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# Solow Sustainablility with Varying Population Levels

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Cover QED Discussion Paper

Two Insertions for

John M. Hartwick, Advanced Introduction to

NATIONAL ACCOUNTING

(Elgar, Cheltenham, 2020)

Insert p. 76

Hartwick, Advanced Introduction to National Accounting, Elgar, 2020.

# Quadratic Approximation

September 16, 2019

Let

$$f(z) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} z_i z_j + \sum_{i=1}^{N} b_i z_i + c,$$

where $a_{ij} = a_{ji}$. Then

$$\begin{aligned} f(z) - f(y) &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(z_i z_j - y_i y_j) + \sum_{i=1}^{N} b_i(z_i - y_i) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(z_i z_j - z_i y_j + z_i y_j - y_i y_j) + \sum_{i=1}^{N} b_i(z_i - y_i) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left[z_i(z_j - y_j) + y_j(z_i - y_i)\right] + \sum_{i=1}^{N} b_i(z_i - y_i). \end{aligned}$$

But because of the symmetry $a_{ij} = a_{ji}$,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} z_i (z_j - y_j) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} z_j (z_i - y_i).$$

So

$$\begin{aligned} f(z) - f(y) &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} z_j (z_i - y_i) + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} y_j (z_i - y_i) + \sum_{i=1}^{N} b_i(z_i - y_i) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(z_i - y_i)(z_j + y_j) + \sum_{i=1}^{N} b_i(z_i - y_i) \\ &= \sum_{i=1}^{N}(z_i - y_i)\left[b_i + \sum_{j=1}^{N} a_{ij}(z_j + y_j)\right]. \end{aligned}$$

Now calculate derivatives. The terms containing $z_i$ in $f(z)$ are

$$a_{ii}\left(z_i\right)^2 + a_{i1}z_iz_1 + \cdots + a_{i,i-1}z_iz_{i-1} + a_{i,i+1}z_iz_{i+1} + \cdots + a_{iN}z_iz_N$$
$$+ a_{1i}z_1z_i + \cdots + a_{i-1,i}z_{i-1}z_i + a_{i+1,i}z_{i+1}z_i + \cdots + a_{Ni}z_Nz_i + b_iz_i.$$

So,

$$\frac{\partial f}{\partial z_i} = 2a_{ii}z_i + a_{i1}z_1 + \cdots + a_{i,i-1}z_{i-1} + a_{i,i+1}z_{i+1} + \cdots + a_{iN}z_N + a_{1i}z_1 + \cdots + a_{i-1,i}z_{i-1} + a_{i+1,i}z_{i+1} + \cdots + a_{Ni}z_N + b_i.$$

But $a_{i1} = a_{1i}$, etc. So

$$\begin{aligned}\frac{\partial f}{\partial z_i} &= 2a_{ii}z_i + 2\left(a_{i1}z_1 + \cdots + a_{i,i-1}z_{i-1} + a_{i,i+1}z_{i+1} + \cdots + a_{iN}z_N\right) + b_i \\ &= 2\sum_{j=1}^{N} a_{ij}z_j + b_i.\end{aligned}$$

Similarly,

$$\frac{\partial f}{\partial y_i} = 2\sum_{j=1}^{N} a_{ij}y_j + b_i,$$

and so

$$\frac{1}{2}\left(\frac{\partial f}{\partial z_i} + \frac{\partial f}{\partial y_i}\right) = \sum_{j=1}^{N} a_{ij}(z_j + y_j) + b_i.$$

Finally,

$$f(z) - f(y) = \frac{1}{2}\sum_{i=1}^{N}(z_i - y_i)\left(\frac{\partial f}{\partial z_i} + \frac{\partial f}{\partial y_i}\right),$$

or, in vector notation,

$$f(z) - f(y) = \langle (z-y), \frac{1}{2}\left(\nabla f(z) + \nabla f(y)\right)\rangle.$$

Replace pp. 138-142 with this material

October 18, 2021

Solow Sustainability with Varying Population Levels


**Abstract**

We take up three variants of Solow [1974], each with population change endogenous. When each model exhibits sustainability the same three conditions are satisfied: (i) investment in produced capital is funded by resource rents plus "extra" saving, (ii) "extra" saving funds the same two gaps related to poppulation increase and (iii) Hotelling's Rule is satisfied. We focus attention on condition (ii) here. The Stollery variant involves warming caused by current hydrocarbon extraction.



