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The All-Gap Phillips Curve

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Abstract

The all-gap Phillips curve (PC) explains inflation by expected inflation and an activity variable such as output or the unemployment rate, but with both inflation and the activity variable measured relative to their stochastic trends and thus as gaps. We study this relationship with minimal auxiliary assumptions and under rational expectations (RE). We show restrictions on an unobserved-components model that identify the Phillips curve parameters, first with an autonomous output gap and second with output and inflation gaps following a VAR. For the US, UK, and Canada both cases yield all-gap PCs with slopes of the expected signs, but there is little support for the restrictions implied by RE.

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1. Introduction

A range of time series models refer to a 'gap' as the difference between a variable and its stochastic trend. Thus the all-gap Phillips curve explains inflation by expected inflation and an activity variable such as output or the unemployment rate, but with inflation, expected inflation, and the activity variable all measured relative to their stochastic trends. For simplicity we refer to the gap in the activity variable as the output gap, though we measure it using either output or the unemployment rate. The trend in inflation usually is interpreted as the slowly evolving target of the central bank, while the trend in the activity variable reflects factors in potential output or the natural rate of unemployment. This version of the Phillips curve (PC) is of interest in part because it features in macroeconomic analyses of Woodford (2008) and Goodfriend and King (2012) for example.

The all-gap PC has two key, appealing features. First it involves no long-run tradeoff between inflation and the output gap. That is because the trend in inflation also coincides with long-run inflation expectations. This PC thus retains this desirable feature of some other Phillips curves, including those specified in levels rather than gaps. Second, it is balanced in terms of order of integration. The output gap or difference between the unemployment rate and its natural rate are generally stationary by construction, yet the inflation rate often is modelled with a slowly evolving stochastic trend over long spans of postwar data. The all-gap PC thus involves explaining one stationary variable by another.

We aim to parametrize the all-gap PC and to test it under rational expectations (RE). While this PC has been included in DSGE models, the idea here is to study it with minimal auxiliary assumptions so as to see whether it can be recommended as a building block in such models. In macroeconomic policy, the slope of the PC matters for estimating the output costs of disinflation, while the weight on expected future inflation matters for measuring the effect of announcements or forward guidance. Moreover, the use of RE restrictions, if valid, can potentially improve measurement of the output gap.

In contrast with the New Keynesian PC in levels, for example, we cannot simply estimate the all-gap PC using instrumental variables, because the trends relative to which gaps are defined are not directly observed. So this minimal approach necessarily involves modelling these trends. Here we draw on the wealth of statistical work on the unobserved components (UC) model, that has been applied to both inflation and to activity variables and sometimes to them jointly. In any case, there is a long tradition in RE econometrics of using the law of motion for the independent variable to add efficiency. Here we use the same idea but also focus on the fact that there are many studies using the UC model for the dependent variable, inflation, as well as for output or the unemployment rate.

We show how the all-gap PC restricts the UC model first under the widely-used assumption that the output gap evolves autonomously. In this case the restrictions are of the familiar Hansen-Sargent variety. Second, we show the restrictions when the inflation gap and the output gap jointly follow a vector autoregression (VAR). Here we directly apply to gaps the recommendations that Kurmann (2007) developed for levels. Each case also yields a test of overidentifying restrictions. Our contribution is to apply these well-known tools to the UC model.

We study two inflation rates (in the CPI and the GDP deflator), two activity variables (the unemployment rate and real GDP), and three countries (the US, the UK, and Canada) for which there are long spans of quarterly data since the 1940s or 1950s. There is evidence of all-gap PCs of the expected slope: negative for unemployment and positive for output. There is less evidence of a well-identified role for expected, future inflation. And the overidentifying restrictions implied by RE find little support so that an alternative model of joint gap dynamics may be needed.

2. The All-Gap Phillips Curve

Denote the inflation rate by π_t and its Beveridge-Nelson trend by $\tau_{\pi,t}$, which follows a martingale. We refer to $\epsilon_{\pi,t} \equiv \pi_t - \tau_{\pi,t}$ as the inflation gap. Suppose that a measure of real activity, such as log output or the unemployment rate, is labelled x_t. And suppose that it has a stochastic trend $\tau_{x,t}$, possibly with drift μ, and a gap $\epsilon_{x,t} \equiv x_t - \tau_{x,t}$.

Consider this Phillips curve:

$$\pi_t - \tau_{\pi,t} = \beta E_t(\pi_{t+1} - \tau_{\pi,t+1}) + \kappa(x_t - \tau_{x,t}) + u_{\pi,t}. \tag{1}$$

Inflation reacts to expected inflation and to real activity, in each case relative to trend or expected trend. The last term $u_{\pi,t}$, is a cost shock that has mean zero and is unpredictable.

Here firms that do not reset their prices in a given period automatically index them to the trend rate of inflation. Meanwhile, firms that do reset their prices may depart from the trend if there is an output gap or a cost shock.

