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Quantile Tool Box Measures for Empirical Analysis and for Testing Distributional Comparisons in Direct Distribution-Free Fashion

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by

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Abstract

This paper provides a set of tool box measures for flexibly describing distributional changes and empirically implementing several dominance criteria for social welfare comparisons and broad income inequality comparisons. Dominance criteria are expressed in terms of vectors of quantile statistics based on income shares and quantile means. Asymptotic variances and covariances of these sample ordinates are established from a Quantile Function Approach that provides a framework for direct statistical inference on these vectors. And practical empirical criteria are forwarded for using formal statistical inference tests to reach conclusions about ranking social welfare and inequality between distributions. Examples include rank dominance, Lorenz dominance, generalized Lorenz dominance, income polarization, and distributional distance dominance between income groups.

1. Introduction

Since about 1980 to the early twenty-first century, income inequality in many developed economies rose dramatically to historic levels (Guvenen et al., 2022). Media attention has focused on issues of "equitable growth" (Drummond, 2021), "fairness in growth" (Lohr, 2022), and "common prosperity" (The Economist, 2021a,b). Over the same while, the theoretical economics literature has been examining the distributional incidence of growth and its social welfare implications (Palmisano and Peragine, 2015) and its related implications for equality of opportunity in a growing economy (Peragine, 2004; Bosmans and Ozturk, 2021). Empirically implementing such concerns has been reviewed, for example, in Cowell (2011) and Duclos and Araar (2006). Meanwhile, recent world events such as the covid pandemic and war in Ukraine have elicited major economic adjustments as supply chains are reworked and much basic and high-tech manufacturing is becoming reshored, and as income support programs burgeon (at least temporarily) and labour markets dramatically tighten – resulting in notable improvements in lower incomes and a new "unstuck middle" (The Economist, 2023). In both Canada and the United States, federal governments have also been focusing policies to better target low-incomes and middle-class families. All these developments call for an empirically implementable set of criteria and procedures for better evaluating if people are indeed better (or worse) off, income inequality has been reduced, or economic opportunities have been improved. This paper offers an empirical approach to evaluate such changes in an easily implementable framework of statistical inference based on well-known disaggregative distributional statistics. The paper is thus written in the spirit of providing a tool box of useful measures for effective analysis of income distributions and their changes (Skuterud et al., 2004; Bellu et al., 2005).

The paper has three objectives. The first is *descriptive*. It seeks to offer a tool box set of standard intuitive measures for the empirical analysis of income distributions (and how they may show differences between population groups and changes over time). Since the measures are disaggregative (e.g., based on deciles or vigantiles), they can flexibly highlight different patterns for various regions of the distribution (e.g., low incomes, middle incomes and high incomes). Conveniently, these different patterns can often be graphically illustrated in terms of simple curves.

The second objective is *inferential*. The paper seeks to provide an easily implementable framework of statistical inference for formal statistical testing and evaluation of various forms of distributional dominance or comparison, for example in terms of normative social welfare and general economic well-being, in terms of income inequality variously measured, and in terms of distributional distance between income groups.

The third objective is *simplification* by showing that such analyses can be carried out in straightforward distribution-free fashion that involves direct variance-covariance estimation (as in Beach, 2023) for the above tool box measures. The paper thus seeks to simplify and unify distributional analysis in a common framework that involves explicit variance-covariance formulas (for distribution-free standard errors) that can be directly estimated without burdensome computational procedures.

The paper makes three main contributions. First, it provides a unified framework in the form of the Quantile Function Approach (QFA) for establishing statistical inference for a set of tool box measures of disaggregative income inequality statistics such as decile income shares or quantile means. This allows one to derive explicit formulas for the (asymptotic) variances and covariances of these measures. Second, it turns out that the (asymptotic) variances and covariances of these quantile-based measures are distribution-free in the sense that they can be directly and consistently estimated from conventional statistics readily calculated from available microdata files without having to know the specific underlying income distribution function itself and thus avoiding computationally burdensome resampling procedures or kernel estimation approaches. In the process of this derivation, the paper corrects or adjusts for previous faulty specifications in the published statistics literature. Third, the paper follows Beach (2022) in providing an empirically implementable set of rules or a practical empirical criterion (PEC) for empirically testing hypotheses of distributional dominance and establishing whether such distributional comparisons are indeed statistically significant. Such comparisons include rules for social welfare dominance, general income inequality comparisons, and changes over time of income polarization or distributional distance (as an aspect of inequality of opportunity). Clearly, this general empirical approach relates to a wide range of topics that are touched upon as the analytical development proceeds below.

The paper is organized as follows. The next section sets out the Quantile Function Approach (QFA) developed in previous work by the author (Beach, 2021) that serves as the basis for the statistical inference framework of the present analysis. Asymptotic variances and standard error results are presented for quantile-based estimates of sets of quantile means – such as provided for income deciles by Statistics Canada or the United States Bureau of the Census – used to distributionally characterize a distribution of income. Section 3 extends the analysis to present the basic analytical results from this approach for quantile-based income shares – such as income deciles also provided by Statistics Canada and the U.S. Bureau of the Census again used to basically characterize a distribution of income. Section 4 outlines the normative perspective to evaluating changes in social welfare and income inequality. Then Section 5 applies the above

QFA framework to testing for rank dominance between distributions. Since each distribution is characterized by a vector of disaggregative distributional statistics (such as decile means), the comparison of two vectors involves setting out a practical empirical criterion (or PEC) for the statistical ranking of vectors of statistics such as decile means. Sections 6 and 7 then apply this empirical framework to Lorenz dominance and generalized Lorenz dominances between income distributions with appropriate PEC_s forwarded for each. Section 8 shows how one can decompose (empirically estimated) social welfare into efficiency and equity contribution components, and develops a PEC for the equity component itself. Sections 9 and 10 consider how to address inequality dominance when the Lorenz curves of two distributions cross one or more times. Section 11 develops distributional distance functions and income polarization curves and applies the PEC procedure to compare vectors of distributional distance and income polarization for two income distributions. Section 12 concludes with some implications of the paper's analysis.

2. The Quantile Function Approach and Quantile Means

Empirical measures of economic well-being and income inequality are built up from disaggregative statistics on percentile mean income levels and percentile income shares. Percentile statistics are those that are expressed in terms of given percentage groups of the ranked or ordered observations in a microdata sample. In the case of income distribution statistics, the data observations in a sample are ordered by income from the lowest income observation to the highest income observation. The ordered observations are then divided into non-overlapping income groups, say, in terms of ten deciles or twenty vigintiles (or generically referred to as quantile income groups or simply quantiles). So the first decile group consists of those observations with the 10 percent lowest income levels, the second decile group consists of the next 10 percent lowest income recipients, and so on to the top or tenth decile income group which includes those 10 percent of income recipients with the highest income levels in the sample. The standard Lorenz curve of (cumulated) income shares, for example, is based around such percentile groups, and quantile mean income levels can also be calculated for each of the percentile groups.

The key feature of such percentile statistics is that the relative sizes of the percentile groups are *given* percentages of the sample or distribution. This turns out to simplify quite dramatically the sampling properties of quantile-based sample statistics. (Contrast this with, say, median-based income groups where the middle-income group consists of those with incomes lying between 50 percent and 150 percent of the median income level – a common designation of the so-called Middle Class. In this case, the size of the group is not given, but a consequence of random sampling. As a results, the sampling properties of median-based statistics are not at all convenient to deal with (Beach, 2021).)

Note also that quantile means and quantile income shares are two examples of what can be referred to as tool box measures (Beach, 2021) for characterizing the disaggregative structure of an income distribution. Both Statistics Canada and the U.S. Bureau of the Census publish annual series on both decile (and quintile) income shares and mean income levels. Such a disaggregative characterization is very flexible and especially useful for highlighting the quite different patterns of distributional change that have occurred since around 1980 in many developed economies (including Canada and the United States) where middle incomes have slipped quite markedly while higher incomes have dramatically risen. Such a disaggregative approach is also useful for comparing different regional income distributions or income distributions between different demographic groups.

To formalize the Quantile Function approach (QFA) taken in this paper, consider first some formal concepts and notation. Suppose the distribution of income Y is divided into K ordered income groups, so that K=10 in the case of deciles and K=5 for quintiles. Let the dividing proportions of recipients be $p_1 < p_z < \ldots < p_{K-1}$ (with $p_o = 0$ and $p_K = 1.0$)[1]. Then in terms of the underlying (population) density of income recipients, the mean income of the i'th quantile is given by

$$\mu_i = \int_{\xi_{i-1}}^{\xi_i} y\, f(y)dy \,/\, \int_{\xi_{i-1}}^{\xi_i} f(y)dy \qquad \text{for } i = 1, \ldots, K \qquad (1)$$

where $f(\cdot)$ is the underlying (population) density function and the ξ_i's are the cut-off income levels corresponding to the proportions $p_1, p_2, \ldots, p_{K-1}$ (with $\xi_o = 0$ where incomes are assumed positive and $\xi_K = \infty$).[2] Similarly, the income share of the i'th income group can be expressed as

$$IS_i = \int_{\xi_{i-1}}^{\xi_i} \left(\frac{1}{\mu}\right) y f(y)dy \qquad \text{for } i = 1, \ldots, K \qquad (2)$$

and μ is the mean of the overall population distribution of income.

The integral expressions – what we'll refer to as quantile functions – link the quantile means μ_i and quantile income shares IS_i to the quantile income cut-offs ξ_i, ξ_{i-1} and the overall mean μ. A broad theorem by C.R. Rao (1965) says that, if one knows the asymptotic distribution of the sample estimates of ξ_i, ξ_{i-1}, and μ as joint normal and if, in the population, functions of ξ_i, ξ_{i-1}, and μ are continuous and differentiable in these parameters, then sample estimates of these functions will also be asymptotically normally distributed with asymptotic means and variances

[1] We assume in what follows that the data samples used are random samples. If the survey records are indeed weighted (as in the case of stratified samples, for example), the formulas can be readily adjusted by replacing sums of observations by sums of the sample weighted observations.

[2] If some incomes do take negative values (such as with capital gains losses in a year or net self-employment income that is negative in a year where illness has prevented the recipient working for a time), then simply define p_o to be the lowest income value in the sample.

(and covariances) that can be calculated in a straightforward fashion. We refer to this as Rao's linkage theorem. From both (1) and (2), it can be seen that one can use this theorem to thus establish the asymptotic distributions of sample estimates of both μ_i and IS_i.

In the case of quantile means, as a simple illustration of the quantile function approach, it has long been established that the sample cut-offs $\hat{\xi}_i$'s are indeed asymptotically normally distributed. More specifically, let $\hat{\xi} = \left(\hat{\xi}_1, \hat{\xi}_2, \ldots \hat{\xi}_{K-1}\right)'$ be a vector of *K-1* sample quantile cut-offs[3] from a random sample of size N drawn from a continuous population density $f(\bullet)$ such that the $\hat{\xi}_i$'s are uniquely defined and $f_i \equiv f(\xi_i) > 0$ for all *i = 1, …, K-1*. Then it can be shown (see, for example, Wilks (1962), p. 273, or Kendall and Stuart, 1969, pp. 237-239) that the vector $\sqrt{N}\,(\hat{\xi} - \xi)$ converges in distribution to a *(K-1)*-variate normal distribution with mean zero and variance-covariance matrix **Λ** where

$$\Lambda = \begin{bmatrix} \frac{p_1(1-p_1)}{f_1^2} & \cdots & \frac{p_1(1-p_{K-1})}{f_1 f_{K-1}} \\ \vdots & & \vdots \\ \frac{p_1(1-p_{K-1})}{f_1 f_{K-1}} & \cdots & \frac{p_{K-1}(1-p_{K-1})}{f_{K-1}^2} \end{bmatrix}.$$

Note how the (asymptotic) variances and covariances of the $\hat{\xi}_i$'s depend on the specific functional form of $f(\bullet)$ in the denominators and hence are not distribution-free.

Then applying a multivariate version of Rao's linkage theorem (Rao, 1965, p. 388), consider the full set of *K* sample quantile means $\hat{m} = (\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_K)'$ corresponding to the vector of population quantile means $m = (\mu_1, \mu_2, \ldots, \mu_K)'$ where μ_i is defined in eq. (1). Then according to Rao's theorem for continuous differentiable functions, the vector $\hat{m}$ is

[3] To estimate the sample quantile cut-offs, order the sample of *N* observations by income level. Then, in the case of deciles, say, $\hat{\xi}_i$ is that income level such that $p_i N$ observations lie below it and the rest at or above. If there is no single observation meeting this condition, simply take the average of the two adjacent observations (below and above) that are closest.

asymptotically joint normally distributed in that $\sqrt{N}(\widehat{m} - m)$ converges in distribution to a joint normal with *KxK* (asymptotic) variance-covariance matrix *V* where

$$Asy.var(\widehat{m}) \equiv V = G\ \Lambda\ G' \tag{3}$$

and the *K x (K-1)* matrix *G* is

$$G = \begin{bmatrix} g_{11} & \cdots & g_{1,K-1} \\ \vdots & & \vdots \\ g_{K,1} & \cdots & g_{K,K-1} \end{bmatrix}$$

$$= \left[\frac{\partial \mu_i}{\partial \xi_j}\right] \qquad \text{with } i = 1, \ldots, K \text{ rows and } j = 1, \ldots, K\text{-}1 \text{ columns.} \tag{4}$$

For convenience, rewrite eq. (1) as

$$\mu_i = \left(\frac{1}{D_i}\right) \bullet N_i(\xi_i, \xi_{i-1}) \qquad \text{for } i = 1, \ldots, K,$$

where N_i is an explicit function of ξ_i and ξ_{i-1} in the numerator of (1) and

$$D_i = \int_{\xi_{i-1}}^{\xi_i} f(y)dy = p_i - p_{i-1}$$

is given.