- key words: sustainability; population increase; funding gaps
- JEL classification: Q010; Q320; Q290


## 1. Introduction

Asheim et. al. (2007) re-worked Solow sustainability (Solow (1974)) with an endogenous population level changing over time. We derive a population-change condition for a variant of that model, an equilibrium condition which "re-appears" for two variants of Solow (1974), namely the model of Stollery (1998) with a global warming level endogenous and that of d'Autume and Schubert (2008) with the resource stock yielding utility at each date. The familiar Solow (1974) reduces to three conditions (i) invest resource rents to accumulate produced capital (ii) invoke Hotelling's Rule for efficient use of exhaustible resource stock capital and (iii) require aggregate consumption to be unchanging. Sustainability in this model is achieved when conditions (i) and (ii) are satisfied.[1] Here with population changing in Solow (1974) with the passing of time, there is a fourth condition, (iv) the efficient use "extra" or "supplementary" savings; that is, savings that is above that from current exhaustible resource rent. [[ This condition on "efficient" use of the supplementary savings reduces to the equation for "equilibrium" population change and is the subject of this contribution.]] We first take up Solow (1974) with the population level changing and observe the new condition for sustainability involving "efficient" use of "extra" saving. We observe how this equation "reduces" to the equation of "equilibrium population change". We then turn to sustainability in the variants of d'Autume-Schubert and Stollery, each now extended, with an endogenous population level changing over time. Population was unchanging in the versions of d'Autume-Schubert and Stollery that have already been published.

## 2. The Model

First, the Solow model with the level of population changing. Current population comprises workers who are also consumers and people can be drawn into the model of tossed out of the model at no direct cost.[2] Extra people are available "on the sidelines" at any date and are drawn in or tossed out by planner, herself in the background. We are of course concerned with admissable population levels for each date. The objective function in the first model is unchanging per capita consumption. Production is governed by a neo-classical production funtion: $Q =$

[1] See Buchholz, Dasgupta and Mitra (2005).

[2] Arrow, Dasgupta and Maler (2003) take up sustainability with population governed by a given growth function. Population in our model is a number of people that a planner can impose at any date. This is analogous to the static problem of Meade in which an optimal number of people for the economy is invoked or lands by helicopter. In our model the Meade approach for a single date operates over an infinity of dates, an "optimal" population analogous to Meade's, but at each date. Of course if the model is asking for population of decline we assume that the "optimally" discarded people can indeed be removed from the model or the model economy "freely". We report on Meade's optimal population problem in Appendix 2.

$F(K, R, N)$ for $Q$ aggregate output in the economy at date $t$, $K$ is the current stock of produced capital, $R$ is the current input of the exhaustible resource ($R = -\dot{S}$, for $S$ the current stock of the homogeneous input remaining), and $N$ is the current population size (all people are working). Current aggregate consumption $C$ satisfies $C = Q - \dot{K}$, for $\dot{K}$ current investment and augmentation in $K$. There is no decay in $K$. Novel here is the form of savings-investment (invest resource rents plus some extra resource, $sQ$, with $s$ and $Q$ varying over time):[3]

$$\dot{K} = RF_R + sQ, \tag{1}$$

i.e rent plus "extra" savings $sQ$ is used for current investment. (Note that capital $S$ refers to the exhaustible stock at a point of time and non-capital $s$ refers to a level of "extra" saving at a point in time going to current investment in $K$.) Observe then

$$d\dot{K}/dt = d[RF_R]dt + d[sQ]/dt. \tag{2}$$

Dynamic efficiency in exhaustible resource use is governed by (Hotelling Rule):

$$F_K F_R = \dot{F}_R. \tag{3}$$

$F_K$ and $F_R$ are partial derivatives for $K$ and $R$ respectively

We proceed to derive our new condition for sustainability with population endogenous. Per capita consumption unchanging defines sustainability for this model. We draw on the equations above in the analysis of per capita consumption evolving, i.e. in

$$d\left[\frac{C}{N}\right]/dt = \frac{\dot{C}}{N} - \frac{C}{N}\frac{\dot{N}}{N}. \tag{4}$$

[3]In Solow (1974) current saving and investment equalled current exhaustible resource rents alone.