This format was introduced by Woodford (2008). He solves a New Keynesian model in all-gap form that shows the source of the inflation trend $\tau_{\pi,t}$: It is inherited from the targeting rule of the central bank. Ireland (2007) shows the same thing in a more elaborate model. Goodfriend and King (2012) adopt this form (1) in their history of US inflation. They note that it involves a short-run, Phillips curve tradeoff if there are no changes in expected inflation, but not a long-run tradeoff. The all-gap model implies that inflation moves 1:1 with long-horizon inflation expectations, as measured by $\tau_{\pi,t}$. We shall measure this trend as part of this exercise. We also model the output gap by allowing for a stochastic trend in output that follows a random walk. After all, many studies using the Beveridge-Nelson trend apply it to output.

Also note that:

$$\begin{aligned} \tau_{\pi,t} &= \lim_{j\to\infty} E_t \pi_{t+j} \\ \tau_{x,t} &= \lim_{j\to\infty} E_t (x_{t+j} - j\mu), \end{aligned} \tag{2}$$

which are the Beveridge-Nelson (BN, 1981) trends. Because the inflation trend follows a random walk, $E_t \tau_{\pi,t+1} = \tau_{\pi,t}$ and so the Phillips curve becomes:

$$\pi_t = \beta E_t \pi_{t+1} + (1-\beta)\tau_{\pi,t} + \kappa \epsilon_{x,t} + u_{\pi,t},$$

or equivalently, in all-gap form:

$$\epsilon_{\pi,t} = \beta E_t \epsilon_{\pi,t+1} + \kappa \epsilon_{x,t} + u_{\pi,t}. \tag{3}$$

The key feature here is that the Phillips curve (3) is balanced in the sense that all three terms are stationary. It is very common to use an output gap or the difference between the unemployment rate and its natural trend to explain inflation. That is what is labelled as $\epsilon_{x,t}$. But that is a statistical reason why (1) also studies inflation relative to trend, which yields a Phillips curve in gaps (3).

Notice that we do not call this a New Keynesian Phillips Curve (NKPC). First, as is well-known, there is no closed-form expression for the NKPC with Calvo pricing around trend inflation. Ascari and Sbordone (2014) provide a complete review. But note that Cogley and Sbordone (2008) find that there is little role for lagged inflation (i.e. a hybrid NKPC) once trend inflation is controlled for in an NKPC based on Calvo pricing. Second, we study an all-gap PC in which x is measured by GDP or the unemployment rate rather than an estimate of marginal cost. However, the timing—in which inflation is partly explained by current expectations of future inflation—is the same as in the NKPC. And this timing is adopted in several related studies reviewed in the next section.

3. Related Research

A number of studies link the stationary components of inflation and either output or the unemployment rate in multivariate UC models. Kuttner (1994), Planas and Rossi (2004), and Planas, Rossi, and Fiorentini (2008) focus on refining measurement of the output gap while Laubach (2001) focuses on measuring the natural rate of unemployment. The studies of Doménech and Gómez (2006), Lee and Nelson (2007), Basistha and Nelson (2007), Piger and Rasche (2008), Harvey (2011), Stella and Stock (2015), Hasenzagl, Pellegrino, Reichlin, and Ricco (2020), and Panovska and Ramamurthy (2022) are important precursors that study Phillips curves with one or more unobserved components, although not of the all-gap, forward-looking variety.

Several studies do consider a PC of the form (1). Kim, Manopimoke, and Nelson (2014) lag the expectation term to create their observation equation and so do not estimate it in this form. Morley, Piger, and Rasche (2015, p 889) show how one might identify β and κ in this case with an autonomous AR(1) model of $\epsilon_{x,t}$, but they do not report estimates of those parameters. Berger, Everaert, and Vierke (2016, p 182) combine their all-gap PC with an autonomous AR(2) model of $\epsilon_{x,t}$ to yield RE restrictions. But they do not report estimates of the underlying parameter β. These recent studies include a number of features we do not include, such as discrete breaks in volatility, stochastic volatility, time-varying parameters, or linkages to other variables such as survey measures of expectations. The resulting state space models often are estimated by Bayesian methods.

Two studies do use RE restrictions. Crump, Eusepi, Giannoni, and Şahin (2019) study a hybrid, all-gap PC in that it also includes lagged inflation and a persistent cost shock. They study this in an environment that includes information from forecast surveys and from the flows in and out of unemployment. Crump et al. (2019) calibrate $\beta = 0.99$ while estimating κ. Hessler (2021) studies a UC model of an entire New Keynesian system, involving inflation, income, consumption, and a nominal interest rate. In his study the inflation and income gaps co-evolve, and he includes RE restrictions in his system. He estimates a large weight on expected inflation but a negligible slope in the Phillips curve. His larger system may aid efficiency, while his findings may depend on the specification of the IS curve and the monetary policy rule.