To illustrate the calculation of the gradients $\frac{\partial \mu_i}{\partial \xi_j}$, make use of Leibnitz's Rule and consider the case of decile income groups (i.e., K = 10 and D_i = .10):

$$g_{11} = \frac{\partial \mu_1}{\partial \xi_1} = 10\frac{\partial N_1}{\partial \xi_1} = 10\ \xi_1 \bullet f(\xi_1)$$

$$g_{1j} = \frac{\partial \mu_1}{\partial \xi_j} = 10\frac{\partial N_1}{\partial \xi_j} = 0 \qquad \text{for } j = 2, \ldots, K\text{-}1\,.$$

$$g_{21} = \frac{\partial \mu_2}{\partial \xi_1} = 10\frac{\partial N_2}{\partial \xi_1} = 10\,(-\xi_1) \bullet f(\xi_1)$$

$$g_{22} = \frac{\partial \mu_2}{\partial \xi_2} = 10\frac{\partial N_2}{\partial \xi_2} = 10\ \xi_2 \bullet f(\xi_2)$$

$$g_{2j} = \frac{\partial \mu_2}{\partial \xi_j} = 10\frac{\partial N_2}{\partial \xi_j} = 0 \qquad \text{for } j = 3, \ldots, K\text{-}1\ .$$

and so on up to

$$g_{Kj} = \frac{\partial \mu_K}{\partial \xi_j} = 10\frac{\partial N_K}{\partial \xi_j} = 0 \qquad \text{for } j = 1, \ldots, K\text{-}2\ .$$

$$g_{K,K-1} = \frac{\partial \mu_K}{\partial \xi_{K-1}} = 10\frac{\partial N_K}{\partial \xi_{K-1}} = 10\ (-\xi_{K-1}) \bullet f(\xi_{K-1}).$$

As a result, the *G* matrix is the banded diagonal-type matrix:

$$G = [\begin{matrix} 10\ \xi_1 \bullet f(\xi_1) & 0 & 0 & 0 & \cdots \\ -10\ \xi_1 \bullet f(\xi_1) & 10\ \xi_2 \bullet f(\xi_2) & 0 & 0 & \cdots \\ 0 & -10\ \xi_2 \bullet f(\xi_2) & 10\ \xi_3 \bullet f(\xi_3) & 0 & \cdots \\ \vdots & 0 & -10\ \xi_3 \bullet f(\xi_3) & 10\ \xi_4 \bullet f(\xi_4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots \end{matrix}$$

$$\begin{matrix} \cdots & 0 & 0 \\ \cdots & 0 & 0 \\ & \vdots & \vdots \\ \cdots & 0 & 0 \\ \cdots & -10\ \xi_8 \bullet f(\xi_8) & 10\ \xi_9 \bullet f(\xi_9) \\ \cdots & 0 & -10\ \xi_9 \bullet f(\xi_9) \end{matrix}\ ]\ . \qquad (5)$$

The (asymptotic) variances, then, are gotten by multiplying the corresponding row of *G* and column of G' (i.e., row of *G*) by the appropriate diagonal element of the variance-covariance matrix. So

$$Asy.var(\hat{\mu}_1) = G(row\ 1) \bullet \Lambda \bullet\ G(row\ 1)'$$

$$= (10)^2\ \xi_1^2\ \bullet f(\xi_1)^2\ \bullet \left[\frac{p_1(1-p_1)}{f(\xi_1)^2}\right]$$

$$= (10)^2\ p_1(1-p_1)\ \xi_1^2\ . \qquad (6a)$$

Similarly,

$$Asy.var(\hat{\mu}_{10}) = G(row\ 10) \bullet \Lambda \bullet G(row\ 10)'$$

$$= (10)^2\, \xi_9^2 \bullet f(\xi_9)^2 \bullet \left[\frac{p_9(1-p_9)}{f(\xi_9)^2}\right]$$

$$= (10)^2\, p_9(1-p_9)\, \xi_9^2 \ . \tag{6b}$$

And for *i = 2, …, 9* ,

$$Asy.var(\hat{\mu}_i) = G(row\ i) \bullet \Lambda \bullet G(row\ i)'$$

$$= (10)^2\, [p_{i-1}(1-p_{i-1})\xi_{i-1}^2 + p_i(1-p_i)\, \xi_i^2$$

$$- 2\, p_{i-1}(1-p_i)\xi_{i-1}\, \xi_i] \ . \tag{6c}$$

More generally, then,

$$Asy.var(\hat{\mu}_1) = \left(\frac{1}{D_1}\right)^2 p_1(1-p_1)\, \xi_1^2 \ , \tag{7a}$$

$$Asy.var(\hat{\mu}_K) = \left(\frac{1}{D_K}\right)^2 p_{K-1}(1-p_{K-1})\, \xi_{K-1}^2 \ ; \tag{7b}$$

and for *i = 2, …, K-1* ,

$$Asy.var(\hat{\mu}_i) = \left(\frac{1}{D_{i-1}}\right)^2 p_{i-1}(1-p_{i-1})\xi_{i-1}^2 + \left(\frac{1}{D_i}\right)^2 p_i(1-p_i)\xi_i^2$$

$$- 2\left(\frac{1}{D_{i-1}}\right)\left(\frac{1}{D_i}\right) p_{i-1}(1-p_i)\xi_{i-1}\, \xi_i \ . \tag{7c}$$

These results on the (asymptotic) variances, then, are sufficient to determine the standard errors of the quantile mean estimates. Since the formulas in eqs. (6)-(7) involve unknown population parameters, one obtains *estimated* (asymptotic) variances by replacing all the unknown parameters by their consistent estimates. So, for example, in (6a),

$$As\hat{y.}var(\hat{\mu}_1) = (10)^2\, p_1(1-p_1)\, \hat{\xi}_1^2$$

where ξ_1 is replaced by its standard sample estimate. Rao (1965, p. 355) has also shown that if $f(\bullet)$ is strictly positive, then the $\hat{\xi}_i{'}$s are indeed (strongly) consistent. The estimated standard error for $\hat{\mu}_i$ is then gotten by adjusting for the sample size of the estimation sample:

$$S.E.(\hat{\mu}_i) = \left[\frac{Asy\hat{.}var(\hat{\mu}_i)}{N}\right]^{1/2} \tag{8}$$

for all *i = 1, …, K* .

Testing for the statistical significance of $\hat{\mu}_i$ can then be undertaken by calculating the conventional "t-ratio"

$$t = \hat{\mu}_i \; / \; S.E.(\hat{\mu}_i)$$

and comparing it to appropriate critical values on the standard normal table (at a given level of significance or confidence).

Note that the asymptotic variances and standard errors of the quantile means for given percentile groups are *distribution-free*. While the elements of matrices G and Λ do indeed depend on the terms $f_i = f(\xi_i)$ and hence are distribution-dependent, when the calculations in eq. (3) are carried out, all thef_i terms cancel out. The calculations in eqs. (7) and (8) are thus very straightforward and easy to calculate directly. This dramatic simplification follows from the definition of income groups in (exogenously designated) percentile terms. Contrast this, for example, with the case of median-based conditional means. If the proportion of the population of income recipients with incomes between 50 percent and 150 percent of the median is taken as a measure of the size of the so-called Middle Class, then it turns out that the mean income of this middle class income group can be calculated, but its (asymptotic) variance is a complicated function of the terms $f(0.5\ \xi_m)$ and $f(1.5\ \xi_m)$ when ξ_m is the median of the underlying income distribution (Beach, 2021) and hence is distribution-dependent. Its standard error thus involves having to empirically evaluate the density function $f(\bullet)$ at different points. Taking a quantile function approach based on percentile income groups thus allows one to avoid having to estimate assumed underlying population density function forms (such as the lognormal in Beach, 2021), or undertake burdensome computer-based estimation techniques for density ordinate evaluation such as bootstrapping procedures (as in Davidson, 2018) or kernel estimation methods (as in Cowell, 2011). It is in this sense that the current approach is said to provide *direct* (asymptotic) variance-covariance estimates.

Note also the flexibility of the quantile function approach applied to percentile income groups. The latter groups do not have to be of uniform size. Depending on the empirical analysis being undertaken, one may wish to have narrower groups (eg., deciles or vigintiles) towards the two ends of the distribution and wider groups (eg., quintile) over the middle range of the distribution. One can also choose how fine a breakdown of percentile groups one wishes. While official statistical agencies publish results for deciles and quintiles, one could – depending on how large microdata sets are available – also examine, say, vigintile income groups as well, again at the two ends of the distribution (Beach, 2023).

Note further that one can apply the above results to look at *differences* in individual quantile means between different population groups – such as quantile mean earnings differences between male and female workers in the labour market – and at *changes* in separate quantile means between time periods. So long as the estimates being compared are from independent samples, the variance of the difference in sample estimates is simply the sum of the separate variances, and the standard error of the difference is given by

$$S.E.\left(\hat{\mu}_i^b - \hat{\mu}_i^a\right) = \left[\frac{Asy\hat{.}var\left(\hat{\mu}_i^b\right)}{N^b} + \frac{Asy\hat{.}var\left(\hat{\mu}_i^a\right)}{N^a}\right]^{1/2} \tag{9}$$

where superscripts *a* and *b* refer to the two separate sample estimates. A quantile analysis thus allows for potentially quite detailed disaggregative examination of differences between distributions. And the range of toolbox measures available furthers the perspective and flexibility of such examinations.

Indeed, one could express these differences in relative or percentage terms – or what The Economist (2021c, p. 24) refers to as Piketty lines of different growth rates of quantile means across the different regions of the income distribution. In this case, it is shown in Beach (2021) that, if

$$\hat{q}_i = \left(\hat{\mu}_i^b - \hat{\mu}_i^a\right) / \hat{\mu}_i^a = \left(\frac{\hat{\mu}_i^b}{\hat{\mu}_i^a}\right) - 1\,,$$

then approximately

$$V\hat{a}r(\hat{q}_i) = \left(\frac{-\hat{\mu}_i^b}{\left(\hat{\mu}_i^a\right)^2}\right)^2 \bullet\; V\hat{a}r(\hat{\mu}_i^a) + \left(\frac{1}{\hat{\mu}_i^a}\right)^2 \bullet V\hat{a}r\left(\hat{\mu}_i^b\right)$$

$$= \left(\frac{-\hat{\mu}_i^b}{\left(\hat{\mu}_i^a\right)^2}\right)^2 \bullet \left[\frac{Asy\hat{\cdot}var\left(\hat{\mu}_i^a\right)}{N^a}\right] + \left(\frac{1}{\hat{\mu}_i^a}\right)^2 \bullet \left[\frac{Asy\hat{\cdot}var\left(\hat{\mu}_i^b\right)}{N^b}\right]$$

and again

$$S.E.(\hat{q}_i) = [V\hat{a}r(\hat{q}_i)]^{1/2}.$$

Again, the standard error estimates are distribution-free.

It would thus be helpful to users of official decile and quintile mean statistics from government statistical agencies if these agencies provided the actual sample sizes (i.e., the *N* in the denominator of (8)) that their survey estimates are based on, and not just the overall survey sample size (i.e., the estimation sample size as well as the survey sample size).

3. Application of the QFA to Incomes Shares

The income share of the i'th income group can be expressed as

$$IS_i \equiv \int_{R_i} \left(\frac{1}{\mu}\right) yf(y)dy \qquad \text{for } i = 1, \ldots, K, \tag{10a}$$

with integration over the region R_i running from ξ_{i-1} to ξ_i , and μ is the mean of the overall (population) distribution of income. The integral in (10a) can, for future notational convenience, be written as

$$IS_i = N_i(\xi_{i-1}, \xi_i, \mu) = \int_{\xi_{i-1}}^{\xi_i} \left(\frac{1}{\mu}\right) yf(y)dy\,. \tag{10b}$$

It can be seen that estimates of IS_i or N_i involve estimates of two sets of parameters – the range of integration cut-offs ξ_{i-1} and ξ_i and the overall population mean μ . To take account of this, we make use of a useful paper by Lin, Wu and Ahmad (1980) (henceforth LWA). LWA establish that, under general regularity conditions, $\hat{\xi}_{i-1}, \hat{\xi}_i$, and $\hat{\mu}$ are asymptotically joint normally distributed with (asymptotic) variance-covariance matrix

$$\Sigma = \left[\sigma_{ij}\right] \tag{11}$$

where $\sigma_{11} = \frac{p_{i-1}(1-p_{i-1})}{[f(\xi_{i-1})]^2}, \quad \sigma_{22} = \frac{p_i(1-p_i)}{[f(\xi_i)]^2}, \quad \sigma_{33} = \sigma^2$

$$\sigma_{12} = \frac{p_{i-1}(1-p_i)}{f(\xi_{i-1})f(\xi_i)} = \sigma_{21}$$

$$\sigma_{13} = \frac{\chi_{i-1} - \mu(1-p_{i-1})}{f(\xi_{i-1})}$$

and $\sigma_{23} = \frac{\chi_i - \mu(1-p_i)}{f(\xi_i)}$,

where σ^2 is the variance of the overall (population) distribution of income, and

$$\chi_{i-1} = \int_{\xi_{i-1}}^{\infty} yf(y)dy \qquad \chi_i = \int_{\xi_i}^{\infty} yf(y)dy\;.$$

(Note that the published journal version of LWA has some serious typos in the statement of their theorem 2.1, but their earlier discussion paper-version presents the theorem correctly. To keep the correct version clear, I have adopted slightly different notation from LWA. I have also translated the statement of their theorem into the context of the current analysis.)

The last two terms can be stated more conveniently for our purposes. Note that

$$\mu = \int_0^\infty y f(y) dy$$

$$= \int_0^{\xi_i} y f(y) dy + \chi_i$$

$$= p_i \bar{\mu}_i + \chi_i$$

where $\bar{\mu}_i$ is the mean income for incomes below ξ_i. So

$$\chi_i = \mu - p_i \bar{\mu}_i$$

and hence

$$\sigma_{23} = p_i(\mu - \bar{\mu}_i) \,/\, f(\xi_i) \qquad \text{(12a)}$$

which, interestingly, is strictly positive. Similarly,

$$\chi_{i-1} = \mu - p_{i-1}\bar{\mu}_{i-1} \quad \text{and} \quad \sigma_{13} = p_{i-1}(\mu - \bar{\mu}_{i-1}) \,/\, f(\xi_{i-1}) \qquad \text{(12b)}$$

which, again, is also strictly positive.[4]

One can now combine this set of LWA results with Rao's linkage theorem. So, if $\hat{\xi}_{i-1}, \hat{\xi}_i$ and $\hat{\mu}$ are asymptotically joint normal with (asymptotic) variance-covariance matrix Σ above,

[4] While the numerators of σ_{13} and σ_{23} can't be ranked, for percentiles above the mode of the income distribution the denominators can be. So one would generally expect $cov(\hat{\xi}_i, \hat{\mu})$ to be stronger for higher percentile levels.

then $\widehat{IS}_i$ is also asymptotically normally distributed with the (asymptotic) variance of $\widehat{IS}_i$ is given by

$$Asy.var\left(\widehat{IS}_i\right) = G' \, \Sigma \, G \tag{13a}$$

where

$$G = \left[\frac{\partial N_i}{\partial \xi_{i-1}}, \frac{\partial N_i}{\partial \xi_i}, \frac{\partial N_i}{\partial \mu}\right]' = [g_1, g_2, g_3]' \,. \tag{13b}$$

So in the case of *i = 1*:

$$g_1 = 0$$

$$g_2 = \left(\frac{1}{\mu}\right) \xi_1 \bullet f(\xi_1)$$

$$g_3 = \frac{-N_1}{\mu} = \frac{-IS_1}{\mu}\,,$$

and

$$Asy.var\left(I\hat{S}_1\right) = g_2^2\,\sigma_{22} + g_3^2\,\sigma_{33} + 2g_2g_3\sigma_{23}$$

$$= \left(\frac{\xi_1}{\mu}\right)^2 p_1(1-p_1) + \left(\frac{IS_1}{\mu}\right)^2 \sigma^2 - 2\left(\frac{\xi_1}{\mu}\right)\left(\frac{IS_1}{\mu}\right) p_1(\mu - \bar{\mu}_1)\,. \tag{14a}$$

In the decile case of *i = 10*:

$$g_1 = -\left(\frac{1}{\mu}\right) \xi_9 \bullet f(\xi_9)$$

$$g_2 = 0$$

$$g_3 = \frac{-N_{10}}{\mu} = \frac{-IS_{10}}{\mu}\,,$$

so

$$Asy.var\left(I\hat{S}_{10}\right) = g_1^2\,\sigma_{11} + g_3^2\,\sigma_{33} + 2g_1g_3\sigma_{13}$$

$$= \left(\frac{\xi_9}{\mu}\right)^2 p_9(1-p_9) + \left(\frac{IS_{10}}{\mu}\right)^2 \sigma^2 + 2\left(\frac{\xi_9}{\mu}\right)\left(\frac{IS_{10}}{\mu}\right) p_9(\mu - \bar{\mu}_9)\,. \tag{14b}$$

And in the decile case of i = *2, …, 9*:

$$g_1 = -\left(\frac{1}{\mu}\right)\ \xi_{i-1}\ \bullet f(\xi_{i-1})$$

$$g_2 = \left(\frac{1}{\mu}\right)\ \xi_i\ \bullet f(\xi_i)$$

and $$g_3 = -\left(\frac{1}{\mu}\right)\ IS_i\ .$$

Therefore,

$$Asy.var\left(I\hat{S}_i\right) = G'\,\Sigma\,G$$

$$= \left(\frac{\xi_{i-1}}{\mu}\right)^2 p_{i-1}(1-p_{i-1}) + \left(\frac{\xi_i}{\mu}\right)^2 p_i(1-p_i) + \left(\frac{IS_i}{\mu}\right)^2 \sigma^2$$

$$-\,2\left(\frac{\xi_{i-1}}{\mu}\right)\left(\frac{\xi_i}{\mu}\right) p_{i-1}(1-p_i)$$

$$+\,2\left(\frac{\xi_{i-1}}{\mu}\right)\left(\frac{IS_i}{\mu}\right) p_{i-1}(\mu - \bar{\mu}_{i-1}) \tag{14c}$$

$$-\,2\left(\frac{\xi_i}{\mu}\right)\left(\frac{IS_i}{\mu}\right) p_i(\mu - \bar{\mu}_i)\ .$$

The standard error of the i'th quantile income share is thus given by

$$S.E.\left(\widehat{IS}_i\right) = \left[\frac{Asy\hat{.}var(\widehat{IS}_i)}{N}\right]^{1/2} .$$

Note, incidentally, that just as IS_i is a ratio and hence units-free, so also is each term of its (asymptotic) variance and hence its standard error. The effect of randomness corresponding to $\hat{\mu}$ operates through the third term of eq. (14c) (corresponding to the simple variance of $\hat{\mu}$) and through the last two terms (corresponding to $\hat{\mu}$'s covariance with $\hat{\xi}_{i-1}$ and $\hat{\xi}_i$, respectively).