We proceed to isolate a population-change function which is associated with $\frac{C}{N}$ unchanging. First we have

$$\begin{aligned}
\dot{C} &= \dot{Q} - d\dot{K}/dt \\
&= F_K\dot{K} + F_R\dot{R} + F_N\dot{N} - [\dot{R}F_R + R\dot{F}_R + d[sQ]/dt] \text{ ... (using (2))} \\
&= F_K F_R R - R\dot{F}_R + F_K sQ + F_N\dot{N} - d[sQ]/dt \text{... (using (1), the savings rule)} \\
&= F_K sQ + F_N\dot{N} - d[sQ]/dt \text{ ... (using (3), the Hotelling rule).}
\end{aligned}$$

Hence, given our expression for $\dot{C}$, (4) becomes

$$d\left[\frac{C}{N}\right]/dt = [F_K sQ - d[sQ]/dt]/N + \left[F_N - \frac{C}{N}\right]\frac{\dot{N}}{N}. \tag{5}$$

Hence, given sustainability ($d\left[\frac{C}{N}\right]/dt = 0$), we see that the right side of (5) must equal zero.[4] We have then sustainability (with the population change) governed by $sQF_K = d[sQ]/dt + \{[\frac{C}{N} - F_N]\dot{N}\}$. This in turn can be written as $sQ = \frac{[\frac{C}{N}-F_N]\dot{N}}{F_K} + \frac{d[sQ]/dt}{F_K}$. Recall that $d[sQ]/dt = d[\dot{K} - RF_R]/dt$. Hence our equation of interest becomes

$$sQ = \frac{[\frac{C}{N} - F_N][dN/dt]}{F_K} + \frac{d[\dot{K} - RF_R]/dt}{F_K}. \tag{6}$$

Equation (6) can be read as: in sustainable development, $sQ$ covers a labor value gap, $[\{\frac{C}{N} - F_N\}\dot{N}]/F_K$ ] plus a resource rent gap, $d[\dot{K} - RF_R]/dt]/F_K$. We might refer to (6) as "the two gap condition". The $F_K$ is the rental price of a unit of $K$. Thus $F_K$ is here translating terms expressed in units of $Q$, into units of $K$. Along the "growth" path, $sQ$ is supplying units of $\dot{K}$ to "close" the two gaps. $\frac{[\frac{C}{N}-F_N]\dot{N}}{F_K}$ and $\frac{d[\dot{K}-RF_R]/dt}{F_K}$ are in units of $K$.[5] The identical two-gap relation re-appears in the same exact form when sustainability prevails in the models of

[4] The temptation to infer that the equilibrium path has $F_N - \frac{C}{N} = 0$ and $F_K sQ - d[pQ]/dt = 0$ is not correct since $\frac{C}{N}$ is unchanging and we cannot contemplate a solution with $F_N$ unchanging.

[5] For the special case of $\dot{N}$ set at zero, we have $sQ = F_K * d[sQ]/dt$, i.e. value $sQ$ unchanging.

d'Autume-Schubert and Stollery, with population endogenous, taken up below. Our inference now is that, for our model of Solow (1974) with population varying, ((a) i.e. $d\left[\frac{C}{N}\right]/dt = 0$), and (b) $\dot{K} = RF_R + sQ$, and (c) Hotelling rule, imply that our new condition, equation (6), is satisfied.[6] These same three conditions imply equation (6) for the d'Autume-Schubert and Stollery variants of Solow (1974) taken up below. (6) expresses how extra saving, $sQ$ is allocated, given population increasing. [[Asheim et. al. (2007) point out that $sQ$ must equal $\dot{N}[K/N]$ for the analysis to move forward. $\dot{N}[K/N]$ is the current capital per worker that must be supplied to $\dot{N}$ currently new workers. One might refer to it as the drag on current $K$ posed by $\dot{N}$ currently new workers.