This brief survey shows that there is a long-standing interest in exploring the links between the inflation gap and the output gap or unemployment-rate gap. These studies generally find that the inflation gap is negatively correlated with the unemployment gap and positively correlated with the output gap. We see if that is true for the countries, time periods, activity measures, and time-series model of gaps here. And we do generally find PCs of the expected slopes. But our main goal is to study the forward-looking, all-gap PC under RE with minimal auxiliary assumptions. We derive the RE restrictions both with an autonomous x-gap and with the two gaps following a VAR. And we explain and calculate the likelihood ratio tests implied by these restrictions.

4. Unobserved Components

Recall that $\pi_t = \tau_{\pi,t} + \epsilon_{\pi,t}$ and $x_t = \tau_{x,t} + \epsilon_{x,t}$. To measure trends and gaps we use this statistical model for trends:

$$\begin{aligned} \tau_{\pi,t} &= \tau_{\pi,t-1} + \nu_{\pi,t} \\ \tau_{x,t} &= \mu + \tau_{x,t-1} + \nu_{x,t} \end{aligned} \tag{4}$$

which allows for drift μ in the activity variable x. The gaps have unconditional mean zero and follow a VAR(2) process:

$$\begin{pmatrix} \epsilon_{\pi,t} \\ \epsilon_{x,t} \end{pmatrix} = B \begin{pmatrix} \epsilon_{\pi,t-1} \\ \epsilon_{x,t-1} \end{pmatrix} + C \begin{pmatrix} \epsilon_{\pi,t-2} \\ \epsilon_{x,t-2} \end{pmatrix} + \begin{pmatrix} \eta_{\pi,t} \\ \eta_{x,t} \end{pmatrix}, \tag{5}$$

where B and C are 2×2 matrices:

$$B \equiv \begin{pmatrix} b_{\pi\pi} & b_{\pi x} \\ b_{x\pi} & b_{xx} \end{pmatrix}$$

and

$$C \equiv \begin{pmatrix} c_{\pi\pi} & c_{\pi x} \\ c_{x\pi} & c_{xx} \end{pmatrix}.$$

The four shocks—$\nu_{\pi,t}$, $\nu_{x,t}$, $\eta_{\pi,t}$, and $\eta_{x,t}$—are mean-zero innovations and are assumed to be normally distributed, with no restrictions on their correlations. This feature allows us to follow in the footsteps of a number of researchers who find negative correlations between shocks to trend and shocks to cycle: Morley, Nelson, and Zivot (2003) for US GDP; Sinclair (2009) for US output and the unemployment rate; Mitra and Sinclair (2012) for output in G7 countries; and Hwu and Kim (2019) for US inflation. The resulting trends are what Morley (2011) calls BN-as-estimate, as opposed to BN-as-definition where the correlation is -1.

The VAR(2) process captures most of the specifications used in previous studies cited here. Trenkler and Weber (2016) show that the UC model is identified—without restricting the shock correlations to be zero—given gap lag lengths greater than one.

Since Stock and Watson's (2007) study of inflation forecasting, unobserved components in US inflation often have been modelled with stochastic volatility (SV) in the innovations. In the applications below, we explore this issue by studying the squared residuals for the inflation components ($\hat{\nu}^2_{\pi,t}$ and $\hat{\eta}^2_{\pi,t}$) and also for the activity components ($\hat{\nu}^2_{x,t}$ and $\hat{\eta}^2_{x,t}$). First, we compute Q statistics and find some evidence of persistence in these squared residuals, though that varies with the series and the time period. Second, however, we do not find persistence in the *ratio* of squared residuals for each series, which is an estimate of the *relative* variance. We also find a positive, contemporaneous correlation between squared shocks to the trend and squared shocks to the gap, for each series. Thus there are occasional, large shocks to realized volatilities and they tend to occur in both components at the same time.

If there were persistent swings in the relative variance of the two shocks then that would lead to bias in our estimates of the gap dynamics when they are estimated assuming constant variances. Intuitively, changes in this ratio lead to changes in the conditional mean of inflation and its gap, by analogy with Stock and Watson (2007). However, we do not find this pattern and so we proceed with estimation assuming constant shock variances. We estimate the state space model by maximum likelihood using the BFGS algorithm with code written in the Julia programming language.

5. RE Restrictions I: An Autonomous Output Gap

The all-gap PC could be studied with a range of models of expectations, but it seems natural to begin with RE. To make that operational, we begin with a widely-used special case of (5) in which the output gap follows an AR(2) process. This time-series model is adopted by Lee and Nelson (2007), Sinclair (2009), Kim, Manopimoke, and Nelson (2014), Morley, Piger, and Rasche (2015), Berger, Everaert, and Vierke (2016), and Crump *et al* (2019). Thus let the x-gap follow this process:

$$\epsilon_{x,t} = b_{xx}\epsilon_{x,t-1} + c_{xx}\epsilon_{x,t-2} + \eta_{x,t}. \tag{6}$$

We refer to this as the autonomous case because $b_{x\pi} = c_{x\pi} = 0$ so that lagged values of the inflation gap do not enter the equation for the activity gap. But the innovations $\eta_{\pi,t}$ and $\eta_{x,t}$ may still be correlated.