Once again, the asymptotic variances and standard error formulas for income shares are also distribution-free because of the way that the $f(\xi_{i-1})$ and $f(\xi_i)$ terms in the G and Σ

components cancel each other out. So, again, conventional statistical inference can be undertaken in straightforward fashion.

Note further that the formulas in eq. (14) include σ^2, so that implementation of these formulas requires an estimate of the variance (or standard deviation) for the estimation sample used to calculate the terms in (14). So again it would be helpful to users of quantile income share statistics from official statistical agencies if these agencies' documentation provided the estimated standard deviations for the actual samples that the reported estimates are based on, and not just the estimated sample means.

Researchers using public use survey microdata sets from statistical agencies to undertake their empirical analysis can view their analysis as occurring in two distinct stages. Stage 1 involves calculating estimates of the underlying parameters ξ_i, μ_i, IS_i, μ and σ (as well as N) directly from the microdata file for a specified estimation sample. Then Stage 2 involves calculating the estimated variances and standard errors of the $\hat{\mu}_i$'s and $\widehat{IS}_i$'s from above formulas (7) and (14), perhaps in a separate purpose-built computer program.

The quantile means and income shares serve as the basis for operationally implementing the evaluation of changes in social welfare and income inequality in the next several sections of this paper.

4. A Normative Perspective for Evaluating Changes in Social Welfare and Inequality

The traditional way of measuring income inequality in an income distribution is in terms of some summary or aggregate measure of inequality such as the Gini coefficient (G), coefficient of variation (C), relative mean (absolute) deviation (M), or the standard deviation of the logs of income (L). But such measures are subject to two basic criticisms. First is the aggregation problem: various summary measures aggregate income differences in different ways, so that different measures can give different results when comparing two distributions. One way (partially) to address this is to identify several desirable properties we may want such summary measures to satisfy. These could include, for example:

i) Symmetry (or Anonymity) – An inequality measure depends only on incomes in a distribution and not on who has which incomes;

ii) Mean Independence – An inequality measure is invariant to proportional changes (e.g., doubling) of all incomes (i.e., it is a relative measure of inequality);

iii) Population Homogeneity – An inequality measure is invariant to replication of the population (e.g., doubling the number of persons in the distribution while keeping the shape of the distribution the same);

iv) Principle of Transfers – Any transfer of $x from a richer person to a poorer person so that $y_i + x < y_j - x$ if initially $y_i < y_j$ should reduce inequality;

v) Transfer Sensitivity – A transfer of $x such as envisioned in (iv) should reduce inequality more if it occurs among a lower-income pair of individuals than if it occurs among a higher-income pair of individuals. This is obviously a stronger form of the Principle of Transfers.

It turns out that, (i), (ii) and (iii) are satisfied by all the above four inequality measures, but (iv) is satisfied only by C and G, and (v) is not satisfied by any of them.

Alternatively, another way to address the aggregation problem is to rely on a disaggregative measure of inequality or a set of multiple values linked up graphically such as a Lorenz curve. A problem here, though, is that two empirical Lorenz curves being compared often, if not typically, cross, so a clear comparison is not straightforward.

The second basic criticism of conventional summary measures of inequality is the implicit value judgement problem. That is, any summary inequality measure involves implicit value judgements or weightings of different persons' incomes (or economic well-being), and thus contains embedded in it an implicit social welfare function (SWF). For example, different inequality measures differently emphasize income differences at the bottom, middle, or upper end of the distribution. Consequently, it can be argued, it would be better to choose desirable SWF properties explicitly and then derive the implied inequality measure from the desired SWF. To do so is to take a *normative* approach to measuring inequality (originated by Atkinson, 1970) rather than the traditional descriptive approach. This is the approach followed in the current paper.

To implement such a normative approach, one first needs to define a social welfare function and its basic properties. For a much more expansive discussion of the normative approach, see, for example, Boadway and Bruce (1984), Blackorby et al. (1990), Lambert (2001) or Cowell (2011). Specifically, a social welfare function $W(\bullet)$ is any function

$$W = f(U_1, \dots, U_N)$$

that has as arguments U_i individual (or household) utility functions and that incorporates social values used to aggregate economic well-being across the population. The $U_i(\bullet)$ can thus be viewed as a "social income valuation function". To do this, we require that:

- the U_i's must be at least cardinal scale measurable in order to be aggregated across persons;
- the U_i's must have at least some degree of comparability across persons in the population (i.e., if utilities are cardinally measured for each individual, the units of measurement must be the same across individuals); and
- for technical convenience, each U_i depends only on incomes and indeed only on individual *i*'s income (i.e., $U_i = U_i(Y_i)$, so there is no envy or altruism).

One can then identify several possible desirable properties for such a social welfare function:

i) <u>(Strong) Pareto Principle</u> – State *X* is socially preferred to state *Y* if at least one person strictly prefers *X* to *Y* and no one prefers state *Y* to *X* (i.e., $\partial U_i \,/\, \partial Y_i > 0$ and social indifference curves in Y_i, Y_j space have negative slopes);

ii) <u>Symmetry or Anonymity</u> – Everyone's incomes are evaluated by using the same $U(\bullet)$ function (i.e., $U_i(\bullet) = U(\bullet)$ for all *i = 1, …, N*);

iii) <u>Population Invariance</u> – If the population is replicated *K* times, then social welfare increases *K*-fold (i.e., $W(Y_1, \dots, Y_{KN}) = K \bullet W(Y_1, \dots, Y_N))$;

iv) <u>Strict Concavity of the SWF or the Principle of Transfers</u> – A strictly concave SWF is such that $\partial^2 U_i \,/\, \partial Y_i^2 < 0$ for all *i* (this implies that social indifference curves are strictly convex to the origin).[5] This is sometimes referred to as an "egalitarian SWF";

v) <u>Transfer Sensitivity or the Principle of Diminishing Transfers</u> – A transfer-sensitive SWF is such that

[5] Actually, strict concavity is sufficient for the Principle of Transfers to hold. A weaker necessary and sufficient property for the Principle of Transfers is referred to as Schur concavity. See general discussion in Lambert (2001).

$$\frac{\partial^3 U_i}{\partial Y_i^3} > 0 \ .$$

Again, this is a stronger version of the Principle of Transfers.

Atkinson (1970) uses this normative approach to show that, under properties (i)-(iv), an empirical proxy of social welfare or economic well-being (SW_p) can be expressed as

$$SW_p = \bar{Y} \bullet (1 - I_A)$$

$$= \bar{Y} \bullet \mathrm{E}$$

where $\bar{Y}$ is the mean income level of a distribution and I_A is a measure of inequality based on the above four properties and where it turns out that $0 \le I_A \le 1$ where higher values indicate greater levels of inequality in the distribution. That is, SW_p can be decomposed into two (multiplicative) components – an efficiency dimension $(\bar{Y})$ or average per capita income and an equity dimension *(E)* where $E = 1 - I_A$.

If one further assumes a specific functional form for $U(\bullet)$ – in the convenient form of an iso-elastic social welfare function – Atkinson (1970) then derives a specific formula for the calculation of I_A . An iso-elastic SWF is general and flexible enough to incorporate a wide range of social attitudes to income inequality from the Benthanite utilitarian SWF to Rawls' maxi-min SWF.

But I_A is still a summary or aggregate measure of income inequality. What the social choice literature since Atkinson's (1970) paper has tried to do is to extend or apply Atkinson's normative perspective to develop a set of disaggregative criteria for comparing different income distributions based on the above properties, so that both criticisms of traditional inequality measures are addressed. The rest of this paper examines several such disaggregative criteria from the theoretical social choice literature and proposes ways to operationalize or empirically implement these criteria in terms of vectors of quantile means and income shares and related disaggregative distributional statistics. The paper also develops inference procedures to allow for formal statistical testing for these criteria. This development is applied to six such criteria in the following sections.

5. Application to Rank Dominance and a Practical Empirical Criterion

One early example of a disaggregative normative ranking criterion for distributions comes from Saposnik (1981) and involves what may be called the quantile curve (Duclos and Araar, 2006, p. 45) which is essentially the inverse of the cumulative distribution function of the distribution. Saposnik's rank dominance theorem says that, for any social welfare function satisfying the properties of symmetry, population invariance and the Pareto principle (i.e., social welfare conditions (i) – (iii)), distribution *A* is socially preferred to distribution *B* if the quantile curve for *A* is everywhere higher than that for *B*. Note that there is no egalitarianism built into this criterion. It essentially says that, if everyone has higher incomes in *A* than in *B*, then they must be better off. This is useful in comparing distributions many years apart, say, for example, the Canadian income distributions for 1961 versus 2021. But in most practical cases faced by empirical researchers, this situation doesn't apply.

Nonetheless, it is useful to begin our application of dominance criteria with this relatively simple criterion. To empirically implement it, one represents the two distributions or quantile curves being compared by their respective vectors of sample quantile means, $\hat{\mu}_i$, for $i = 1, \ldots, K$ quantiles. The actual decision rule for determining the outcome of the comparison of vectors requires some practical empirical criterion (henceforth a PEC) based on the principles of statistical inference.

5.1 A Practical Empirical Criterion for Quantile Means

Following Beach, Davidson and Slotsve (1994) and Davidson and Duclos (2000), one can set out a two-step test procedure for the PEC. It is assumed that the data samples for the two distributions being compared are independent and hence do not overlap. Examples are, say, two different years of data (with non-overlapping samples) being compared or two different (non-overlapping) population groups such as age, racial, or sex groups.

Step 1 – Test the joint null hypothesis of equality of the two (population) quantile mean vectors versus the alternative hypothesis of non-equality. This can be done by a standard (but asymptotic) chi-square test with K degrees of freedom, where K is the number of quantiles. For a meaningful disaggregative analysis, it makes sense to let $K = 10$ or 20, say, rather than a small number such as 5. If the null hypothesis is not rejected, then the two sets of quantile means can be said to be not statistically significantly different, and further comparison is not pursued. This is taken as an empirical proxy for comparison of the two underlying quantile curves.

Step 2 – If, however, the null hypothesis in Step 1 is rejected – which is the typical case when using large microdata sets for the sample distributions – then proceed to calculate separate t-statistics for differences for each of the individual quantile means. These K individual t-statistics, however, are correlated, and hence comparing each test statistic to the critical value on an (asymptotic) normal distribution would not be appropriate. One has to recognize that this Step 2 involves correlated multiple comparisons. Following the work of Beach and Richmond (1985) and Bishop, Formby and Thistle (1989, 1992) on multiple comparison testing, one should compare the K separate t-statistics (for differences in quantile means) to critical values on the Studentized Maximum Modules (or SMM) distribution. If at least one of the quantile mean differences t-statistics has the appropriate sign and is statistically significant (based on the SMM distribution) and none of the t-statistics of the remaining quantile mean differences has the wrong sign and is significant, then conclude that the quantile curve with the higher sample quantile means rank dominates (or is socially preferred to) that with the lower quantile means. If not, then one can say only the two quantile curves are statistically significantly different and not reach a preferred or dominance conclusion. Note that this is an asymptotic test and critical values from the SMM distribution correspond to *K* and infinite degrees of freedom. Typical usefully critical values from the SMM distribution are:

	$\propto = .01$	$\propto = .05$	$\propto = .10$
K = 5 –	3.289	2.800	2.560
K = 10 –	3.691	3.254	3.043
K = 20 –	4.043	3.643	3.453

Source: Stoline and Ury (1979), Tables 1-3.

5.2 Full Variance-Covariance Matrix for Quantile Means

The first step in the above practical empirical criterion (PEC) involves a joint test of the difference between two vectors or sets of estimated quantile means. If the two quantile curves being compared are designated *A* and *B*, then the vectors of quantile means can be represented as

$$\hat{\mu}^a = (\hat{\mu}_1^a, \dots, \hat{\mu}_K^a)', \quad \mu^a = (\mu_1^a, \dots, \mu_K^a)'$$

and $$\hat{\mu}^b = \left(\hat{\mu}_1^b, \dots, \hat{\mu}_K^b\right)', \quad \mu^b = \left(\mu_1^b, \dots, \mu_K^b\right)'.$$

A standard result from statistics, then, shows that if the random vector $\hat{\mu}^a$ is normally distributed with mean μ^a and variance-covariance matrix V^a, $\hat{\mu}^b$ is normally distributed with mean μ^b and variance-covariance matrix V^b, and $\hat{\mu}^a$ and $\hat{\mu}^b$ are statistically independent, then $\hat{\mu}^b - \hat{\mu}^a$ is also normally distributed with mean $\mu^b - \mu^a$ and variance-covariance matrix $V^a + V^b$. Under the null hypothesis that the two vectors μ^a and μ^b are the same (i.e., $\mu^b - \mu^a = 0$), then the quadratic form

$$(\hat{\mu}^b - \hat{\mu}^a)'[V^a + V^b]^{-1}(\hat{\mu}^b - \hat{\mu}^a)$$

is distributed as a chi-squared random variable with K degrees of freedom. If V^a and V^b are estimated consistently, then the test statistics for step 1 of the PEC,

$$(\hat{\mu}^b - \hat{\mu}^a)'\left[\hat{V}^a + \hat{V}^b\right]^{-1}(\hat{\mu}^b - \hat{\mu}^a) \tag{15}$$

is asymptotically distributed as a chi-square variate again with *K* degrees of freedom.

In order to implement the chi-square test in (15), however, one needs to know how to estimate the full variance-covariance matrices $\hat{V}^a$ and $\hat{V}^b$ of $\hat{\mu}^a$ and $\hat{\mu}^b$, respectively. The development in Section 2 above showed how to obtain the estimated variances (the square of the estimated standard errors of the various individual quantile means). But, in order to perform the joint chi-square test in Step 1, one also needs estimates for all the covariances in $\hat{V}^a$ and $\hat{V}^b$ as well. The approach followed to obtain them, however, is the same as for the variances.

Argumentation is expressed in terms of asymptotic variances and covariances. Again, let $\hat{m} = (\hat{\mu}_1, \dots, \hat{\mu}_K)'$ generically represent the vector of sample quantile mean estimates for a given income distribution, so it has been shown that

$$Asy.var(\hat{m}) \equiv V_S = G\,\Lambda\,G' \tag{16}$$

where Λ is the asymptotic variance-covariance matrix of the sample quantile cut-off levels, the $\hat{\xi}_i$'s, and G is a *K x (K-1)* matrix of partial derivatives

$$G = \left[g_{ij}\right] \qquad \text{where } g_{ij} = \frac{\partial \mu_i}{\partial \xi_j} \tag{17a}$$

for *i = 1, …, K* rows and *j = 1, …, K-1* columns.

