum of the quantity and price terms is 34/55=0.618181818, which is $\Gamma - 1$. The price component of the Fisher multiplicative decomposition is

$$I^P = \{[\frac{p_1^1 q_1^0 + p_2^1 q_2^0 + p_3^1 q_3^0}{p_1^0 q_1^0 + p_2^0 q_2^0 + p_3^0 q_3^0}] \times [\frac{p_1^1 q_1^1 + p_2^1 q_2^1 + p_3^1 q_3^1}{p_1^0 q_1^1 + p_2^0 q_2^1 + p_3^0 q_3^1}]\}^{(1/2)}$$

which works out to be 1.58554755. (This compares with the relevant term for our linear decomposition, namely 1.5909091.) And the product of I^Q and I^p is 1.618181818, the value of Γ. □

2 Alternative Expressions

We can express $\Gamma - 1$ in a Divisia-like formulation. We have $\Gamma - 1 = \frac{\Sigma p_i^1 q_i^1 - \Sigma p_i^0 q_i^0}{\Sigma p_i^0 q_i^0}$ and we get

$$\begin{aligned}\Gamma - 1 &= \frac{\overline{q}_1 p_1^0 q_1^0}{q_1^0 \Sigma p_i^0 q_i^0}[\frac{p_1^1 - p_1^0}{p_1^0}] + \frac{\overline{q}_2 p_2^0 q_2^0}{q_2^0 \Sigma p_i^0 q_i^0}[\frac{p_2^1 - p_2^0}{p_2^0}] + ... + \frac{\overline{q}_n p_n^0 q_n^0}{q_n^0 \Sigma p_i^0 q_i^0}[\frac{p_n^1 - p_n^0}{p_n^0}] \\ &+ \frac{\overline{p}_1 p_1^0 q_1^0}{p_1^0 \Sigma p_i^0 q_i^0}[\frac{q_1^1 - q_1^0}{q_1^0}] + \frac{\overline{p}_2 p_2^0 q_2^0}{p_2^0 \Sigma p_i^0 q_i^0}[\frac{q_2^1 - q_2^0}{q_2^0}] + ... + \frac{\overline{p}_n p_n^0 q_n^0}{p_n^0 \Sigma p_i^0 q_i^0}[\frac{q_n^1 - q_n^0}{q_n^0}];\end{aligned}$$

which we write as

$$\begin{aligned}\Gamma - 1 &= \frac{\overline{q}_1}{q_1^0} s_{q1}[\frac{p_1^1 - p_1^0}{p_1^0}] + \frac{\overline{q}_2}{q_2^0} s_{q2}[\frac{p_2^1 - p_2^0}{p_2^0}] + ... + \frac{\overline{q}_n}{q_n^0} s_{qn}[\frac{p_n^1 - p_n^0}{p_n^0}] \\ &+ \frac{\overline{p}_1}{p_1^0} s_{p1}[\frac{q_1^1 - q_1^0}{q_1^0}] + \frac{\overline{p}_2}{p_2^0} s_{p2}[\frac{q_2^1 - q_2^0}{q_2^0}] + ... + \frac{\overline{p}_n}{p_n^0} s_{pn}[\frac{q_n^1 - q_n^0}{q_n^0}],\end{aligned}$$

for s_{qi} and s_{pi} value shares for the base period. It is still correct to interpret the first line as the price-change term, same as above, and the second as the quantity-change term, same as above, in a decomposition of $\Gamma - 1$. Novel of course are the weights $\frac{\overline{q}_i}{q_i^0}$ and $\frac{\overline{p}_i}{p_i^0}$, $i = 1, ..., n$. A naive version of decomposing $\Gamma - 1$ is simply

$$\begin{aligned}& s_{q1}[\frac{p_1^1 - p_1^0}{p_1^0}] + s_{q2}[\frac{p_2^1 - p_2^0}{p_2^0}] + ... + s_{qn}[\frac{p_n^1 - p_n^0}{p_n^0}] \\ &+ s_{p1}[\frac{q_1^1 - q_1^0}{q_1^0}] + s_{p2}[\frac{q_2^1 - q_2^0}{q_2^0}] + ... + s_{pn}[\frac{q_n^1 - q_n^0}{q_n^0}]\end{aligned}$$

which is presumably not too bad an approximation to the result with the correct expression.

We can consider productivity increase by substituting $q_i^1 = (1 + g_i^0 + \lambda_i^0) q_i^0$ $(i = 1, ..., n)$ with g_i^0 the growth in i attributable to raw input increase and λ_i^0 the growth in i attributable to technical progress. Then we get

$$\begin{aligned}\Gamma - 1 &= [1 + \frac{g_1^0 + \lambda_1^0}{2}] s_{q1}[\frac{p_1^1 - p_1^0}{p_1^0}] + [1 + \frac{g_2^0 + \lambda_2^0}{2}] s_{q2}[\frac{p_2^1 - p_2^0}{p_2^0}] + ... \\ &+ [1 + \frac{g_n^0 + \lambda_n^0}{2}] s_{qn}[\frac{p_n^1 - p_n^0}{p_n^0}] + s_{p1}[g_1^0 + \lambda_1^0] + s_{p2}[g_2^0 + \lambda_2^0] + ... + s_{pn}[g_n^0 + \lambda_n^0]. \\ &+ s_{pn}[g_n^0 + \lambda_n^0].\end{aligned}$$

This is of course a statement and not a formula for obtaining a value for $\Gamma - 1$, since one now requires values for g_i^0 and λ_i^0 in order to calculate a value for $\Gamma - 1$.

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