To derive RE restrictions we use the guess-and-verify method, also known as the method of undetermined coefficients. We guess a linear form for the solution which gives $\epsilon_{\pi,t}$ as a function of current and lagged $\epsilon_{x,t}$. We use that guess to replace both $\epsilon_{\pi,t}$ and $E_t\epsilon_{\pi,t+1}$ in the all-gap PC and then use the actual law of motion for the output gap to replace $E_t\epsilon_{x,t+1}$. It is the fact that these unobserved expectations coincide with forecasts from the law of motion that leads to the resulting restrictions being labelled RE.

Using this method with the PC (3) and the law of motion (6) gives:

$$\epsilon_{\pi,t} = \frac{\kappa}{1-\beta b_{xx} - \beta^2 c_{xx}}\epsilon_{x,t} + \frac{\beta\kappa c_{xx}}{1-\beta b_{xx} - \beta^2 c_{xx}}\epsilon_{x,t-1}. \tag{7}$$

Substituting for $\epsilon_{x,t}$ using its law of motion (6) gives:

$$\epsilon_{\pi,t} = b_{\pi x}\epsilon_{x,t-1} + c_{\pi x}\epsilon_{x,t-1} + \eta_{\pi,t} \tag{8}$$

where

$$\begin{aligned} b_{\pi x} &= \frac{\kappa b_{xx} + \beta\kappa c_{xx}}{1-\beta b_{xx} - \beta^2 c_{xx}} \\ c_{\pi x} &= \frac{\kappa c_{xx}}{1-\beta b_{xx} - \beta^2 c_{xx}}. \end{aligned} \tag{9}$$

The error term $\eta_{\pi,t}$ includes both a term in $\eta_{x,t}$ from the substitution and the cost shock $u_{\pi,t}$. Section 4 noted that we allow for correlated shocks in the UC model and this example gives a theoretical reason to do so.

This statistical model gives a VAR (6) and (8) in the two gaps in which only lagged output gaps enter. The RE restrictions (9) just identify the PC parameters β and κ. But the autonomous model also implies that $\epsilon_{\pi,t-1}$ and $\epsilon_{\pi,t-2}$ are excluded from the law of motion for $\epsilon_{\pi,t}$ (8) i.e. $b_{\pi\pi} = c_{\pi\pi} = 0$. We test this restriction with a likelihood ratio test below.

6. RE Restrictions II: An All-Gap VAR

We next generalize this example by allowing the two gaps to follow a bivariate VAR(2). In linear RE models it is natural to forecast with lagged endogenous variables, perhaps because of the superior information of price-setters relative to econometricians. As noted above, a number of UC studies use second-order dynamics, in part to capture hump-shaped reactions. Trenkler and Weber (2016) show that a lag order greater than one is necessary to identify a UC model without imposing a zero covariance between the trend and gap innovations.

Moreover, Woodford's (2008) New Keynesian model shows that it may make theoretical sense to treat the inflation gap and the output gap as co-evolving. There the unobserved cost shock, $u_{\pi,t}$, and a shock to the natural real rate of interest combine to induce shocks to this bivariate system. He shows the VAR resulting from a three-equation New Keynesian model. But the specifications of the IS curve and the monetary policy rule are just as controversial as that of the Phillips curve (3) so we study only the PC restrictions so as to test them under weak assumptions. In sum, we could do this with any VAR as the forecasting model (of course as long as it included $\epsilon_{\pi,t}$ and $\epsilon_{x,t}$), but this one fits the information set of the simplest NK model and is consistent with the dynamics of a number of UC studies.

We can write the VAR (5) in companion form, with state vector labelled z_t:

$$z_t \equiv \begin{pmatrix} \epsilon_{\pi,t} \\ \epsilon_{x,t} \\ \epsilon_{\pi,t-1} \\ \epsilon_{x,t-1} \end{pmatrix} = \begin{pmatrix} B & C \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} \epsilon_{\pi,t-1} \\ \epsilon_{x,t-1} \\ \epsilon_{\pi,t-2} \\ \epsilon_{x,t-2} \end{pmatrix} + \begin{pmatrix} \eta_{\pi,t} \\ \eta_{x,t} \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

with 4×4 transition matrix:

$$A \equiv \begin{pmatrix} B & C \\ I_2 & 0 \end{pmatrix}. \tag{11}$$

Suppose that the cost shock, $u_{\pi,t}$, is an innovation with respect to past values of the gaps. We next find RE-PC restrictions on this all-gap VAR. Project the PC (3) onto the lagged state vector using the law of iterated expectations and the fact that $E(u_{\pi,t}|z_{t-1}) = 0$:

$$E(\epsilon_{\pi,t}|z_{t-1}) = \beta E(\epsilon_{\pi,t+1}|z_{t-1}) + \kappa E(\epsilon_{x,t}|z_{t-1}). \tag{12}$$