Note that, in this development, the asymptotic variance-covariance matrix of $\hat{m}$ is V_S with a subscript S (for asymptotic) to distinguish it from matrix $\hat{V}$ which refers to the estimated variance-covariance matrix of $\widehat{m}$ which is gotten by rescaling the estimated asymptotic variance-covariance matrix by the inverse of the sample size. We also use a slightly more general notation than before with

$$\mu_i = \left(\frac{1}{D_i}\right) \bullet \int_{\xi_{i-1}}^{\xi_i} y\, f(y)dy$$

$$= \left(\frac{1}{D_i}\right) \bullet N_i(\xi_{i-1}, \xi_i) \qquad \text{for } i = 1, \ldots, K$$

where $D_i = p_i - p_{i-1}$ and $\xi_0 = 0$. Then

$$g_{ij} = \left(\frac{1}{D_i}\right) \bullet \frac{\partial N_i}{\partial \xi_j}. \tag{17b}$$

The *i,j*'th element of V_S, then, is obtained by premultiplying the matrix Λ by the *i*'th row of *G* treated as a row vector and postmultiplying by the *j*'th row of *G* (written as a column vector):

$$v_s(i,j) = (i'\text{th row of } G) \bullet \Lambda \bullet (j'\text{th row of } G)'. \tag{18}$$

In the case of variances *i = j*, the calculations lead to

$$Asy.var(\hat{\mu}_1) \equiv Asy.var(\widehat{m}_1)$$

$$= p_1(1 - p_1)\left(\frac{\xi_1}{D_1}\right)^2 \tag{19a}$$

and $$Asy.var(\hat{\mu}_K) \equiv Asy.var(\widehat{m}_K)$$

$$= p_{K-1}(1 - p_{K-1})\left(\frac{\xi_{K-1}}{D_K}\right)^2. \tag{19b}$$

For *i = 2, …, K-1*:

$$Asy.var(\hat{\mu}_i) = p_{i-1}(1 - p_{i-1}) \left(\frac{\xi_{i-1}}{D_i}\right)^2 + p_i(1 - p_i) \left(\frac{\xi_i}{D_i}\right)^2$$

$$- 2p_{i-1}(1 - p_i) \left(\frac{\xi_{i-1}\xi_i}{D_i{}^2}\right) \quad . \qquad (19c)$$

Equations (19a)-(19c) determine the elements on the principal diagonal of V_S already presented in eqs. (6) and (7).

Now consider the off-diagonal elements in the first row of V_S . For $1 < j < K$:

$$Asy.cov(\hat{\mu}_1, \hat{\mu}_j) = -p_1(1 - p_{j-1}) \left(\frac{\xi_1}{D_1}\right)\left(\frac{\xi_{j-1}}{D_j}\right) + p_1(1 - p_j) \left(\frac{\xi_1}{D_1}\right)\left(\frac{\xi_j}{D_j}\right). \qquad (19d)$$

For elements along the last column of V_S , i.e., for $1 < i < K$:

$$Asy.cov(\hat{\mu}_i, \hat{\mu}_K) = p_{i-1}(1 - p_{K-1}) \left(\frac{\xi_{i-1}}{D_i}\right)\left(\frac{\xi_{K-1}}{D_K}\right) + p_i(1 - p_{K-1}) \left(\frac{\xi_i}{D_i}\right)\left(\frac{\xi_{K-1}}{D_K}\right). \quad (19e)$$

For the top right-hand corner element,

$$Asy.cov(\hat{\mu}_1, \hat{\mu}_K) = -p_1(1 - p_{K-1}) \left(\frac{\xi_1}{D_1}\right)\left(\frac{\xi_{K-1}}{D_K}\right). \qquad (19f)$$

For all remaining above-diagonal elements of V_S ; i.e., for $1 < i < j < K$:

$$Asy.cov(\hat{\mu}_i, \hat{\mu}_j) = p_{i-1}(1 - p_{j-1}) \left(\frac{\xi_{i-1}}{D_i}\right)\left(\frac{\xi_{j-1}}{D_j}\right) - p_{i-1}(1 - p_j) \left(\frac{\xi_{j-1}}{D_i}\right)\left(\frac{\xi_j}{D_j}\right)$$

$$- p_i(1 - p_{j-1}) \left(\frac{\xi_i}{D_i}\right)\left(\frac{\xi_{j-1}}{D_j}\right) + p_i(1 - p_j)\left(\frac{\xi_i}{D_i}\right)\left(\frac{\xi_j}{D_j}\right). \tag{19g}$$

Since a variance-covariance matrix is symmetric about its principal diagonal, all below-diagonal elements can be obtained as

$$v_S(i,\ j) = v_S(j,\ i) \qquad \text{for } i > j\,. \tag{19h}$$

Note also that all terms in the V_S matrix – both (asymptotic) variances and covariances – are also distribution-free in that they do not depend on $f(\bullet)$ evaluations, and thus can all be readily estimated consistently and directly. Thus consistent estimates of the actual variances and covariances of the μ_i's can be obtained as

$$\hat{v}(i,\ j) = \hat{v}_S(i,\ j)\ /\ N \tag{20}$$

where *N* is the size of the estimation sample.

In order to perform Step 1 of the PEC for comparing the two quantile mean vectors $\hat{\mu}^a$ and $\hat{\mu}^b$, then, calculate estimates of all asymptotic variances and covariances ($\hat{V}_S^a$ and $\hat{V}_S^b$) for the two samples using the formulas in equations (19a)-(19h) by replacing population parameters by their consistent sample estimates, rescale the (asymptotic) variance-covariance estimates to the actual variance-covariance estimates $\hat{V}^a$ and $\hat{V}^b$ as in (20), and then calculate the joint chi-square test statistic in (15).

To perform the individual tests in Step 2 of the PEC, compute the standard "t-statistic" ratio for the difference between two independent random variables ($\hat{\mu}_i^a$ and $\hat{\mu}_i^b$) as

$$t_i = \frac{\hat{\mu}_i^b - \hat{\mu}_i^a}{\left[\hat{v}^a(i,i) + \hat{v}^b(i,i)\right]^{1/2}}$$

and compare this to the appropriate critical value on the SMM distribution.

Note that, in the above test procedure, the quantile income curve is approximated or represented by a vector of quantile means ($\hat{\mu}_i$'s) and not by a vector of quantile cut-off levels ($\hat{\xi}_i$'s). This is because the latter have a variance-covariance structure that is distribution-dependent (i.e., involving *f(•)*'s), while the variance-covariance structure of the former has been shown to be distribution-free. So the use of quantile means is convenient and sensible.

It could also be argued that the test reliance on a set of quantile points is arbitrary and doesn't provide adequate coverage of the income distribution as a whole. But such a choice of quantiles is quite conventional and even standard in the income distribution literature – as witness by published official distribution statistics in terms of deciles and quintiles. This is in contrast, say, to the ranking of investment opportunities on financial portfolios where all alternatives need to be examined. The availability of large microdata files nowadays also allows considerable disaggregative detail (such as vigintiles, say, or even percentiles in large census or administrative files) as well as a flexible differentiated focus (such as vigintiles in the tails of the distribution and quintiles or deciles over the mid-range of the distribution).

6. Application to Lorenz Dominance

The same approach can be applied to an inequality-based dominance criterion (for more extensive discussions of Lorenz curve comparisons, see Maasoumi, 1998; Lambert, 2001; and Aaberge, 2000, 2001.) Atkinson, in his famous 1970 paper, forwarded what has come to be known as the Lorenz dominance theorem. For any (summary) inequality measure satisfying symmetry, mean independence, population homogeneity and the principle of transfers (i.e., essentially inequality criteria (i)-(iv) above), if the Lorenz curve for distribution A lies everywhere above the Lorenz curve for distribution B, then all inequality measures satisfying these properties will indicate that (summary) inequality in A is less than in B. Note that this theorem does not say anything about social welfare; it refers only to inequality. It also does not say anything if the two Lorenz curves cross. Interestingly, comparing quantile curves and Lorenz curves can also serve as the basis for the measurement of first- and second-order earnings discrimination, such as between men and women (Le Breton et al., 2012).

To empirically implement this dominance criterion, one can again represent a Lorenz curve or cumulative income-shares by a vector of its estimated ordinates. Testing between Lorenz curves then amounts to tests of differences between the estimated ordinate vectors. Again, if the two distributions whose inequality is being compared are designated A and B, then the vectors of Lorenz curve ordinates can be represented by

$$\hat{l}^a = \left(\hat{l}_1^a, \ldots, \hat{l}_{K-1}^a\right)', \quad l^a = (l_1^a, \ldots, l_{K-1}^a)'$$

and $$\hat{l}^b = \left(\hat{l}_1^b, \ldots, \hat{l}_{K-1}^b\right)', \quad l^b = \left(l_1^b, \ldots, l_{K-1}^b\right)',$$

and their respective variance-covariance matrices by Φ^a and Φ^b . The ordinates $l_1, \ldots, l_{K-1}$ correspond to the given (cumulative) proportions $p_1, \ldots, p_{K-1}$. Since the two end points on a Lorenz curve are fixed at $p_0 = 0$ and $p_K = 1$, only *K-1* ordinates are random variables.

The actual decision rule or PEC for comparing the vectors of Lorenz curve ordinates again involves two steps. And again it is assumed that the two sets of ordinate estimates are statistically independent and based on two quite separate samples.

<u>Step 1</u> – Test the joint null hypothesis of equality of the two ordinate vectors (i.e., $l^b - l^a = 0$) versus the alternative hypothesis of non-equality. In this case, the test statistic is

$$(\hat{l}^b - \hat{l}^a)'\left[\hat{\Phi}^a + \hat{\Phi}^b\right]^{-1}(\hat{l}^b - \hat{l}^a) \tag{21}$$

which is distributed asymptotically as a chi-square random variable with *K-1* degrees of freedom. If the null hypothesis is not rejected, then the two estimated Lorenz curves can be said to be not statistically significantly different, and further comparison is not pursued.

Step 2 – If, however, the null hypothesis in Step 1 is rejected, then undertake separate "t-statistic" calculations for differences on each of the individual estimated Lorenz curve ordinates. If at least one of the t-statistics has the appropriate sign and is statistically significant compared to critical values on the SMM distribution with *K-1* and infinite degrees of freedom and none of the t-statistics (if any) that has the wrong sign is statistically significant (again based on the SMM distribution), then one can conclude that one set of estimated ordinates statistically dominates the other. If statistical dominance is found, this implies dominance for all summary inequality measures satisfying inequality properties (i)-(iv). Again, typical useful SMM critical values are:

	$\propto = .01$	$\propto = .05$	$\propto = .10$
K-1 = 4	3.430	2.631	2.378
K-1 = 9	3.634	3.190	2.976
K-1 = 19	4.018	3.615	3.425

Source: Stoline and Ury (1979), Tables 1-3.

This leaves two problems to be resolved: (i) how does one determine the statistical properties of the estimated Lorenz curve ordinates in order to make statistical inference decisions, and (ii) how to establish the full variance-covariance matrix of the vector of estimated ordinates. These are addressed in the next two subsections.

6.1 Inference for Lorenz Curve Ordinates

Recall that Lorenz curve ordinates are simply cumulative income shares (which have already been considered in Section 3 above). Let the K-vector of individual income share statistics be

$$\hat{n} = (\hat{n}\cdot, \dots, \hat{n}_K)'$$

with corresponding population shares $n = (n_1, \dots, n_K)'$. Then it can be seen that

$$\hat{l} = U \bullet \hat{n} \tag{22}$$

where U is a $(K-1)\ x\ K$ matrix with ones on its principal diagonal and below, and zeros above the diagonal. U is given and non-random. Since (22) is a linear transformation, if $\hat{n}$ is (asymptotically) joint normally distributed with mean vector n and variance-covariance matrix W_S, then $\hat{l}$ is also (asymptotically) joint normally distributed with mean $l = U \bullet n$ and asymptotic variance-covariance matrix

$$\Phi_S = U \bullet W_S \bullet U' \qquad \text{and hence} \qquad \Phi = U \bullet W \bullet U' . \tag{23}$$

So if *W*, the actual variance-covariance matrix of the estimated income shares can be established, so also can that of the vector of implied Lorenz curve ordinates.

6.2 Full Variance-Covariance Matrix for Income Shares

In order to obtain estimates of variance-covariance matrix elements for sample income shares, it is again useful to work out asymptotic variances and covariances for $\hat{n}$. It has been established in Section 3 that

$$Asy.var(\hat{n}_i) = G_i' \, \Sigma_i \, G_i$$

where Σ_i is the asymptotic variance-covariance matrix of the triplet $\hat{\xi}_{i-1}$, $\hat{\xi}_i$, and $\hat{\mu}$ and

$$G_i = \left[\frac{\partial N_i}{\partial \xi_{i-1}}, \frac{\partial N_i}{\partial \xi_i}, \frac{\partial N_i}{\partial \mu}\right]'$$

where income share $n_i \equiv IS_i = \int_{R_i} \left(\frac{1}{\mu}\right) y\, f(y)dy \equiv N_i\,(\hat{\xi}_{i-1}, \hat{\xi}_i, \mu)$. More generally, one can use a multivariate version of the Rao linkage theorem to establish that

$$Asy.var(\hat{n}) \equiv W_S = [w_S(i,j)] = G'\,\Sigma\, G \tag{24}$$

where now Σ is the *KxK* asymptotic variance-covariance matrix of the full set of sample quantile cut-offs, the $\hat{\xi}_i$'s, and the overall sample mean, $\hat{\mu}$. Σ thus consists of $\mathbf{\Lambda}$ in the upper-left *K-1* rows and columns, the (asymptotic) covariances of the $\hat{\xi}_i$ and $\hat{\mu}$ along the bottom row and right-hand column, and the (asymptotic) variance of $\hat{\mu}$ $(=\sigma^2)$ in the bottom right-hand corner. The *KxK* matrix *G* of partial derivatives then has as its *i*'th row all zeros except for the three elements $\frac{\partial N_i}{\partial \xi_{i-1}}, \frac{\partial N_i}{\partial \xi_i}, \frac{\partial N_i}{\partial \mu}$. Thus it is more convenient to work out the terms of W_S element by element where $w_S(i,j) = (i\text{'th row of } G) \bullet \Sigma \bullet (j\text{'th row of } G)'$. (25)

In the case of variances, *i=j*, which works out to the results:

$$Asy.var\left(\widehat{IS}_1\right) \equiv Asy.var(\hat{n}_1) = w_S(1,1)$$

$$= p_1(1-p_1)\left(\frac{\xi_1}{\mu}\right)^2 + \left(\frac{IS_1}{\mu}\right)^2 \sigma^2 \tag{26a}$$

$$-2\left(\frac{\xi_1}{\mu}\right)\left(\frac{IS_1}{\mu}\right) p_1(\mu - \bar{\mu}_1)\,,$$

$$Asy.var\left(\widehat{IS}_K\right) \equiv Asy.var(\hat{n}_K) = w_S(K,K)$$

$$= \; p_{K-1}(1 - \; p_{K-1}) \left(\frac{\xi_{K-1}}{\mu}\right)^2 + \; \left(\frac{IS_K}{\mu}\right)^2 \; \sigma^2 \qquad (26b)$$

$$+2 \; \left(\frac{\xi_{K-1}}{\mu}\right)\left(\frac{IS_K}{\mu}\right) p_{K-1}(\mu - \; \bar{\mu}_{K-1}) \; .$$

And for $i = j = 2, \ldots, K\text{-}1$:

$$Asy.var\left(\widehat{IS}_i\right) \; \equiv Asy.var(\hat{n}_i) = \; w_S(i,i)$$

$$= \; p_{i-1}(1 - \; p_{i-1}) \left(\frac{\xi_{i-1}}{\mu}\right)^2 + \; p_i(1 - \; p_i) \left(\frac{\xi_i}{\mu}\right)^2 + \; \left(\frac{IS_i}{\mu}\right)^2 \; \sigma^2$$

$$-2 \; p_{i-1}(1 - \; p_i) \left(\frac{\xi_{i-1}}{\mu}\right)\left(\frac{\xi_i}{\mu}\right) \qquad (26c)$$

$$+2 \left(\frac{\xi_{i-1}}{\mu}\right)\left(\frac{IS_i}{\mu}\right) \; p_{i-1}(\mu - \; \bar{\mu}_{i-1})$$

$$-2 \left(\frac{\xi_i}{\mu}\right)\left(\frac{IS_i}{\mu}\right) p_i(\mu - \; \bar{\mu}_i) \; .$$