A MAIN RESULT

Given $sQ \equiv \dot{N}[K/N]$, $Q = K^{\alpha}R^{\beta}N^{1-\alpha-\beta}$ with $\alpha$ and $\beta$ positive and $\alpha+\beta = 1$, and $K(t)/Q(t) = x(0)[1+\mu t]$, $(x(0) = \sigma/\mu)$, equation (6) reduces to

$$nKF_K = [C - (1-\alpha-\beta)Q]n + d[nK]/dt,$$

which can be written as

$$n\alpha Q = [C - Q]n - (-\alpha-\beta)Qn + d[nK]/dt,$$

which can be written as

$$\dot{K}n = \beta Qn + \dot{n}K + n\dot{K}$$

Also, $sQ = \dot{K} - RF_R = \dot{K} + \dot{S}F_R$, the value of net investments. Then our equation (6) means that the present value of net investments is constant, but not necessarily equal to zero. This is the rule discussed in Dixit, Hammond and Hoel (1980): Thanks to Geir Asheim for this instructive observation. The two gaps have a certain mutual independence.

[6] For the basic Solow (1974) model, Buchholz, Swapan Dasgupta, and Mitra (2005) establish that any two of (a) $\dot{K} = RF_R$, (b) Hotelling Rule and (c) $dN/dt = 0$ imply the third relation.

For the special case of $\dot{N}$ set at zero, we have $sQ = F_K * d[sQ]/dt$, i.e. value $sQ$ unchanging. Also, $sQ = \dot{K} - RF_R = \dot{K} + \dot{S}F_R$, the value of net investments. Then our equation (6) means that the present value of net investments is constant, but not necessarily equal to zero. This is the rule discussed in Dixit, Hammond and Hoel (1980). Thanks to Geir Asheim for this instructive observation.

which becomes

$$\begin{aligned}\frac{\dot{n}}{n} &= \frac{-\beta Q}{K} \\ &= \frac{-\beta}{x(0)[1+\mu t]} \\ &= \frac{-\beta}{(\sigma/\mu)[1+\mu t]}.\end{aligned}$$

We now hypothesize that $N(t) = N(0)(1+\mu t)^{\varphi}$, $\mu$ and $\varphi$ positive, or that population motion is quasi-arithmetic. This implies that $\frac{\dot{n}}{n} = \frac{-\mu}{(1+\mu t)}$. Hence the right side equals the left side of our equation if $\beta/\sigma$ equals unity. This is true, as in equation (12) of Asheim et. al. (2007).

Hence the inference that the population motion is quasi-arithmetic when (a) the production function is Cobb-Douglas (b) "extra" investment $sQ$ is taken to be $nK$ or $\dot{N}[K/N]$, and (c) the $K(t)/Q(t)$ is linear in time. Rougly speaking, our new equation, namely (6), is defining "equilibrium population dynamics". ]]

## 3. The Resource Stock in the Utility Function

We turn to a Solow model with population varying and with the current stock of the exhaustible resource yielding utility. That is, we are dealing with d'Autume and Schubert (2008) with population varying. The utility function is then $u(\frac{C}{N}, S)$. The stock placed here might be standing for a conservation interest by households. Investment is funded by resource rent plus extra saving ((1) above). For this model, the Hotelling Rule takes the form (see the Appendix):

$$F_K F_R - \dot{F}_R = [Nu_S]/[u_{[\frac{C}{N}]}] \tag{7}$$

or

$$F_K F_R - \dot{F}_R = u_S/Z \quad \text{for } Z \equiv \frac{1}{N}\{\partial u/\partial[\frac{C}{N}]\}. \tag{8}$$

We proceed to locate a condition on population that is compatible with $du(\frac{C}{N}, S)/dt = 0$, sustainability per person in effect. First we have

$$\begin{aligned}
\dot{C} &= \dot{Q} - d\dot{K}/dt \\
&= F_K F_R R - R\dot{F}_R + \dot{N}F_N - d[sQ]/dt + sQF_K \text{ ... (using (1))} \\
&= \frac{RNu_S}{u_{[\frac{C}{N}]}} + \dot{N}F_N - d[sQ]/dt + sQF_K \text{ ... (using (7)).}
\end{aligned}$$