Define selection vectors $s_\pi = (1\,0\,0\,0)$ and $s_x = (0\,1\,0\,0)$. Then use the VAR (10) to replace forecasts in the projection (12):

$$s_\pi A z_{t-1} = \beta s_\pi A^2 z_{t-1} + \kappa s_x A z_{t-1},$$

so that:

$$s_\pi A = \beta s_\pi A^2 + \kappa s_x A. \tag{13}$$

Notice that from the definition of A (11):

$$A^2 = \begin{pmatrix} B^2 + C & BC \\ B & C \end{pmatrix}. \tag{14}$$

In turn one can write out the top left corner of the A^2 matrix:

$$B^2 + C = \begin{pmatrix} b_{\pi\pi}^2 + b_{\pi y} b_{x\pi} + c_{\pi\pi} & b_{\pi\pi} b_{\pi x} + b_{\pi x} b_{xx} + c_{\pi x} \\ b_{x\pi} b_{\pi\pi} + b_{xx} b_{x\pi} + c_{x\pi} & b_{x\pi} b_{\pi x} + b_{xx}^2 + c_{xx} \end{pmatrix}. \tag{15}$$

And the top right corner of the A^2 matrix is:

$$BC = \begin{pmatrix} b_{\pi\pi} c_{\pi\pi} + b_{\pi x} c_{x\pi} & b_{\pi\pi} c_{\pi x} + b_{\pi x} c_{xx} \\ b_{x\pi} c_{\pi\pi} + b_{xx} c_{x\pi} & b_{x\pi} c_{\pi x} + b_{xx} c_{xx} \end{pmatrix}. \tag{16}$$

The selection vector s_π just extracts the top row of the relevant 4×4 matrix, while the selection vector s_x extracts the second row. Drawing on (14)–(16), system (13) thus yields four equations:

$$\begin{aligned} b_{\pi\pi} &= \beta(b_{\pi\pi}^2 + b_{\pi x} b_{x\pi} + c_{\pi\pi}) + \kappa b_{x\pi} \\ b_{\pi x} &= \beta(b_{\pi\pi} b_{\pi x} + b_{\pi y} b_{xx} + c_{\pi x}) + \kappa b_{xx} \\ c_{\pi\pi} &= \beta(b_{\pi\pi} c_{\pi\pi} + b_{\pi x} c_{x\pi}) + \kappa c_{x\pi} \\ c_{\pi x} &= \beta(b_{\pi\pi} c_{\pi x} + b_{\pi x} c_{xx}) + \kappa c_{xx}. \end{aligned} \tag{17}$$

In this case there are 4 restrictions on the VAR, which thus overidentify the two PC parameters β and κ, yielding a test with 2 degrees of freedom. We report a likelihood ratio test of these restrictions below.

Kurmann (2007) recommends writing the restrictions on the law of motion for the $\epsilon_{x,t}$ variable. We next see how to restrict $b_{x\pi}$, b_{xx}, $c_{x\pi}$ and c_{xx} i.e. the line for $\epsilon_{x,t}$ in the VAR (5). There is only one of these coefficients in each equation so rearranging (17) isolates those terms:

$$\begin{aligned} b_{x\pi} &= \frac{b_{\pi\pi}(1-\beta b_{\pi\pi}) - \beta c_{\pi\pi}}{\beta b_{\pi x} + \kappa} \\ b_{xx} &= \frac{b_{\pi x}(1-\beta b_{\pi\pi}) - \beta c_{\pi x}}{\beta b_{\pi x} + \kappa} \\ c_{x\pi} &= \frac{c_{\pi\pi}(1-\beta b_{\pi\pi})}{\beta b_{\pi x} + \kappa} \\ c_{xx} &= \frac{c_{\pi x}(1-\beta b_{\pi\pi})}{\beta b_{\pi x} + \kappa}. \end{aligned} \tag{18}$$

We use these formulas to implement the restrictions.

Whichever form they are written in, these restrictions include the special case of a VAR(1) when $C = 0$ in which case $A = B$. They also include the special case of an autonomous output gap outlined in section 5, in which case $b_{x\pi} = c_{x\pi} = 0$ and the result is that $b_{\pi\pi} = c_{\pi\pi} = 0$. In that case the system (17) simplifies to (9) while the Kurmann-form restrictions (18) become:

$$\begin{aligned} b_{xx} &= \frac{b_{\pi x} - \beta c_{\pi x}}{\beta b_{\pi x} + \kappa} \\ c_{xx} &= \frac{c_{\pi x}}{\beta b_{\pi x} + \kappa}. \end{aligned} \tag{19}$$

Higher-order dynamics of course would induce further restrictions. For example a VAR(p) in the gaps implies $2p - 2$ overidentifying restrictions yielding a test with this many degrees of freedom.

7. Estimates and Tests

We focus on countries with (a) long postwar spans of quarterly data on inflation, the unemployment rate, and real GDP and (b) inflation measured using both the CPI and the GDP deflator. Data sources for these countries—the US, UK, and Canada—are described in the appendix. In unrestricted UC models the AIC does not suggest lag lengths greater than 2, so we use that lag length to be consistent with this finding and with the use of the AR(2) model in some previous studies noted in section 5.