Now address the (asymptotic) covariances in the first row of W_S . For $1 \; < j \; < K$:

$$Asy.cov\left(\widehat{IS}_1, \widehat{IS}_j\right) = \; w_S(1,j)$$

$$= \; -p_1\left(1 - \; p_{j-1}\right)\left(\frac{\xi_1}{\mu}\right)\left(\frac{\xi_{j-1}}{\mu}\right) + \; p_1\left(1 - \; p_j\right)\left(\frac{\xi_1}{\mu}\right)\left(\frac{\xi_j}{\mu}\right) + \left(\frac{IS_1}{\mu}\right)\left(\frac{IS_j}{\mu}\right) \; \sigma^2$$

$$- \; \left(\frac{\xi_1}{\mu}\right)\left(\frac{IS_j}{\mu}\right) \; p_1(\mu - \; \bar{\mu}_1) \qquad (26d)$$

$$+ \left(\frac{\xi_{j-1}}{\mu}\right)\left(\frac{IS_1}{\mu}\right) \; p_{j-1}(\mu - \; \bar{\mu}_{j-1})$$

$$- \left(\frac{\xi_j}{\mu}\right)\left(\frac{IS_1}{\mu}\right) \; p_j(\mu - \; \bar{\mu}_j) \; .$$

For elements down the last column of W_S ; i.e., for $1 \; < i \; < K$:

$$Asy.cov\left(\widehat{IS}_i, \widehat{IS}_K\right) = \; w_S(i,K)$$

$$= \; p_{i-1}(1 - \; p_{K-1}) \left(\frac{\xi_{i-1}}{\mu}\right)\left(\frac{\xi_{K-1}}{\mu}\right) - \; p_i(1 - \; p_{K-1}) \left(\frac{\xi_i}{\mu}\right)\left(\frac{\xi_{K-1}}{\mu}\right) + \left(\frac{IS_i}{\mu}\right)\left(\frac{IS_K}{\mu}\right) \; \sigma^2$$

$$+ \left(\frac{\xi_{i-1}}{\mu}\right)\left(\frac{IS_K}{\mu}\right) p_{i-1}(\mu - \bar{\mu}_{i-1}) \tag{26e}$$

$$- \left(\frac{\xi_i}{\mu}\right)\left(\frac{IS_K}{\mu}\right) p_i(\mu - \bar{\mu}_i)\ .$$

$$+ \left(\frac{\xi_{K-1}}{\mu}\right)\left(\frac{IS_i}{\mu}\right) p_{K-1}(\mu - \bar{\mu}_{K-1})\ .$$

For the top right-hand corner element of W_S,

$$Asy.cov\left(\widehat{IS}_1, \widehat{IS}_K\right) = w_S(1, K)$$

$$= -p_1(1 - p_{K-1})\left(\frac{\xi_1}{\mu}\right)\left(\frac{\xi_{K-1}}{\mu}\right) + \left(\frac{IS_1}{\mu}\right)\left(\frac{IS_K}{\mu}\right)\sigma^2$$

$$- \left(\frac{\xi_1}{\mu}\right)\left(\frac{IS_K}{\mu}\right) p_1(\mu - \bar{\mu}_1) \tag{26f}$$

$$+ \left(\frac{\xi_{K-1}}{\mu}\right)\left(\frac{IS_1}{\mu}\right) p_{K-1}(\mu - \bar{\mu}_{K-1}).$$

Finally, for all remaining above-diagonal elements of W_S ; i.e., for $1 < i < j < K$:

$$Asy.cov\left(\widehat{IS}_i, \widehat{IS}_j\right) = w_S(i, j)$$

$$= p_{i-1}\left(1 - p_{j-1}\right)\left(\frac{\xi_{i-1}}{\mu}\right)\left(\frac{\xi_{j-1}}{\mu}\right) - p_{i-1}\left(1 - p_j\right)\left(\frac{\xi_{i-1}}{\mu}\right)\left(\frac{\xi_j}{\mu}\right) + \left(\frac{IS_i}{\mu}\right)\left(\frac{IS_j}{\mu}\right)\sigma^2$$

$$- p_i\left(1 - p_{j-1}\right)\left(\frac{\xi_i}{\mu}\right)\left(\frac{\xi_{j-1}}{\mu}\right) + p_i\left(1 - p_j\right)\left(\frac{\xi_i}{\mu}\right)\left(\frac{\xi_j}{\mu}\right)$$

$$+ \left(\frac{\xi_{i-1}}{\mu}\right)\left(\frac{IS_j}{\mu}\right) p_{i-1}(\mu - \bar{\mu}_{i-1}) \tag{26g}$$

$$- \left(\frac{\xi_i}{\mu}\right)\left(\frac{IS_j}{\mu}\right) p_i(\mu - \bar{\mu}_i)$$

$$+ \left(\frac{\xi_{j-1}}{\mu}\right)\left(\frac{IS_i}{\mu}\right) p_{j-1}(\mu - \bar{\mu}_{j-1})$$

$$- \left(\frac{\xi_j}{\mu}\right)\left(\frac{IS_i}{\mu}\right) p_j(\mu - \bar{\mu}_j)\ .$$

In the above expressions, recall that $\bar{\mu}_i$ is the cumulative mean up to ξ_i . That is, $\bar{\mu}_i = d_i' \mu$ where d_i is a vector whose first i elements are $D_1 / p_i, \ldots, D_i / p_i$ and zeros elsewhere, and $\mu = (\mu_1, \ldots, \mu_K)'$.

Again, since a variance-covariance matrix is symmetric about its principal diagonal, all below-diagonal covariance terms can be obtained as

$$w_S(i,j) = w_S(j,i) \qquad \text{for } i > j . \tag{26h}$$

Note also that all terms in W_S are distribution-free, and thus can be readily estimated consistently and directly. More specifically, consistent estimates of the actual variances and covariances of the $\widehat{IS}_i$ can thus be obtained as

$$\hat{w}(i,j) = \hat{w}_S(i,j) / N \tag{27}$$

where N is the estimation sample size.

Once again to perform Step 1 of the PEC for comparing the two vectors of estimated Lorenz curve ordinates $\hat{l}^a$ and $\hat{l}^b$, first calculate estimates of all the asymptotic variances and covariances ($\widehat{W}_S^a$ and $\widehat{W}_S^b$) for the two estimation samples from equations (26a)-(26h) by replacing population parameters by their consistent sample estimates, rescale the (asymptotic) variance and covariance estimates to the actual variance and covariance estimates ($\widehat{W}^a$ and $\widehat{W}^b$) as in equation (27), calculate the Lorenz curve ordinates by $\hat{l} = U \bullet \hat{n}$ from equation (22) and Lorenz curve ordinate estimated variances and covariances from

$$\hat{\Phi} = U \bullet \widehat{W} \bullet U'$$

following equation (23), and then finally calculate the joint chi-square test statistic in equation (21).

To perform the individual tests in Step 2 of the PEC, again use the standard "t-statistic" ratio for the difference between two independent random variates ($\hat{l}_i^a$ and $\hat{l}_i^b$) as

$$t_i = \frac{\hat{l}_i^b - \hat{l}_i^a}{\left[\hat{w}^a(i,i) + \hat{w}^b(i,i)\right]^{1/2}}$$

and compare this to the relevant critical value on the SMM tables.

As an aside, it can be noted that the Gini coefficient of overall income inequality can be approximated by (twice) the area between the estimated Lorenz curve ordinate segments and the 45 degree equality diagonal. If one represents each of these segments as a quadrilateral, one can calculate

$$\hat{G} = \sum_{i=1}^{K}(1/K)\left[\left((i/K) - \hat{l}_i\right) + \left((i-1/K) - \hat{l}_{i-1}\right)\right].$$

Since this is a linear function of the ordinates $\hat{l}_i$, one can calculate the asymptotic variance of $\hat{G}$ as a simple quadratic form in the (asymptotic) variance-covariance matrix W_s of the estimated Lorenz curve ordinates. Once again, the standard error of $\hat{G}$ is the square root of the rescaled estimated (asymptotic) variance of $\hat{G}$, and once again the standard error is distribution-free.

7. Application to Generalized Lorenz Dominance

A blending of the first two dominance criteria is provided in a third application of empirically implementing curve-based dominance criteria. Shorrocks (1983) uses a transformed Lorenz curve as the basis for social welfare inferences, not just inequality conclusions. The generalized Lorenz dominance theorem of Shorrocks (1983) says that, for any additively separable social welfare function essentially satisfying social welfare conditions (i)-(iv) including the principle of transfers, distribution *A* is socially preferred to distribution *B* if the generalized Lorenz curve for *A* lies everywhere above the generalized Lorenz curve for *B*. The generalized Lorenz curve ordinates for an income distribution are obtained by scaling up the Lorenz curve ordinates of the distribution by the distribution's overall mean income level:

$$g_i = \mu \bullet l_i \quad \text{and } \hat{g}_i = \hat{\mu} \bullet \hat{l}_i \,. \tag{28}$$

Essentially, the argument is that, if the mean income of the distribution *A* is sufficiently higher than that in distribution *B*, this can compensate for some greater degree of inequality in *A* than in *B*, so that social welfare will still be greater in distribution *A* than in *B*. It turns out that this rule is very convenient for ranking social welfare among quite disparate countries, or for ranking income distributions in a given country (or group) over long periods of time (e.g., the Canadian income distribution across the decades of 1950, 1960, 1970 and 1980). This use of generalized Lorenz curves for dominance comparisons has also been applied to earnings discrimination analysis over an entire distribution (Jenkins, 1994; del Rio et al., 2011; and Salas et al., 2018).

To implement this dominance criterion, one can again represent a generalized Lorenz curve by a vector of its estimated ordinates:

$$g = (g_1, \dots, g_{K-1})' \qquad \text{and } \hat{g} = (\hat{g}_1, \dots, \hat{g}_{K-1})' \,.$$

Testing between generalized Lorenz curves then amounts to tests of differences between the estimated ordinate vectors $\hat{g}^a$ and $\hat{g}^b$. The respective generalized Lorenz curve ordinate variance-covariance matrices may be labelled Ψ^a and Ψ^b .

The corresponding decision rule or PEC for comparing vectors $\hat{g}^a$ and $\hat{g}^b$ once again involves two steps (where, as above) the estimation samples are statistically independent.

Step 1 – Test the joint null hypothesis of equality of two estimated generalized Lorenz curve ordinate vectors (i.e., $g^b - g^a = 0$) versus the alternative hypothesis of non-equality. In this case, the test statistics is

$$(\hat{g}^b - \hat{g}^a)' \left[\hat{\Psi}^a + \hat{\Psi}^b\right]^{-1} (\hat{g}^b - \hat{g}^a) \tag{29}$$

which, under the null hypothesis, is asymptotically distributed as a chi-square random variable with *K-1* degrees of freedom. If the null hypothesis is not rejected, then the two generalized Lorenz curves can be said to be not statistically significantly different, and further comparison is not warranted.

Step 2 – If, however, the null hypothesis in Step 1 is rejected, then proceed to compute separate "t-statistics" for differences on each of the individual generalized Lorenz curve ordinates. If at least one of the t-statistics has the appropriate sign and is statistically significant compared to critical values on the SMM distribution with *K-1* and infinite degrees of freedom and none of the t-statistics (if any) that have the wrong sign is statistically significant (again on the SMM critical values), then one can conclude that the distribution with the higher sample generalized Lorenz curve ordinates rank dominates (or is socially preferred to) the distribution with the corresponding lower ordinates. If not, then one can say only that the social welfare of the two distributions are statistically significantly different, but not reach a preferred or dominance conclusion.

7.1 Variance-Covariance Structure of Generalized Lorenz Curve Ordinates

Since Lorenz curve ordinates are calculated from income shares, it makes sense to consider the relationship of generalized Lorenz curve ordinates to these underlying income shares as well. To go back to first principles, consider $\mu \bullet IS_i$ as the dollar contribution of the i'th income group to the overall mean income of the distribution. So we can represent it by the "contribution"

$$c_i = \mu \bullet IS_i \tag{30a}$$

$$= \mu \bullet \int_{R_i} \left(\frac{1}{\mu}\right) y\, f(y)dy$$

$$= \int_{R_i} y\, f(y)dy \qquad \equiv \quad N_i\,(\xi_{i-1},\ \xi_i)$$

$$= D_i \bullet \left(\frac{1}{D_i}\right) N_i(\xi_{i-1},\ \xi_i)$$

$$= D_i \bullet \mu_i \tag{30b}$$

where μ_i is the quantile mean of the i'th income group and $D_i = p_i - p_{i-1}$. So c_i is simply a scalar transform of the quantile mean μ_i . Consequently, $\hat{c}_i = D_i \bullet \hat{\mu}_i$, and the elements of the variance-covariance matrix of the vector of sample contributions $\hat{c} = (\hat{c}_1, \dots ,\ \hat{c}_K)'$ are simply scalar transforms of the corresponding elements of the variance-covariance matrix of the vector of quantile means $\hat{\mu} = (\hat{\mu}_1, \dots ,\ \hat{\mu}_K)'$. More specifically, all the D_i terms in equations (19a)-(19h) for the asymptotic variances and covariances drop out. Alternatively viewed, since $c_i = N_i(\xi_{i-1},\ \xi_i)$, all the partial derivatives in equations (17a) and (17b) now involve simply $\frac{\partial N_i}{\partial \xi_j}$ without the term $\frac{1}{D_i}$. For ease of reference, these may be set out explicitly for the asymptotic variance-covariance matrix Γ_S of $\hat{c}$:

$$Asy.var(\hat{c}_1) = p_1(1 - p_1)\xi_1^2 \tag{31a}$$

$$Asy.var(\hat{c}_K) = p_{K-1}(1 - p_{K-1})\xi_{K-1}^2 \tag{31b}$$

$$Asy.var(\hat{c}_i) = p_{i-1}(1 - p_{i-1})\xi_{i-1}^2 + p_i(1 - p_i)\xi_i^2 \tag{31c}$$

$$- 2\, p_{i-1}(1 - p_i)\xi_{i-1}\xi_i \qquad \text{for } i = 2, \ldots, K\text{-}1\ .$$

For $1 < j < K$:

$$Asy.cov\left(\hat{c}_1, \hat{c}_j\right) = -\, p_1\left(1 - p_{j-1}\right)\xi_1\xi_{j-1} + p_1\left(1 - p_j\right)\xi_1\xi_j\ . \tag{31d}$$

For $1 < i < K$:

$$Asy.cov(\hat{c}_i, \hat{c}_K) = p_{i-1}(1 - p_{K-1})\xi_{i-1}\xi_{K-1} - p_i(1 - p_{K-1})\xi_i\xi_{K-1}\ . \tag{31e}$$

$$Asy.cov(\hat{c}_1, \hat{c}_K) = -\, p_1(1 - p_{K-1})\xi_1\xi_{K-1}\ . \tag{31f}$$

For $1 < i < j < K$:

$$Asy.cov\left(\hat{c}_i, \hat{c}_j\right) = p_{i-1}\left(1 - p_{j-1}\right)\xi_{i-1}\xi_{j-1} - p_{i-1}\left(1 - p_j\right)\xi_{i-1}\xi_j \tag{31g}$$

$$-\, p_i\left(1 - p_{j-1}\right)\xi_i\xi_{j-1} + p_i\left(1 - p_j\right)\xi_i\xi_j\ .$$

And for all below-diagonal elements of $\Gamma_S = [\gamma_S(i,j)]$,

$$\gamma_S(i,j) = \gamma_S(j,i) \qquad \text{for } i > j\ . \tag{31h}$$

Thus consistent estimates of the actual variances and covariances of the $\hat{c}_i$'s can be obtained as

$$\hat{\gamma}(i,j) = \hat{\gamma}_S(i,j)\ /\ N \tag{32}$$

where here N is the estimation sample size.