Then $du(\frac{C}{N}, S)/dt$ becomes

$$u_{[\frac{C}{N}]}[\frac{u_S R}{u_{[\frac{C}{N}]}} + \frac{\dot{N}}{N}F_N - \frac{C}{N}\frac{\dot{N}}{N} + \frac{1}{N}\{-d[sQ]/dt + sQF_K\}] + u_S\dot{S}. \tag{9}$$

Since $\dot{S} = -R$, (9) reduces to $\frac{\dot{N}}{N}u_{[\frac{C}{N}]}\left[F_N - \frac{C}{N}\right] + u_{[\frac{C}{N}]}\frac{1}{N}\{-d[sQ]/dt + sQF_K\} = 0$ or $sQ = \frac{[\frac{C}{N}-F_N][dN/dt]}{F_K} + \frac{d[\dot{K}-RF_R]/dt}{F_K}$, the same as equation (6) above. We infer that $sQ = \frac{[\frac{C}{N}-F_N][dN/dt]}{F_K} + \frac{d[\dot{K}-RF_R]/dt}{F_K}$ when (1) $du(\frac{C}{N}, S)/dt = 0$ (2) $\dot{K}$ is funded with resource rents plus extra saving and (3) the appropriate "Hotelling Rule" prevails.

[[ We proceed to specify $sQ$ as $nK$ and our production function as $K^{\alpha}R^{\beta}N^{(1-\alpha-\beta)}$. W substitute in $sQ = \frac{[\frac{C}{N}-F_N][dN/dt]}{F_K} + \frac{d[\dot{K}-RF_R]/dt}{F_K}$ to get

$$nKF_K = [C - (1-\alpha-\beta)Q]n + n\dot{K} + \dot{n}K.$$

Since $C - Q = \dot{K}$ and $KF_K = \alpha Q$, we have

$$\alpha Qn = -\dot{K}n + \alpha Qn + \beta Qn + n\dot{K} + \dot{n}K$$

which gives us

$$\frac{\dot{n}}{n} = \frac{-\beta}{(K/Q)}.$$

This equation is the same as its counterpart above for the Solow (1974) model with population. We cannot go farther at this point because we do not have an expression for $K/Q$. We have no reason to believe that the population dynamics are characterized by a quasi-arithmetic function. We do infer that his equation is defining "equilibrium population change" for the d'Autume-Schubert model. ]]

## 4. Extraction and Warming (Stolley (1998))

Another Solow variant has current extraction, $R(t)$ causing the current temperature, $W(t)$ to rise (Stollery (1998)). Here we introduce population-change to the Hartwick-Mitra (2020) version of the Stollery model. Higher temperature affects utility, $u(\frac{C}{N}, W)$ negatively, as well as current output, $Q = F(K, R, N, W)$. The given temperature-rise relation is specified as

$$\frac{\dot{W}}{W} = \xi R \tag{10}$$

with $R$ current extraction from the depleting stock, $S$. That is $R = -\dot{S}$. $\xi$ is a positive constant. As above, we assume that current resource rents $RF_R$ plus "extra" investment $sQ$ constitute payment for current investment, $\dot{K}$. That is

$$\dot{K} = sQ + RF_R \tag{11}$$

As above, we also use the derivative $d\dot{K}/dt = \dot{R}F_F + R\dot{F}_R + d[sQ]/dt$.

The dynamic efficiency condition on resource use (Hotelling Rule[7]) is

$$F_K F_R - \dot{F}_R + \left[\frac{u_Z F_W + N u_W}{u_Z}\right] * \xi W = 0 \text{ for } Z \equiv C/N. \tag{12}$$

We proceed to examine $d[u(\frac{C}{N}, W)]/dt$ and search for a current condition on population dynamics and $s$ that "solves" the model.

We start with

$$\dot{C} = \dot{Q} - d\dot{K}/dt.$$

Using the production function and (11) we get

$$\begin{aligned}
\dot{C} &= F_K\dot{K} + F_R\dot{R} + F_N\dot{N} + F_W\dot{W} - \dot{R}F_R - R\dot{F}_R - d[sQ]/dt \\
&= F_K R F_R - R\dot{F}_R + \dot{N}F_N + \dot{W}F_W - d[sQ]/dt + F_K sQ.
\end{aligned}$$

[7] See Hartwick and Mitra (2020) for this condition.

and $d\left[\frac{C}{N}\right]/dt = \frac{1}{N}[RF_K F_R - R\dot{F}_R + \dot{N}F_N + \dot{W}F_W - d[sQ]/dt + F_K sQ] - (\dot{N}/N^2)C]$.