Identifying parameters like β has sometimes proved difficult in linear RE models. As noted earlier, Crump et al. (2019) calibrate it, perhaps for this reason. We do not calibrate β because we want to see if the difficulty in identifying it carries over to the all-gap UC environment. However, to constrain our search for a maximum likelihood, we write $\beta \equiv \exp(\omega)/[1 + \exp(\omega)]$ and do not restrict the value of the parameter ω. This transformation yields $\beta \in (0, 1)$. Although we do not refer to the all-gap PC as New Keynesian, in the NKPC Woodford (2003, proposition 3.5) shows this parameter coincides with the average discount factor applied to future profits by price-setting firms. In practice, we sometimes find $\hat{\beta}$ near the edges of this range but we do not find larger likelihood values when $\hat{\beta}$ is instead allowed beyond this range.

7.1 Unemployment Rate Results

Table 1 contains results with the unemployment rate as the activity variable. There is little evidence of a drift μ in this activity variable, so that parameter is excluded. The first column lists the country, the time span, and the price index used to measure inflation. We begin with findings for the model with an autonomous process for the unemployment rate, so that $b_{x\pi} = c_{x\pi} = 0$. The next two columns give the PC parameter estimates $\hat{\beta}$ and $\hat{\kappa}$ and their standard errors. The PC applied to the autonomous model implies that $b_{\pi\pi} = c_{\pi\pi} = 0$ so that lagged output gaps are sufficient to predict the current inflation gap, with the law of motion (8). The table next shows a likelihood ratio statistic (labelled RE:AR) that tests that restriction, along with its degrees of freedom (2) and p-value.

The central column (labelled AR:VAR) provides a likelihood ratio test of the restrictions that give the autonomous gap model as a special case of the VAR, in each case without any RE restrictions. Later columns then show estimates of the PC parameters in the VAR case. As noted in section 6, here too there are two overidentifying restrictions, and so the likelihood ratio test (labelled RE:VAR) contrasts the RE version with the unrestricted VAR.

For the US the results are similar for both measures of inflation. Estimates $\hat{\kappa}$ are negative, so that there is a downward-sloping Phillips curve in inflation-unemployment gap space. The precision of these estimates varies depending on the inflation measure and model. Estimates $\hat{\beta}$ are not near the edges of the parameter space. The restriction to the autonomous AR model is not rejected at conventional levels of significance. However, both within the AR model and the VAR model the tests of the RE restrictions yield low p-values, which is evidence against the RE version of the all-gap PC.

For the UK we find negative estimates $\hat{\kappa}$ using the CPI but small positive values using the GDP deflator which is available for a shorter time period since 1956. We also find $\hat{\kappa} < 0$ using the CPI if its sample begins in 1956 too. For the UK the test statistics labelled AR:VAR are much larger, supporting the VAR model rather than the AR special case. In the VAR models the parameter β is precisely estimated and well away from the boundaries of the parameter space. But within the VAR models the test statistics labelled RE:VAR again are quite large which does not support the RE restrictions.

For Canada we find $\hat{\kappa} < 0$ for each inflation measure and time-series model, and it is sometimes estimated with considerable precision. The estimates $\hat{\beta}$ are often near the edges of the parameter space though. In this case the evidence favours the AR model as it did for the US. But, again, the tests of the RE restrictions yield very low p-values for either model and inflation measure. In other words, the tests support modelling the unemployment rate gap as an autoregression, so that it is not Granger-caused by the inflation gap, but they reject treating the unemployment rate gap as the sole cause of $\epsilon_{\pi,t}$, instead finding a role for lagged inflation gaps.

Table 1 includes results that vary by country, inflation measure, and time-series model. We also calculated results with samples starting later, for example in 1956 to avoid the price swings of the Korean War or in 1984 to coincide with the Great Moderation. These also coincide with break dates in some previous studies cited in section 3. The aim here also is to see whether there is evidence of a flattening PC over time. These results defy easy summary. There is some support for the RE-PC, for example for the US GDP deflator after 1956. But in most cases the RE restrictions still are unlikely.

7.2 Real GDP Results

Table 2 reports results when the measure of real activity is 100 times the log of real GDP. The estimated drift $\hat{\mu}$ in this measure is positive but is not shown.

There are four main findings. First, in 11 of 12 cases we find a positive $\hat{\kappa}$, indicating a positive relationship between the inflation gap and the output gap. The exception occurs in the case of the UK CPI with the VAR model. The parameter $\hat{\kappa}$ is sometimes estimated quite precisely, so that there is evidence of an all-gap PC. Second, for the UK $\hat{\beta}$ ranges from 0.889 to 0.916 and is estimated with some precision. But $\hat{\beta}$ is not well-identified for the US and is at the edges of the parameter space for Canada. Third, the test statistic labelled RE:AR suggests that the RE restrictions in the autonomous case do not hold. Across countries and inflation measures the largest p-value is 0.03. Fourth, the test statistic labelled RE:VAR, which tests the RE restrictions in the VAR case, sometimes has p-values above conventional levels such as 0.05 or 0.10, suggesting some support for the RE-PC. Overall, though, there is no case in which we find a large p-value for this statistic along with a positive $\hat{\kappa}$ and a well-identified $\hat{\beta}$.