The ordinates of the generalized Lorenz curve can be readily obtained from the $\hat{c}_i$ by straightforward cumulation:

$$\hat{g}_i = \sum_{j=1}^{i} \hat{c}_j \qquad \text{and} \qquad g_i = \sum_{j=1}^{i} c_j$$

or more generally,

$$\hat{g} = U \bullet \hat{c} \qquad \text{and} \qquad g = U \bullet c \tag{33}$$

where again U is a *(K-1)xK* non-random matrix with ones on the principal diagonal and below, and zeros above the diagonal.

Since the $\hat{c}_i$'s are proportional functions of the $\hat{\mu}_i$'s , and the $\hat{\mu}_i$'s are asymptotically joint normal, then $\hat{c}_i$'s are also asymptotically joint normal with means c_i's and full (asymptotic) variance-covariance matrix Γ_S given by equations (31a)-(31h) and estimated actual variance-covariance matrix $\hat{\Gamma}$. Similarly, since the $\hat{g}_i$'s are linear functions of the $\hat{c}_i$'s , the $\hat{g}_i$'s are also asymptotically joint normally distributed with means g_i's and full (asymptotic) variance-covariance matrix

$$\Psi_S = U \bullet \Gamma_S \bullet U' \tag{34}$$

and estimated actual variance-covariance matrix

$$\hat{\Psi} = U \bullet \hat{\Gamma} \bullet U' \tag{35}$$

where the elements in $\hat{\Gamma}$ are given by equation (32). Once again, all terms in Γ_S are distribution-free, and thus can be readily estimated consistently and directly.

To perform Step 1 of the PEC for comparing the two vectors of generalized Lorenz curve ordinates $\hat{g}^a$ and $\hat{g}^b$, first calculate estimates of all the asymptotic variances and covariances ($\hat{\Gamma}_S^a$ and $\hat{\Gamma}_S^b$) for the two estimation samples from equations (31a)-(31h) by replacing population parameters by their consistent sample estimates, rescale the (asymptotic) variance and covariance estimates to actual variance and covariance estimates ($\hat{\Gamma}^a$ and $\hat{\Gamma}^b$) as in equation (32), calculate the generalized Lorenz curve ordinates by $\hat{g} = U \bullet \hat{c}$ from equation (33) and the generalized Lorenz curve ordinate estimated variances and covariances from equation (35), and then finally calculate the joint chi-square test statistic in equation (29).

To perform the individual tests in Step 2 of the PEC, again use the standard "t-statistic" ratio for the difference between two independent variates ($\hat{g}_i^a$ and $\hat{g}_i^b$) as

$$t_i = \frac{\hat{g}_i^b - \hat{g}_i^a}{\left[\hat{\gamma}^a(i,i) + \hat{\gamma}^b(i,i)\right]^{1/2}}$$

and compare this to the relevant critical value on the SMM distribution.

8. Decomposition of Generalized Lorenz Curve Ordinates into Efficiency and Equity Components

As stated in Section 4 above, an Atkinson-type inequality index, I_A , has the property – under various conditions – that an empirical proxy for social welfare can be decomposed into the product of an efficiency measure and an equity indicator:

$$\widehat{SW} = \hat{\mu} \bullet (1 - \hat{I}_A) .$$

A similar decomposition appears in Jorgenson (1990) as well. Can such an intuitively appealing decomposition also be applied more generally to entire dominance condition curves? The answer is yes. If the $\hat{\mu}_i$'s can be viewed as a disaggregative indicator of economic well-being or social welfare, then

$$\hat{\mu}_i = \hat{\mu} \bullet \left(\frac{\hat{\mu}_i}{\hat{\mu}}\right) \qquad \text{or alternatively} \quad = \hat{\mu}\left[1 - \left(\frac{\hat{\mu} - \hat{\mu}_i}{\hat{\mu}}\right)\right] \tag{36}$$

where $\hat{\mu}$ is again a measure of overall efficiency and $\left(\frac{\hat{\mu}_i}{\hat{\mu}}\right)$ can be viewed as an indicator of disaggregative equity for income group i. The term $\left(\frac{\mu_i}{\mu}\right)$ may be referred to as the relative-mean income gap for quantile group *i* and a vector of such terms as the relative-mean income curve for an income distribution (see Beach, 2021, for further discussion and interpretation of this curve).

When comparing two income distributions, say A and B, it is obviously of interest to look at their differences in overall means, $\hat{\mu}^a$ and $\hat{\mu}^b$. But it is also of interest to consider the relative-mean income gaps across the various quantile groups and how these differ between the distributions. That is, consider the differences in the gaps, say, $\frac{\hat{\mu}_i^b}{\hat{\mu}^b} - \frac{\hat{\mu}_i^a}{\hat{\mu}^a}$, across all of the individual quantile groups as a reflection of the disaggregative equity differences between the two distributions.

Indeed, it turns out that performing formal statistical tests of these relative-mean income gaps is quite straightforward using the above development. For convenience, designate the relative-mean income gap for quantile group *i* by r_i . Then, from first principles,

$$\begin{aligned} r_i &= \left(\frac{\mu_i}{\mu}\right) \\ &= D_i^{-1} \bullet \int_{R_i} y\, f(y) dy \,/\, \mu \\ &= D_i^{-1} \bullet \int_{R_i} \left(\frac{y}{\mu}\right) f(y) dy \\ &= \left(\frac{1}{D_i}\right) \bullet IS_i = \left(\frac{1}{D_i}\right) \bullet n_i \qquad (37) \end{aligned}$$

where, as before, $D_i = p_i - p_i$ and R_i is the appropriate range of integration. That is, r_i is simply a scalar transform of quantile *i*'s income share. And similarly,

$$\hat{r}_i = \left(\frac{\hat{\mu}_i}{\hat{\mu}}\right) = \left(\frac{1}{D_i}\right) \bullet \widehat{IS}_i = \left(\frac{1}{D_i}\right) \bullet \hat{n}_i \,.$$

Thus the vector $r = (r_1, \ldots, r_K)$ is such that

$$r = D^{-1} \bullet n \qquad \text{and similarly } \hat{r} = D^{-1} \bullet \hat{n} \tag{38}$$

where D^{-1} is a *KxK* matrix with elements D_i^{-1} along its principal diagonal and zeros elsewhere. Thus, since $\hat{r}$ is a linear transform of $\hat{n}$ and $\hat{n}$ is asymptotically joint normally distributed, so also is $\hat{r}$ with mean r and asymptotic variance-covariance matrix

$$R_S = D^{-1} \bullet W_S \bullet D^{-1} \tag{39}$$

where W_S is the asymptotic variance-covariance matrix of the estimated income share vector $\hat{n}$. Thus the asymptotic variance of $\hat{r}_i$ (for *i=2, …, K-1*), for example, is given by equation (26c) where each term is divided by D_i^2 . A consistent estimate of the actual variance-covariance matrix of $\hat{r}$ is then given by

$$\hat{R} = D^{-1} \bullet \widehat{W} \bullet D^{-1} = [\hat{r}(i,j)] \tag{40}$$

and the elements of $\widehat{W}$ are given by equation (27).

An asymptotic test of the difference in relative-mean income gaps for quantile group *i* between two independent distributions *A* and *B*, then, is done with the standard “t-statistic”

$$t_i = \frac{\hat{r}_i^b - \hat{r}_i^a}{\left[\hat{r}^a(i,i) + \hat{r}^b(i,i)\right]^{1/2}}$$

and this statistic is then compared to a critical value on the standard normal distribution. Note that the SMM distribution critical values are not used here since this test is not part of a PEC joint test criterion.

9. Inequality Dominance with a Single Lorenz Curve Crossing

What can one infer if Lorenz curves cross? More often than not this is the empirical situation when comparing two estimated Lorenz curves, and the Lorenz dominance criterion above is of no help in such situations. However, Shorrocks and Foster (1987) have come up with an extension of the latter criterion to cover just such situations. What may be called the transfer sensitivity dominance theorem states that, if the Lorenz curve for distribution *A* crosses the Lorenz curve for distribution *B* once from above, then all inequality measures satisfying the inequality properties (i)-(iv) plus property (v) – transfer sensitivity – will indicate that (summary) inequality in *A* is less than in *B* if the coefficient of variation for distribution *A* is lower than that for distribution *B*. The coefficient of variation for a distribution is the ratio of the standard deviation of the distribution to the mean, i.e.: $\hat{\sigma} / \hat{\mu}$ in the estimation sample. Thus, by adding the one further property of transfer sensitivity, one can get a stronger practical result that helps rank aggregate income inequality across distributions even when their Lorenz curves cross (once). Again, this provides a ranking of overall income inequality between distributions, and not of social welfare more generally.

Implementing this stronger dominance rule is indeed feasible in light of the above development in this paper. All it requires is some revision of the Lorenz dominance PEC of Section 6.

The practical empirical criterion (PEC) for inequality dominance can now be revised as follows:

Step 1 – Same as before. Test the joint null hypothesis of equality of the two Lorenz curve ordinate vectors (i.e., $l^b - l^a = 0$) versus the alternative hypothesis of non-equality. In this case, the test statistic is, as before,

$$(\hat{l}^b - \hat{l}^a)'\left[\hat{\Phi}^a + \hat{\Phi}^b\right]^{-1}(\hat{l}^b - \hat{l}^a) .$$

If the null hypothesis is not rejected, the two Lorenz curves can be said to be not statistically significantly different, and further comparison is not pursued.

Step 2 – If the null hypothesis in Step 1 is rejected and there is a single crossing of Lorenz curve ordinates, then undertake separate "t-statistic" calculations for differences on each of the individual estimated Lorenz curve ordinates. If at least one of the t-statistics has the appropriate sign and is statistically significant compared to critical values on the SMM distribution with *K-1* and infinite degrees of freedom and none of the t-statistics (if any) that has the wrong sign is statistically significant (again based on the SMM distribution), then proceed to Step 3. Otherwise, do not draw any dominance inference.

Step 3 – Compare the estimated coefficients of variation for the two distributions. If the coefficient of variation for the distribution with the initially higher Lorenz curve ordinates ($\hat{C}^a$, corresponding to distribution *A*, say) is smaller than the coefficient of variation for the other distribution ($\hat{C}^b$)[6], then one can conclude that inequality in distribution *A* statistically dominates inequality in distribution *B*. This implies dominance for all summary inequality measures – that is, they are smaller in distribution *A* than in distribution *B* – satisfying inequality properties (i)-(v).

Note that in this version of the PEC for inequality dominance, comparison of the coefficients of variation is done simply by inspection. A stronger version of Step 3 (and hence of the PEC) could involve a formal statistical test on $\hat{C}^b - \hat{C}^a$. Since the standard error of the sample coefficient of variation has been found to be

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[6] Note that the capital letter C for coefficient of variation here is quite different from the lower case c_i for mean income contribution in Section 7 above.

$$S.E.(\hat{C}) = 100C\left[\frac{1+2C^2}{2N}\right]^{1/2} \tag{41}$$

where $\hat{C}$ is expressed as a proportion (Ahn and Fessler, 2003), the estimated variance of $\hat{C}^b - \hat{C}^a$ for independent samples is

$$V\hat{a}r(\hat{C}^b - \hat{C}^a) = S.E.(\hat{C}^a)^2 + S.E.(\hat{C}^b)^2 .$$

Since $\hat{C}$ is asymptotically normally distributed (Ahn and Fessler, 2003), one can do an (asymptotic) normal test on the "t-ratio" test statistic

$$t = \frac{\hat{C}^b - \hat{C}^a}{\left[V\hat{a}r(\hat{C}^b - \hat{C}^a)\right]^{1/2}} . \tag{42}$$

And given that one is interested in a one-sided alternative test hypothesis, it makes sense to perform a one-tailed test on the standard normal where $H_1 : C^b - C^a > 0$ (i.e., distribution *A* has a smaller coefficient of variation).

10. Inequality Dominance with Multiple Lorenz Curve Crossings

But what if we have a situation where two Lorenz curves cross more than once? The Shorrocks and Foster (1987) approach has indeed been extended by Davies and Hoy (1994) to handle just this situation and can be viewed as a generalization of the former. In this case, instead of a single crossing and single coefficient of variation test, Davies and Hoy (1994) allow for possibly multiple Lorenz curve crossings – in the current author's experience two crossings is the most ever seen and the typical number of crossings is one – and posit a coefficient of variation condition for each cross-over point (including the top right-hand (1,1) point on the Lorenz curves).

More specifically, where two distributions A and B are being compared, Davies and Hoy (1994 and 1995) argue that the following statements are essentially equivalent:

1) For all summary measures of inequality, I, satisfying inequality properties (i)-(v) – i..e., including transfer sensitivity – $I^a < I^b$; and
2) For all cross-over points $k = 1, 2 \ldots,$ of the Lorenz curves for two distributions, the cumulative coefficients of variation at point k are smaller in distribution A than B.[7]

We will refer to this as transfer sensitivity dominance.

To empirically implement this, again represent the two Lorenz curves being compared by vectors of their (sample) ordinates. Consider also what we will call cumulative or conditional coefficients of variation corresponding to each of the quantile cut-offs, $\xi_1, \ldots, \xi_{K-1}$, and for the full sample as well. In terms of notation, let the cumulative coefficients of variation be Cc_i , where

$$Cc_i^2 = E((Y - \mu c_i)^2 \mid Y \leq \xi_i) \,/\, [E(Y \mid Y \leq \xi_i)]^2 ,$$

$$\mu c_i = E(Y \mid Y \leq \xi_i) \text{ is the cumulative mean (up to } \xi_i\text{)},$$

and $$\sigma c_i^2 = E((Y - \mu c_i)^2 \mid Y \leq \xi_i) \text{ is the cumulative variance.}$$

So $Cc_i = \sigma c_i \,/\, \mu c_i$. The unconditional coefficient of variation for the full set of observations can be viewed as the case of i=K (i.e., $Cc_K = C$). Then a PEC for the Davies-Hoy situation can be stated as follows.

<u>Step 1</u> – Same as before. Test the joint null hypothesis of equality of the two Lorenz curve ordinate vectors (i.e., $l^b - l^c = 0$) versus the alternative hypothesis of non-equality. In this case, the test statistic is again

[7] Actually the statement 2 is only sufficient for statement 1 to hold. More precise explication of the necessary and sufficient conditions can be found in Chiu (2007) and Davies, Hoy and Lin (2022).

$$(\hat{l}^b - \hat{l}^a)'\left[\hat{\Phi}^a + \hat{\Phi}^b\right]^{-1}(\hat{l}^b - \hat{l}^a)\ .$$

If the null hypothesis is not rejected, the two Lorenz curves can be said to be not statistically significantly different, and further comparison is not pursued.

Step 2 – Essentially the same as for the single-crossing case. If the null hypothesis in Step 1 is rejected and there are one or more crossings of Lorenz curve ordinates, then undertake separate "t-statistic" calculations for differences on each of the individual estimated Lorenz curve ordinates. If at least one of the t-statistics has the appropriate sign and is statistically significant compared to critical values on the SMM distribution with *K-1* and infinite degrees of freedom and none of the t-statistics (if any) that has the wrong sign is statistically significant (again based on the SMM distribution), then proceed to Step 3. Otherwise, do not draw any inequality dominance inference.

Step 3 – Compare the cumulative coefficients of variation for the two distributions. As for Step 2, undertake separate "t-statistic" calculations for differences on each of the individual estimated cumulative coefficients of variation (as well as the estimated standard coefficient of variation). If at least one of the t-statistics has the appropriate sign and is statistically significant compared once again to critical values on the SMM distribution with *K* and infinite degrees of freedom and none of the t-statistics (if any) that have the wrong sign is statistically significant (again based on the SMM distribution), then one can conclude that inequality in the distribution with the initially higher Lorenz curve ordinates (distribution *A*, say) statistically dominates inequality in distribution *B*. Once again, this implies dominance for all summary inequality measures – that is, they are smaller in distribution *A* than in distribution *B* – satisfying inequality properties (i)-(v).