Thus, using (10), we get

$$\begin{aligned}
d[u(\frac{C}{N}, W)]/dt &= u_Z \frac{1}{N}[RF_K F_R - R\dot{F}_R + \dot{N}F_N + \dot{W}F_W - d[sQ]/dt + F_K sQ \\
&\quad -(\dot{N}/N)C] + u_W \dot{W}. \\
&= u_Z \frac{1}{N}[\{RF_K F_R - R\dot{F}_R\} + \{\dot{N}F_N - (\dot{N}/N)C - d[sQ]/dt + F_K sQ\}] \\
&\quad +\{u_Z \frac{1}{N} F_W + u_W\}\dot{W} \\
&= u_Z \frac{1}{N}[\{RF_K F_R - R\dot{F}_R\} + \{\dot{N}F_N - (\dot{N}/N)C - d[sQ]/dt + F_K sQ\} \\
&\quad +\{F_W + Nu_W/u_Z\}\dot{W}] \\
&= u_Z \frac{1}{N}[\{RF_K F_R - R\dot{F}_R\} + \{\dot{N}F_N - (\dot{N}/N)C - d[sQ]/dt + F_K sQ\} \\
&\quad +\{[u_Z F_W + Nu_W]/u_Z\} * \xi WR] \\
&= u_Z \frac{1}{N}[\{RF_K F_R - R\dot{F}_R\} + \{[u_Z F_W + Nu_W]/u_Z\} * \xi WR] \\
&\quad +u_Z \frac{1}{N}\{[F_N - (C/N)]\dot{N} - [d[sQ]/dt + F_K sQ]\}.
\end{aligned}$$

Now $[\{RF_K F_R - R\dot{F}_R\} + \{[u_Z F_W + Nu_W]/u_Z\} * \xi WR] = 0$ because it is the Hotelling Rule for this model. Hence we have $d[u(\frac{C}{N}, W)]/dt = 0$ when the Hotelling Rule holds and when

$$sQ = \frac{[\frac{C}{N} - F_N][dN/dt]}{F_K} + \frac{d[\dot{K} - RF_R]/dt}{F_K}. \tag{13}$$

This last condition is familiar as equation (6) above.

[[ We now specify $sQ$ as $nK$ and the production function as $Q = K^\alpha R^\beta N^\gamma W^{-\delta}$

with $\alpha, \beta, \gamma$ and $\delta$ each positive and less than unity. Equation (13) becomes

$$\begin{aligned}
nKF_K &= [C - \gamma Q]n + \dot{n}K + n\dot{K} \\
\text{or } n\alpha Q &= nC - \gamma nQ + \dot{n}K + n\dot{K} \\
\text{or } n\alpha Q &= nC - nQ + nQ - \gamma nQ + \dot{n}K + n\dot{K} \\
\text{or } n\alpha Q &= nQ - \gamma nQ + \dot{n}K \\
\text{or } \frac{\dot{n}}{n} &= -(Q/K)[1 - \alpha - \gamma].
\end{aligned}$$

This expression for $\dot{n}/n$ is similar to its counterparts for our two models above, but is not identical. We infer that this equation if defining the population dynamics for the Stollery model, with population, but we are not able to fill in the details. ]]

## 5. Concluding Remarks

We have observed that when the basic Solow (1974) model of sustainable consumption is extended to incorporate varying population levels, a new equilibrium condition (equation) emerges that captures the allocation of extra investment needed to counter various population-change or population "drag" effects. In place of the familiar three conditions defining sustainability associated with Solow (1974), we observe four conditions when population levels are changing in the model. The new equation relates "extra" saving to current population increase. We observe the same general equation for allocating "extra" savings in three variants of Solow (1974), each with the population level changing. When the expression for "extra saving" is specified, we are able to "reduce" our general equation to an equation characterizing population dynamics for each model. Note that we have, above, worked with sustainability implying a new condition for "maintaining sustainability". A different theorem remains to be worked on, namely how our new equation, accompanied by the relevant Hotelling Rule and the amended invest resource rents rule, implies Solow sustainability.