As with the unemployment rate, we also studied other time periods. We focused on the 1984–2019 period, to explore the possibility that the rejection of the RE restrictions is caused by a shift in the parameter values. The results are not shown but generally are similar to those in Table 2. The exception is the case of the US and inflation measured with the GDP deflator. In that case there are three aspects of the post-1984 results worth noting. First, $\hat{\beta}$ is near one and precisely estimated. Second, $\hat{\kappa}$ is positive with a lower value for each model than the full-sample values in Table 2, indicating a flatter all-gap PC. However, this slope is not always estimated with great precision. Third, p-values for the RE tests are 0.20 in the AR case and 0.25 in the VAR case. In contrast, the p-values for RE tests with US CPI inflation or with either inflation measure for the UK and Canada remain very low.

Recall that one of the original aims of studying an all-gap PC was to refine measurements of the output gap. The underlying idea of course is that if the all-gap PC holds then that should help define the output gap, for example, as the thing that explains the inflation gap. However, we do not graph measured trends and gaps from the RE-PC or comment on how they differ from the trends and gaps in unrestricted models, for two reasons. First, the RE restrictions generally receive little support. Second, there is no general pattern to the effect of the RE restrictions on the output gap. For example, these restricted gaps are not systematically smoother across countries, inflation measures, or activity measures.

8. Conclusion

This paper seeks to estimate the all-gap PC in a limited-information setting under rational expectations. In the LIML tradition, the aim is to see whether it can be useful as a building block in macroeconomic models. In the case with an autonomous, AR(2) x-gap, RE predicts that the inflation gap can be explained by lagged x-gaps, with no role for lagged π-gaps. In the case where the two gaps follow a VAR(2), there are two-overidentifying restrictions on the joint dynamics. Overall, there is little support for these restrictions in long spans of quarterly data for the US, UK, and Canada. There *is* evidence of a PC in that the inflation gap is negatively related to the unemployment rate gap and positively related to the output gap. But this PC does not appear to be of the RE variety.

RE seems a useful benchmark and a good place to start in studying the all-gap PC. As Reis (2021) notes, there is not yet a widely-accepted, plug-in, alternative model of expectations. But, given our findings, it may be fruitful to study the all-gap PC under different assumptions about these expectations.

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Data Appendix

All series are quarterly. The inflation rate is measured as the annualized quarterly percent growth rate in the listed price index.

United States, 1947Q2–2019Q4: CPI (CPIAUCSL) GDP deflator (growth rate A191RI1Q225SBEA); civilian unemployment rate (from 1948Q1) (UNRATE); from FRED, Federal Reserve Bank of St. Louis.

United Kingdom, 1947Q1/1955Q1–2019Q24 The CPI and unemployment rate (16 and over) are from Ryland and Thomas (2017) updated after 2016 from ONS series D7BT and MGSX respectively. The GDP deflator is ONS series L8GG and real SA GDP is ONS series ABMI, both of which begin in 1955Q1.

Canada, 1947Q1–20191Q4: CPI seasonally adjusted (v41690914), GDP deflator (v62305783/v62305752), unemployment rate 15 and over (v2062815). The original source is Statistics Canada with their codes in brackets, spliced to historical data in Stephen Gordon's Project Link (`https://worthwhile.typepad.com/stephen/link/`)

Table 1. Phillips Curve Estimates and Tests
x_t: unemployment rate

$$\epsilon_{\pi,t} = \beta E_t \epsilon_{\pi,t+1} + \kappa \epsilon_{x,t} + u_{\pi,t}$$

Model→	AR $\{\epsilon_{x,t}\}$				VAR $\{\epsilon_{\pi,t}, \epsilon_{x,t}\}$		
Test→			RE:AR	AR:VAR			RE:VAR
Economy: Price Time ↓	$\hat{\beta}$ (se)	$\hat{\kappa}$ (se)	$\chi^2(2)$ (p)	$\chi^2(2)$ (p)	$\hat{\beta}$ (se)	$\hat{\kappa}$ (se)	$\chi^2(2)$ (p)
US: CPI 1948Q1–2019Q4	0.901 (0.001)	-0.114 (0.014)	16.11 (0.001)	2.25 (0.33)	0.783 (0.285)	-0.461 (0.308)	6.98 (0.03)
US: Deflator 1948Q1–2019Q4	0.522 (0.364)	-0.285 (0.129)	14.18 (0.001)	2.55 (0.28)	0.898 (0.005)	-0.612 (0.231)	6.88 (0.03)
UK: CPI 1948Q1–2019Q4	0.005 (0.093)	-1.51 (0.55)	3.66 (0.16)	18.33 (0.001)	0.937 (0.004)	-0.225 (0.059)	6.42 (0.04)
UK: Deflator 1956Q1–2019Q4	0.999 (0.002)	0.013 (0.004)	12.21 (0.002)	10.89 (0.004)	0.935 (0.003)	0.019 (0.005)	19.36 (0.000)
Canada: CPI 1948Q1–2019Q4	0.002 (0.012)	-1.46 (0.356)	15.43 (0.001)	3.30 (0.19)	0.955 (0.016)	-2.36 (1.12)	9.28 (0.01)
Canada: Deflator 1948Q1–2019Q4	0.999 (0.063)	-0.274 (0.111)	13.71 (0.001)	1.26 (0.53)	0.999 (0.031)	-0.864 (0.571)	13.88 (0.001)