To implement this PEC, then, involves doing statistical inference on the $\hat{C}c_i$'s and specifically establishing the variance structure of the set of cumulative coefficients of variation.

10.1 Variance Structure of the Cumulative Coefficients of Variation

Since the coefficient of variation is the ratio of first and second moments, it makes sense that its (cumulative) sample estimates would be asymptotically normally distributed. And since we are interested only in "t-ratios" of differences, we need focus just on the variance structure of the (cumulative) sample estimates rather than the full variance-covariance structure.

We begin by recognizing that the (cumulative) coefficients of variation are continuous differentiable function of the ξ_i's and we know the (asymptotic) distribution of the sample quantile cut-off estimates, $\hat{\xi}_i, i = 1, \dots, K-1$. Since $\hat{\xi}_i, \dots, \hat{\xi}_{K-1}$ are asymptotically joint normal and the Cc_i's are continuous differentiable functions of the ξ_i's, then Rao's linkage theorem says that the set of $\hat{C}c_i$'s are also asymptotically joint normally distributed. Indeed, since each Cc_i is a function of only a single ξ_i, the (asymptotic) variance of $\hat{C}c_i$ is given by simply the single derivative

$$Asy.var(\hat{C}c_i) = \left(\frac{\partial Cc_i}{\partial \xi_i}\right)^2 \bullet Asy.var(\hat{\xi}_i) \qquad \text{for } i = 1, \dots, K\text{-}1\,, \tag{43}$$

where we have already seen that

$$Asy.var(\hat{\xi}_i) = \frac{p_i(1-p_i)}{[f(\xi_i)]^2}$$

where $f(\bullet)$ is the underlying population density function of incomes from which the estimation sample is drawn.

To establish the derivative $\left(\frac{\partial Cc_i}{\partial \xi_i}\right)$, it is convenient to express the (cumulative) variance as

$\sigma c_i^2 = E(Y^2 \mid Y \leq \xi_i) - \mu c_i^2$

and consider the derivative of Cc_i^2 :

$$\frac{\partial Cc_i^2}{\partial \xi_i} = \frac{\partial(\sigma c_i^2 \,/\, \mu c_i^2)}{\partial \xi_i}$$

$$= \left(\frac{1}{\mu c_i^2}\right) \bullet \left[\frac{\partial \sigma c_i^2}{\partial \xi_i}\right] + \sigma c_i^2 \bullet \left[\frac{\partial \mu c_i^{-2}}{\partial \xi_i}\right]$$

i.e., $2Cc_i \bullet \left[\frac{\partial Cc_i}{\partial \xi_i}\right] = \left(\frac{1}{\mu c_i^2}\right) \bullet \left[\frac{\partial \sigma c_i^2}{\partial \xi_i}\right] - 2\left(\frac{\sigma c_i^2}{\mu c_i^2}\right)\left(\frac{1}{\mu c_i}\right) \bullet \left[\frac{\partial \mu c_i}{\partial \xi_i}\right]$.

So,

$$\frac{\partial Cc_i}{\partial \xi_i} = \left(\frac{1}{2}\right)\left(\frac{1}{\sigma c_i \bullet \mu c_i}\right) \bullet \left[\frac{\partial \sigma c_i^2}{\partial \xi_i}\right] - \left(\frac{Cc_i}{\mu c_i}\right) \bullet \left[\frac{\partial \mu c_i}{\partial \xi_i}\right]. \tag{44}$$

By Leibritz's rule, the two component derivatives are then, if $\mu c_i = \left(\frac{1}{p_i}\right) \int_0^{\xi_i} y\, f(y)dy$,

$$\frac{\partial \mu c_i}{\partial \xi_i} = \left(\frac{1}{p_i}\right) \xi_i \bullet f(\xi_i), \tag{45}$$

and if $\sigma c_i^2 = E(Y^2 \mid Y \leq \xi_i) - \mu c_i^2$

$$= \left(\frac{1}{p_i}\right) \int_0^{\xi_i} y^2 f(y)dy - \mu c_i^2 ,$$

$$\frac{\partial \sigma c_i^2}{\partial \xi_i} = \left(\frac{1}{p_i}\right) \xi_i^2 \bullet f(\xi_i) - 2\mu c_i \bullet \left[\left(\frac{1}{p_j}\right) \xi_i \bullet f(\xi_i)\right]$$

$$= \left(\frac{1}{p_i}\right) \xi_i \bullet f(\xi_i) \bullet [\xi_i - 2\mu c_i]. \tag{46}$$

Substituting (45) and (46) into (44) leads to

$$\frac{\partial Cc_i}{\partial \xi_i} = \left(\frac{1}{2}\right)\left(\frac{1}{\sigma c_i \bullet \mu c_i}\right) \bullet \left[\left(\frac{1}{p_i}\right) \xi_i \bullet f(\xi_i)(\xi_i - 2\mu c_i)\right]$$

$$- \left(\frac{Cc_i}{\mu c_i}\right) \bullet \left[\left(\frac{1}{p_i}\right) \xi_i \bullet f(\xi_i)\right]$$

$$= \left(\frac{1}{p_i}\right) \xi_i \bullet f(\xi_i) \bullet \left\{\left(\frac{1}{2}\right)\frac{(\xi_i - 2\mu c_i)}{\sigma c_i \bullet \mu c_i} - \left(\frac{Cc_i}{\mu c_i}\right)\right\}.$$

Thus,

$$Asy.var(\hat{C}c_i) = \left(\frac{\partial Cc_i}{\partial \xi_i}\right)^2 \bullet \frac{p_i(1-p_i)}{[f(\xi_i)]^2}$$

$$= \left(\frac{1-p_i}{p_i}\right)\left(\frac{\xi_i}{\mu c_i}\right)^2 \bullet \left\{\left(\frac{1}{2}\right)\frac{(\xi_i - 2\mu c_i)}{\sigma c_i} - Cc_i\right\}^2$$

$$= \left(\frac{1-p_i}{p_i}\right)\left(\frac{\xi_i}{\mu c_i}\right)^2 \bullet \left\{\left(\frac{1}{2}\right)\left(\frac{1}{Cc_i}\right) \bullet \left[\left(\frac{\xi_i}{\mu c_i}\right) - 2\right] - Cc_i\right\}^2. \quad (47)$$

Note that, once again, the (asymptotic) variance is distribution-free, and each term in (47) can be consistently estimated directly from the available sample.

The standard error of the sample $\hat{C}c_i$, then, is gotten as

$$S.E.(\hat{C}c_i) = \left[\frac{Asy\hat{.}var(\hat{C}c_i)}{N}\right]^{1/2}$$

where, as usual, *N* is the size of the estimation sample. The estimated variance of the difference in (cumulative) coefficients of variation for independent samples from distribution *A* and *B* is then

$$V\hat{a}r({\hat{C}c_i}^b - {\hat{C}c_i}^a) = S.E.({\hat{C}c_i}^a)^2 + S.E.({\hat{C}c_i}^b)^2$$

and the t-ratio statistic of the difference is

$$t = \frac{{\hat{C}c_i}^b - {\hat{C}c_i}^a}{\left[V\hat{a}r({\hat{C}c_i}^b - {\hat{C}c_i}^a)\right]^{1/2}}$$

that is used in Step 3 of the PEC test criterion.

Again, it would be helpful to practitioners using official published statistics on income shares if the official statistical agencies also provided quantile standard deviations along with their quantile means, so users can undertake statistical inference on crossing Lorenz curves if they wish.

It should be noted that the Stage 3 tests take place over the pre-specified quantile intervals (eg., over each decile group) and not at estimated cross-over points. This is done for convenience, based on current knowledge, because of the known and distribution-free (asymptotic) variance-covariance structure of the quantile $\hat{C}c_i$ statistics. The (asymptotic) variance-covariance structure of $\hat{C}c$'s at estimated cross-over points is not known and is likely quite messy. In the experience of this researcher, one cross-over point is quite common and even typical two such points have been observed on rare occasions, but more than two have never been encountered when using decile intervals. I would thus argue that, for the purpose of inequality ranking for income distributions as a whole, decile breakdowns are quite sufficient to pick up relevant Lorenz curve cross-overs.

Note that quantile dominance, Lorenz curve dominance, and transfer sensitivity dominance essentially correspond to first-order, second-order, and third-order stochastic dominance of one distribution by another. This analytical framework has generated a huge literature, especially in the finance field where testing for stochastic dominance efficiency is applied over a class of portfolios based on portfolio returns and investor preference toward risk (Post, 2003; Scaillet and Topaloglou, 2010; and Linton et al., 2014). These test procedures typically involve heavy computational techniques such as computer programming optimization and (block) bootstrapping methods. Application of the stochastic dominance framework to income distributions and income inequality comparisons are found in Anderson (1996) based on

Pearson goodness-of-fit tests and in Maasoumi and Heshmati (2000), Barrett and Donald (2003), and Linton et al. (2005) based on forms of Kolmogorov-Smirnov goodness-of-fit tests and on resampling/simulation procedures and bootstrapping methods. The stochastic dominance approach also underlies Davidson and Duclos' (2000) application of statistical inference to poverty rankings along with welfare and inequality comparisons in a unified analytical framework. While they rely on bootstrapping for inference results, they note that use of (asymptotic) variance information as pivot points is known to improve inference results (i.e., make tests more powerful). Applications of their test procedures are found in Duclos and Araar (2006).

11. Distributional Distance Dominance and Income Polarization

One aspect of concern about rising income inequality is the implied growing economic and social distance between income groups, economic inclusion, and the potential political fracturing this may bring about. The literature and media have focussed on the widening gap between top incomes and the rest of the distribution and the increasing difficulty of lower-income workers to pull ahead into stable middle-income status – the sense of belonging to the Middle Class may be weakening. So this raises the question of whether there is a way to measure, in general fashion, the growing economic distances between different income groups across a distribution?

The analysis of this paper suggests just such a measure – a "distributional distance function". It can perhaps be most conveniently pictured as a graph with deciles or percentiles measured along the base or horizontal axis (e.g., $i = 1, 2, \ldots, K$) and incremental quantile mean income gaps $(\hat{\mu}_i - \hat{\mu}_{i-1})$ measured along the vertical or left-hand axis. This relationship or curve may be referred to as the distributional distance function for a given income distribution.

Its ordinates show the distance or income gap between adjacent quantile groups in a distribution. So for some distributions, the gaps may be relatively wide between lower and middle-class quantile groups, suggesting it is more difficult to move up to middle-income status. Obviously, gaps could be combined to show the distance between, say, bottom and middle income groups. While for other situations – such as over recent decades in the Canadian and U.S. economies – the widening gaps have been most dramatic at the upper end of the distributions.[8]

Indeed, one can compare such curves between two income distributions and argue that the uniformly lower curve is said to distance dominate or show distributional distance dominance over the higher such curve. Comparing such curves would also allow one to identify which regions of a distribution are showing widening income distance gaps over time. Such comparisons can be easily done from decile mean income figures published annually by official statistical agencies such as Statistics Canada and the U.S. Bureau of the Census.

Employing the analytical machinery of the present paper also allows one to extend formal statistical inference and hypothesis testing to such a comparison. The key is to represent each (sample) distributional distance function by the vector[9]

$$\hat{d} = (\hat{\mu}_1,\ \hat{\mu}_2 - \hat{\mu}_1, \ldots,\ \hat{\mu}_K - \hat{\mu}_{K-1}\ )'$$

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[8] Distributional distances could also be expressed in proportion terms – such as $(\hat{\mu}_i - \hat{\mu}_{i-1})\ /\ \hat{\mu}_{i-1}$ – and application of Rao's linkage theorem would still carry through. But for intuitive appeal and convenience of linear analysis, we'll express distributional distances in dollar or level terms (though see the Appendix on income polarization proportional gaps).

[9] This is to be distinguished from the d_i vectors in Section 6.2.

where the first element can be thought of as $\hat{\mu}_1 - \hat{\mu}_0$ where $\hat{\mu}_0 = 0$. Testing and inference then depend on the statistical properties of these quantile mean differences. One can also develop a formal PEC rule for comparing income distributions in terms of overall distance dominance.

11.1 Statistical Inference for the Distributional Distance Function

The sample vector $\hat{d}$ is a linear function of $\hat{\mu}$, the vector of quantile means:

$\hat{d} = D \bullet \hat{\mu}$ and $d = D \bullet \mu$

where, for purposes of this section,[10]

$$D = \begin{bmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{bmatrix} \tag{48}$$

is a *KxK* non-random banded matrix with ones along the principal diagonal, minus ones just below the principal diagonal, and zeros elsewhere. As has already been seen, $\hat{\mu}$ is asymptotically joint normally distributed with mean vector μ and (asymptotic) variance-covariance matrix V_S . It then follows from Rao's linkage theorem that $\hat{d}$ is also asymptotically joint normally distributed with mean d and (asymptotic) variance-covariance matrix

$$Asy.var(\hat{d}) = \Delta_S = D \bullet V_S \bullet D' = [\delta_S(i,j)] , \tag{49}$$

where the elements of V_S are worked out in section 5.2 above. Thus, for example, for *i = 2, …, K*,

$$Asy.var(\hat{d}_i) = v_S(i-1, i-1) + v_S(i,i) - 2v_S(i-1, i) ,$$

[10] This is to be distinguished from the *D* matrix in Section 8.

where $v_S(i,j)$ is the *i,j*'th element of V_S . Thus, the standard error of $\hat{d}_i$ is

$$S.E.(\hat{d}_i) = \left[\frac{Asy\hat{.}var(\hat{d}_i)}{N}\right]^{1/2} \tag{50}$$

for estimation sample size *N*. One can then use (50) to formally test for the statistical significance of any individual quantile difference based on an (asymptotic) standard normal test.

11.2 A PEC for Distributional Distance Dominance

To rank overall distance dominance between two (independent) income distributions, one can adopt a practical empirical criterion similar to that for establishing rank dominance.

Step 1 – Test the joint null hypothesis of equality of the two distributional distance vectors, d^a and d^b (corresponding to distributions *A* and *B*), versus the alternative hypothesis of non-equality. This can be done with the test statistic

$$(\hat{d}^b - \hat{d}^a)' \left[\hat{\Delta}^a + \hat{\Delta}^b\right]^{-1} (\hat{d}^b - \hat{d}^a) \tag{51}$$

where $\hat{\Delta}^a = \left[\hat{\delta}^a(i,j)\right]$, $\qquad \hat{\Delta}^b = \left[\hat{\delta}^b(i,j)\right]$,

$\hat{\delta}^a(i,j) = \hat{\delta}^a_S(i,j) \,/\, N^a$,

and $\quad \hat{\delta}^b(i,j) = \hat{\delta}^b_S(i,j) \,/\, N^b$.

$\hat{\Delta}^a$ and $\hat{\Delta}^b$ are thus the estimated variance-covariance matrices of $\hat{d}^a$ and $\hat{d}^b$, respectively, and are obtained by (i) rescaling the elements of the asymptotic variance-covariance matrices Δ^a_S and Δ^b_S by their respective sample sizes and (ii) replacing all unknown population terms by their consistent sample estimates. Under the null hypothesis of equality of the two distance vectors, statistic (51) is asymptotically chi-square with *K* degrees of freedom. If the null hypothesis is not rejected, then the two distributions can be said to have distributional distance functions that are not statistically significantly different, and further comparison is not pursued.