APPENDIX 1: The Dynamic Efficiency Condition (Hotelling Rule) for the d'Autume-Schubert Model

We set out an optimal savings problem with exhaustible resource extraction "built in". The Hamiltonian for the problem is

$$H = u(\frac{C}{N}, S) + \lambda[F(K, R, S) - C] - \phi R$$

The necessary conditions defining an optimum are

$$\begin{aligned}
\frac{\partial H}{\partial C} &= 0 \Rightarrow \frac{1}{N} u_Y = \lambda, \text{ for } Y \equiv \frac{C}{N}; \\
\frac{\partial H}{\partial N} &= 0 \Rightarrow - \left[\frac{C}{N^2}\right] u_Y + \lambda F_N = 0; \text{ or } - \left[\frac{C}{N^2}\right] + \left[\frac{F_N}{N}\right] = 0; \\
\frac{\partial H}{\partial R} &= 0 \Rightarrow \lambda F_R = \phi; \text{ or } Z = \phi, \text{ for } Z \equiv \left[\frac{F_R}{N} u_Y\right]; \\
-\frac{\partial H}{\partial K} &= \dot{\lambda} - \rho\lambda \Rightarrow -\lambda K = \dot{\lambda} - \rho\lambda; \text{ or } - Z F_K = F_R d[\frac{u_Y}{N}]/dt - Z\rho; \\
-\frac{\partial H}{\partial S} &= \dot{\phi} - \rho\phi \Rightarrow -u_S = \dot{Z} - \rho Z; \text{ or } - u_S = \dot{Z} - Z F_K - F_R d\left[u_Y/N\right] dt, \\
\text{or } - u_S &= F_R d\left[u_Y/N\right] dt + \left[u_Y/N\right] \dot{F}_R - Z F_K - F_R d\left[u_Y/N\right] dt.
\end{aligned}$$

From the last expression we deduce that the "Hotelling Rule" is

$$F_K F_R - \dot{F}_R = \frac{N u_S}{u_Y}.$$

APPENDIX 2: The Sidgewick-Meade Rule for Optimal Population

We have an island that sustains, without labor or capital, a flow of consumables and the annual flow harvest is $\Gamma$. We add people to the island so that $NU(\Gamma/N)$ is maximized. $N$ is the current number of people on the island. There are an unlimited number of persons, "in the wings" available to be put on the island with no cost of moving them on. Persons can be added at zero "moving cost". We have $Z \equiv \Gamma/N$. The first order condition is

$$U(Z) + NU_Z \frac{dZ}{dN} = 0.$$

This reads: given current optimal value $N^*$, adding one more person, adds $U(Z)$ directly to the objective function, but congests the existing $N^*$ people and cuts into each incumbant's current $Z/N$ for total "damage", $NU_Z \frac{dZ}{dN}$.

This first order condition reduces to

$$\frac{U(Z)}{U_Z} = \frac{\Gamma}{N}.$$

This equation is defining the optimizing value of $N$. This is the Sidgewick-Meade Rule for optimal population, $N^*$. The rule says that the "dollar" payoff of the marginal person joining equals the payoff in utils, $U(Z)$, translated back to "dollars" with factor, $U_Z$.

An extension to this island problem is to have people do work so that in place of $\Gamma$ we have $g(N)$ being produced on the island each period. So now the optimal population must maximize $NU(g(N)/N)$. The first order condition is

$$U(Z) + NU_Z \frac{dZ}{dN} = 0 \text{ for } Z = g(N)/N.$$

This has the same interpretation as its analogue above. This condition can be written as

$$U(Z)/U_Z = \{g(N)/N - g_N\}.$$

The interpretation of this "rule" is only slightly different from the classic rule above. Adding a person involves NET payoff to the new person in "dollars", $\{g(N)/N - g_N\}$, where $g_N(N^*)$ is her marginal product. The utility payoff to the entrant is $U(Z)$, and this has value in "dollars" of $U(Z)/U_Z$.

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