Notes: The table shows maximum likelihood estimates of the all-gap PC parameters from the UC model of $\{\pi_t, x_t\}$, with π_t measured with the CPI or GDP deflator and x_t measured with the unemployment rate. For the autonomous model $\epsilon_{x,t}$ follows an AR(2) model (6). The first $\chi^2(2)$ statistic, in the column labelled RE:AR, provides a likelihood ratio test of the RE restrictions that exclude $\epsilon_{\pi,t-1}$ and $\epsilon_{\pi,t-2}$ from the evolution of $\epsilon_{\pi,t}$ (8). The second $\chi^2(2)$ statistic, in the column labelled AR:VAR; provides a likelihood ratio test of the autonomous gap model against the VAR model. The final $\chi^2(2)$ statistic, in the column labelled RE:VAR, provides a likelihood ratio test of the RE restrictions (17) applied to the unrestricted VAR (5). Standard errors are calculated as the square root of the diagonal element of the inverse of the Hessian matrix.

Table 2. Phillips Curve Estimates and Tests
x_t: 100 × ln GDP

$$\epsilon_{\pi,t} = \beta E_t \epsilon_{\pi,t+1} + \kappa \epsilon_{x,t} + u_{\pi,t}$$

Model→	AR $\{\epsilon_{x,t}\}$				VAR $\{\epsilon_{\pi,t}, \epsilon_{x,t}\}$		
Test→			RE:AR	AR:VAR			RE:VAR
Economy: Price Time ↓	$\hat{\beta}$ (se)	$\hat{\kappa}$ (se)	$\chi^2(2)$ (p)	$\chi^2(2)$ (p)	$\hat{\beta}$ (se)	$\hat{\kappa}$ (se)	$\chi^2(2)$ (p)
US: CPI	0.013	0.495	17.07	20.17	0.648	0.513	14.66
1948Q1–2019Q4	(0.048)	(0.397)	(0.00)	(0.00)	(0.636)	(0.332)	(0.00)
US: Deflator	0.408	0.471	7.59	8.71	0.426	0.632	5.36
1948Q1–2019Q4	(3.865)	(0.155)	(0.02)	(0.01)	(2.421)	(0.204)	(0.07)
UK: CPI	0.916	0.304	8.94	19.2	0.912	-0.579	3.18
1956Q1–2019Q4	(0.097)	(0.187)	(0.01)	(0.00)	(0.168)	(0.195)	(0.20)
UK: Deflator	0.889	0.635	10.26	0.82	0.896	0.801	10.12
1956Q1–2019Q4	(0.014)	(0.382)	(0.006)	(0.66)	(0.005)	(0.429)	(0.006)
Canada: CPI	0.999	0.318	10.27	1.55	0.007	1.523	4.58
1948Q1–2019Q4	(0.011)	(0.175)	(0.006)	(0.46)	(0.241)	(0.567)	(0.10)
Canada: Deflator	0.999	0.338	7.18	2.58	0.999	0.767	6.44
1948Q1–2019Q4	(3.197)	(0.223)	(0.03)	(0.27)	(0.821)	(0.527)	(0.04)

Notes: The table shows maximum likelihood estimates of the all-gap PC parameters from the UC model of $\{\pi_t, x_t\}$, with π_t measured with the CPI or GDP deflator and x_t measured with $100 \times$ ln GDP. The trend component for x_t now includes a drift term (not shown). For the autonomous model $\epsilon_{x,t}$ follows an AR(2) model (6). The first $\chi^2(2)$ statistic, in the column labelled RE:AR, provides a likelihood ratio test of the RE restrictions that exclude $\epsilon_{\pi,t-1}$ and $\epsilon_{\pi,t-2}$ from the evolution of $\epsilon_{\pi,t}$ (8). The second $\chi^2(2)$ statistic, in the column labelled AR:VAR; provides a likelihood ratio test of the autonomous gap model against the VAR model. The final $\chi^2(2)$ statistic, in the column labelled RE:VAR, provides a likelihood ratio test of the RE restrictions (17) applied to the unrestricted VAR (5). Standard errors are calculated as the square root of the diagonal element of the inverse of the Hessian matrix.