Step 2 – If, however, the null hypothesis in Step 1 is rejected, then proceed to calculate separate "t-statistics" for differences on each of the individual quantile distance elements:

$$t_i = \frac{\hat{d}_i{}^b - \hat{d}_i{}^a}{\left[\hat{\delta}^a(i,i) + \hat{\delta}^b(i,i)\right]^{1/2}}, \qquad i = 1, \ldots K, \tag{52}$$

where $\hat{\delta}(i,i) = \left[S.E.(\hat{d}_i)\right]^2$ is the estimated variance of $\hat{d}_i$ – disregarding the superscripts *a* and *b* for convenience. Then compare these individual t-statistics to critical values on the SMM distribution with *K* and infinite degrees of freedom. If at least one of these individual "t-tests" is statistically significant of one sign and none of the other individual t-statistics are statistically significant of the other sign, then one can conclude that the distributional distance function with the lower $\hat{d}_i$'s – say distribution *A* – dominates that of the other distribution (i.e., *B*). If not, one can say that the two distributional distance functions are statistically significantly different, but not reach a conclusion as to whether one distance dominates the other overall.

11.3 Income Polarization

An alternative way of looking at distributional distance is in terms of the gap separating lower or higher incomes from middle incomes. This better highlights income distances at the two ends of the distribution from middle class incomes and hence the magnitude of economic exclusion. This could also serve as a measure of the degree of polarization or pulling apart in an income distribution.

In general, the intuitive concept of polarization in an income distribution can be viewed as having two quite distinct dimensions or aspects. One is the size dimension or the relative concentration of income recipients at the two ends of the distribution. This could be labelled *tail frequency polarization*. This could be captured, for example, by the proportion of recipients in lower or higher income groups (Wolfson, 1994), perhaps as captured by the proportions of recipients below 50 percent or above 200 percent of the median income level. The other is the distance dimension or the size of the income gap separating top and middle-class incomes or bottom and mid-range incomes. This may be referred to as *income polarization* and could be captured, say, by the gaps $\mu_K - \mu_M$ and $\mu_M - \mu_1$ where μ_M is some measure of mean middle incomes. These two dimensions correspond exactly to the size of the Middle Class within a distribution and the mean middle-class income level (Beach, 2016). Both provide useful insights to understanding what has been happening to the so-called Middle Class in many countries since about 1980.

However, the statistical properties of these two sets of measures of polarization are quite different. Tail frequency statistics, we have already seen, have (asymptotic) variances and covariances which are distribution-dependent (Beach, 2021). While quantile-based mean income statistics turn out to have (asymptotic) variances and covariances which are distribution-free and hence are easy to estimate directly. For this reason, we focus in this section on income polarization measures that are based on quantile means.

It is useful, then, to consider what may be called an income polarization curve $|\mu_i - \mu_M|$ over different quantile values of *i* and where μ_M is the mean of middle quantile-group incomes. More specifically, in the case of deciles, say, let $\mu_M = (.5)(\mu_5 + \mu_6)$ and define a *Kx1* income polarization ordinate vector to be

$$q = (\mu_M - \mu_1, \dots \mu_M - \mu_4;\ 0, 0;\ \mu_7 - \mu_M, \dots \mu_{10} - \mu_M)' \ . \tag{52}$$

For vigintiles, use $\mu_M = (.5)(\mu_{10} + \mu_{11})$ and the vector q has 18 non-zero elements. The vector q thus be seen[11] as a linear transform of the quantile mean vector (representing the quantile curve of Section 5):

$$q = Q\mu$$

where Q is a (KxK) matrix with (in the case of deciles) – I_4 in its top left portion, I_4 in the bottom right portion, 0.5 elements in rows 1-4 and columns 5 and 6, -0.5 elements in rows 7-10 and columns 5 and 6, and zeros elsewhere (where I_4 is a 4x4 identity matrix).[12] The income polarization curve thus has two separate arms – one showing the degree of income polarization over the lower portion of the distribution and the other illustrating polarization over the upper portion of the distribution. Since the latter is unbounded, it is likely to be much higher or more extreme than the former. For various purposes, one may wish to focus on just one arm or the other of the curve.

This formulation has clear similarities to the Foster and Wolfson's (1992) concept of a polarization curve presented in Kovacevic and Binder (1997, p. 50):

$$B(p) = \frac{|\xi_p - \xi_{50}|}{\xi_{50}} \tag{53}$$

where ξ_{50} is the median income level and ξ_p is the p'th percentile income cut-off level. But, while the intuition is similar between (52) and (53), the latter has an (asymptotic) variance-

[11] See the q_i terms at the end of Section 2.

[12] The income polarization curve could obviously also be expressed in proportional terms by dividing each term in (52) by μ_M . But to keep the analysis as simple as possible, we will work here only with the curve expressed in level or dollar terms. Asymptotic variances and standard errors for vector elements in terms of relative gaps are provided in the Appendix of the paper.

covariance structure that is again distribution-dependent. The corresponding structure of (52) is known and distribution-free.

Indeed, since $\hat{q} = Q\hat{\mu}$ is a linear transform of $\hat{\mu}$, it can be seen that $\hat{q}$ is also asymptotically normally distributed with mean q and (asymptotic) variance-covariance matrix

$$Asy.var(\hat{q}) = H = [\eta_s(i,j)] = Q\, V_s\, Q' \,. \tag{54a}$$

Actually, the (asymptotic) variances can be shown to be

$$Asy.var(\hat{q}_i) = asy.var(|\hat{\mu}_i - \hat{\mu}_M|) = \eta_s(i,j) \quad \text{for } i = 1, \ldots, 4; 7, \ldots, 10$$

$$= v_s(i,i) + (.25)v_s(5,5) + (.25)v_s(6,6) \tag{54b}$$

$$+ (.5)v_s(5,6) - v_s(i,5) - v_s(i,6).$$

Thus, the standard error for each (non-random) curve ordinate is

$$S.E.(\hat{q}_i) = \left[\frac{Asy\hat{.}var(\hat{q}_i}{N}\right]^{1/2} \tag{55}$$

for estimation sample size N, and formal tests on these individual income polarization curve ordinates can be based on (55).

11.4 A PEC for Income Polarization Curve Dominance

A formal PEC procedure for comparing and ranking income polarization dominance can be undertaken along the same 2-step approach as in Section 11.2 for distributional distance dominance. However, quite different factors may be operating over the lower and higher regions of the distribution, resulting in different patterns of change over the two tail regions of the distribution. It thus probably makes more sense to focus income polarization curve dominance tests on only one tail or one arm of the income polarization curve, or on both curves separately. If so, the above 2-step PEC procedure can be applied to only one arm of the curve at a time. The only things that change in the test procedure are the appropriate degrees of freedom for each arm and the relevant variance-covariance matrix used in the calculations. In the case of deciles, the degrees of freedom in Step 2 are (10-2)/2 = 4 and infinite, or more generally (for even numbers of quantile groups) (K-2)/2 and infinite degrees of freedom on the SMM test distribution.

In the case of relevant variance-covariance matrix to use in the calculations, Step 1 of the above procedure involves elements of

$$H = Asy.var(\hat{q}) = Q\ V_s\ Q' \ .$$

For one-arm application of the PEC procedure, one uses only the top portion of H or the bottom portion of H. In either case, it can be shown that

$$\begin{aligned}\eta_s(i,j) &= Asy.cov\left(\hat{q}_i, \hat{q}_j\right) \\ &= v_s(i,j) - [(.5)v_s(i,5) + (.5)v_s(i,6)] \\ &\qquad -[(.5)v_s(j,5) + (.5)v_s(j,6)] \\ &\qquad +[(.25)v_s(5,5) + (.25)v_s(6,6) + (.5)v_s(5,6)] \ .\end{aligned} \tag{56}$$

It can be readily seen that, if $i = j$, eq. (56) reduces to eq. (54) as one would expect.

12. Review and Conclusions

The theoretical literature on social choice and economic welfare evaluation has offered several dominance criteria for ranking key aspects of income distributions – such as rank dominance, Lorenz dominance or generalized Lorenz dominance across different distributions – and based on comparing curves such as quantile mean curves or Lorenz curves. This paper provides the statistical tools and procedures for actually implementing these dominance criteria empirically with microdata sets that can be readily obtained from statistical agencies such as Statistics Canada and the U.S. Bureau of the Census. The approach followed thus advances the statistical inference framework for a tool box of disaggregative income inequality measures (such as quantile means and income shares) published by these agencies.

More specifically, the analysis of this paper suggests a statistical tool box of five set of disaggregative inequality measures that have either simple descriptive value or use in the inferential procedures developed in this paper:

- summary parameters: $\hat{\mu}, \hat{\sigma}$, and $\hat{\sigma} / \hat{\mu}$
- basic quantile statistics: $\hat{\xi}_i, \hat{\mu}_i,$ and $\widehat{IS}_i$ (Section 2)
- Lorenz curve-related statistics: $\hat{l}_i, \hat{g}_i,$ and $\hat{r}_i$ (Sections 4-8)
- crossing Lorenz curve statistics: $\widehat{\mu c}_i, \widehat{\sigma c}_i,$ and $\widehat{Cc}_i$ (Sections 9-10)
- income polarization statistics: $\hat{d}_i, \hat{q}_i,$ and $\widehat{qr}_i$ (Section 11 and Appendix).

The second group consists of quantile means and variances (and the cut-off income levels separating these). The third group includes the Lorenz curve ordinates (or cumulative shares), relative mean incomes $\hat{r}_i = \hat{\mu}_i / \hat{\mu}$, and ordinates of the generalized Lorenz curve. The cumulative coefficient of variation statistics useful for comparing crossing Lorenz curves are listed in the fourth group. And the distributional distance and income polarization curve statistics are featured in the fifth group of tool box disaggregative statistics. These statistics provide the basis for possible computer programs to usefully describe and assess the statistical reliability of distributional change (Jenkins, 1999; and Jann, 2016) or differences between population groups.

The process for implementing the developments in this paper involves three stages. The first stage consists of representing a dominance curve by a vector of the curve's estimated ordinates for a set of specified quantile points (such as deciles or percentiles). This transforms a theoretical problem into a statistical one. The second stage involves establishing the statistical properties of this vector of sample ordinates through use of recent developments on quantile-based inferences that are distribution-free and thus very straightforward to implement directly. This transforms the statistical problem into an inferential one by providing a framework for basing comparisons on formal statistical inference and testing. The third stage of implementing dominance comparisons involves proposing specific practical empirical criteria (or PECs) – one can think of these as a type of decision tree – for using formal statistical inference tests to reach empirical conclusions about the ranking of the key aspects of income distributions between different distributions based on the theoretical dominance criteria. This converts a series of statistical test outcomes to conclusions with respect to the possible ranking of these key features of income distributions being compared.

This approach is applied to several dominance rules for ranking social welfare or income inequality between distributions:

- rank dominance for comparing social welfare (Section 5)
- Lorenz dominance for comparing income inequality (Section 6)
- generalized Lorenz dominance for comparing social welfare (Section 7)
- Lorenz dominance for comparing income inequality when Lorenz curves cross (Sections 9, 10)

- distributional distance dominance and income polarization for comparing distances between income groups (Section 11 and Appendix).

These dominance criteria can all be expressed in terms of income shares and quantile means, and hence their statistical properties can be easily established. Since statistical inference for quantile-based income shares and quantile means has been shown to be distribution-free (in the sense of not depending on any specific underlying income distribution function), so also is statistical inference for these above statistics, and hence test statistics for these can be readily obtained as well.

The analytical results in the paper have several implications. First, they show that quite broad inferences can be drawn as to social welfare and inequality comparisons that do not rely on single aggregate or summary measures and can be much more general. Thus there should be a shift in focus from specific summary measures of inequality to whole sets of disaggregative measures that are readily available in official statistical sources. Since these disaggregative measures are all quantile-based, the analysis thus highlights this disaggregative quantile-based approach to characterizing and measuring income inequality.

Second, the analysis of the paper shows that these disaggregative income inequality statistics can – when used jointly – provide not just descriptive information on changing patterns of inequality, but also (under fairly broad and reasonable conditions) *normative* insights and inferences as well. The paper also shows how these readily available disaggregative measures of income inequality – with only a bit more information – can also provide the basis for formal *statistical inference* and standard statistical testing protocols.

Third, this paper presents a simple and straightforward test procedure for statistical dominance in situations where ordering of observations into quantile groups makes sense and interest focuses on only first-, second-, and third-order dominance. This is the situation, for example, in the empirical analysis of distributions of income, earnings, or wealth.

Fourth, the disaggregative statistics used in this paper provide a useful complement to what government statistical agencies such as Statistics Canada and the U.S. Bureau of the Census already provide. Along with their income shares, quantile means and quantile cut-off values the agencies publish annually, they should also include information on (i) the sample sizes of the estimation samples the above statistics are based on, as well as (ii) cumulative means and standard deviations by quantile group (including for the full samples). This would allow empirical users to calculate relevant test statistics for formal statistical inference on the above published statistics as part of their empirical analysis.

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Appendix

Standard Errors of Relative Income Polarization Curve Ordinates

Consider the relative income polarization curve ordinates

$\widehat{qr}_i = |\hat{\mu}_i - \hat{\mu}_M| \,/\, \hat{\mu}_M$

for, in the case of deciles, *i* =1, …, 4; 7, …, 10, and the middle group mean income is $\hat{\mu}_M = (\hat{\mu}_5 + \hat{\mu}_6) \,/\, 2$.

Then, by the application of Rao's linkage theorem, it can be shown that

$$Asy.var(\widehat{qr}_i) = \left(\frac{1}{\mu_M}\right)^2 \bullet Asy.var(\hat{\mu}_i) + \left(\frac{\mu_i}{\mu_M^2}\right)^2 \bullet Asy.var(\hat{\mu}_M)$$

$$- 2\left(\frac{1}{\mu_M}\right)\left(\frac{\mu_i}{\mu_M^2}\right) \bullet Asy.cov(\hat{\mu}_i,\ \hat{\mu}_M\ ) \qquad \text{(A1)}$$

where

$$Asy.var(\hat{\mu}_M) = (.25)Asy.var(\hat{\mu}_5) + (.25)Asy.var(\hat{\mu}_6) + (.5)Asy.cov(\hat{\mu}_5,\ \hat{\mu}_6)$$

and

$$Asy.cov(\hat{\mu}_i,\ \hat{\mu}_M) = (.5)Asy.cov(\hat{\mu}_i,\ \hat{\mu}_5) + (.5)Asy.cov(\hat{\mu}_i,\ \hat{\mu}_6)\ .$$

Therefore,

$$Asy.var(\widehat{qr}_i) = \left(\frac{1}{\mu_M}\right)^2 \bullet Asy.var(\hat{\mu}_i)$$

$$- \left(\frac{1}{\mu_M}\right)\left(\frac{\mu_i}{\mu_M^2}\right)[Asy.cov(\hat{\mu}_i,\ \hat{\mu}_5) + Asy.cov(\hat{\mu}_i,\ \hat{\mu}_6)] \qquad \text{(A2a)}$$

$$+ \left(\frac{\mu_i}{\mu_M^2}\right)^2 [(.25)Asy.var(\hat{\mu}_5) + (.25)Asy.var(\hat{\mu}_6) + (.5)Asy.cov(\hat{\mu}_5,\ \hat{\mu}_6)]$$

$$= \left(\frac{1}{\mu_M}\right)^2 v_S(i,i) - \left(\frac{1}{\mu_M}\right)\left(\frac{\mu_i}{\mu_M^2}\right)[v_S(i,5) + v_S(i,6)] \qquad \text{(A2b)}$$

$$+\left(\frac{\mu_i}{\mu_M^2}\right)^2 [(.25)v_S(5,5) + (.25)v_S(6,6) + (.5)v_S(5,6)].$$

It then follows that

$$S.E.(\widehat{qr}_i) = \left[\frac{Asy\hat{.}var(\widehat{qr}_i}{N}\right]^{1/2}. \quad \text{(A3)}$$

These formulas hold for both i = 1, …, 4 as well as for i = 7, …, 10; i.e. for both arms of the relative income polarization curve ordinates.