Queen's Economics Department Working Paper No. 1528

Universal Choice Spaces and Expected Utility: A Banach-type Functorial Fixed Point

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2-2025

Universal Choice Spaces and Expected Utility: A Banach-type Functorial Fixed Point

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Abstract

This paper utilizes a Banach-type fixed point theorem in a functorial context to develop Universal Choice Spaces for addressing decision problems, focusing on expected utility and preference uncertainty. This generates an infinite sequence of optimal selection problems involving probability measures on utility sets. Each solution at a given stage addresses the preference ambiguity from the previous stage, enabling optimal choices at that level. The Universal Choice Space is characterized as a collection of finite-dimensional vectors of probability distributions, with the m^{th} component being an arbitrary probability measure relevant to the m^{th} stage of the problem. The space is derived as the canonical fixed point of a suitable endofunctor on an enriched category and simultaneously as the colimit of the sequence of iterations of this functor, starting from a suitable object.
JEL Codes: D81.
Keywords: Expected utility, ambiguity of preferences, infinite regress, enriched category, endofunctor, canonical fixed point, initial algebra, colimit, universal choice space.

1 Introduction

The underlying decision problem in the framework of expected utility is about choosing optimally from a set of probability distributions, that represent forms of uncertainty about a phenomenon at hand, given the underlying preferences of the decision maker

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on this set. Preferences are represented by a utility function, termed von Neumann-Morgenstern (vNM) utility, that assumes the form of the integral of an underlying Bernoulli utility function w.r.t. the probability measure at hand. Conditions for the existence of a Bernoulli function are well known (see for example Ok and Weaver (2013) (17) and the references therein). Ambiguity of the decision maker about her preferences is possible-see for example Schulz (2020) (21). Suchlike uncertainty could in principle be represented by a set of probability distributions defined now on the set of possible Bernoulli utilities. Resolution of ambiguity could arise by choosing optimally among them; convexity of the set of the Bernoulli utilities implies that the convex combination of those obtained via integrating w.r.t. the optimal measure would result in the optimal choice of a Bernoulli utility which could then be used in order to solve the original first order problem. Within this framework, optimal selection of this second order measure would necessitate preferences among the underlying set of measures, and thus a second order set of Bernoulli utilities. But then preference ambiguity could also accrue in this level. This reasoning thereby forms an infinite sequence (regress) of problems of optimal selection of probability measures on sets of utilities so that any solution at each stage, resolves the ambiguity of preferences at the previous stage making possible optimal choice there. The aim of this note is to show that this infinite regress is resolvable; this means that there exists a set, called the Universal Choice Space, that incorporates all uncertainty that appears in the infinite regress and also contains all possible ways in which the agent can deal with this uncertainty.

The analysis will be based on methods from category theory, a branch of theoretical and applied mathematics used in the past to analyze problems in economics. In most cases, the common element of such problems is that they have a recursive structure and it is precisely the complexity of this recursive-ness that allows or even necessitates the use of category theory. The classic example here comes from game theory. In the context of games with incomplete information, Mertens and Zamir (1985) (15) proved the existence of a space that contains all possible hierarchies of beliefs that the agents can form in a game. This space is frequently called the Universal Type Space, and it is constructed as a (what is termed inverse) limit of a suitable sequence that accommodates hierarchies of beliefs and belief interactions between agents. The aforementioned result essentially allows the formulation of the problem through category theory. Indeed, many authors have used category theory to provide comprehensive accounts regarding the problem of existence of the Universal Type Space. See for example Vassilakis (1992) (24), Moss and Viglizzo (2004) (16), Pinter (2010) (18), Blumensath and Winschel (2013) (5), Heinsalu (2014) (11), Fukuda (2024) (7), Guarino (2024) (10).

In the context of single agent decision problems, suchlike recursive structures forming infinite regresses similar to the above, were were considered by Vassilakis (1989) (22) and Lipman (1991) (12), who gave sufficient conditions for the existence of a Universal Choice Space; Vassilakis (1991) (23) showed that this was realizable as a minimal fixed point of a functor defined on a suitable category. A plethora of such fixed point results are now available in the literature-see for example Adamek, Milius, and Moss (2018) (1)-hereafter AMM18, potentially making the insights of Vassilakis (1991) (23) applicable in more general contexts. Indicative literature that uses this language in the context of decision is Galeazzi and Marti (2023) (8) or Pivato (2024) (19).

There is a huge literature that could serve as excellent source of formal introductions to category theory and its useful tools in the above contexts. Classic textbooks are Arbib and Manes (1986) (6) and MacLane (1978) (14). Categories provide a general framework for describing mathematical structures and their relationships. An informal description of tools and notions that will be used below is the following: A category consists of objects and morphisms (arrows) between these objects that satisfy two main properties; composition of morphisms is associative, and each object has an identity morphism. Two objects are considered isomorphic in the category at hand if there exists a morphism from the first to the second, and a morphism from the second to the first, that when composed produce the relevant identity. A category is considered locally small if for every pair of objects the collection of morhisms from the first object to the second is a set, usually termed as hom-set. A locally small category is enriched over a second category equipped with a particular algebraic structure, if every hom-set is also an object of the second category. Thereby in an enriched category, the sets of morhisms between objects have additional structure reflecting the properties of the enriching category.

A colimit is a way to "glue together" a collection of objects in a category based on specified morphisms between them. It generalizes constructions like the direct sum of vector spaces or the union of sets. Formally, it is the universal co-cone over a diagram of objects, capturing the idea of a "coherent merging" in the direction of the morphisms. In other words, given any arbitrary sequence of morphisms between objects of some category, an ordered pair that constitutes of an object and a collection of morphisms from each object of the sequence to that object, is considered a colimit of the sequence if, firstly, the composition of morphisms satisfies a certain naturality condition, and secondly, there exists a unique mapping from the colimit to any other ordered pair that satisfies the naturality condition. Colimits are unique up to isomorphism. Vassilakis (1991) (23) characterizes the Universal Choice Space over a fixed set of actions, among others, as a colimit of a sequence of objects in a suitable category.

A (covariant) functor is a mapping between two categories that preserves their structure. It assigns to each object and morphism in one category a corresponding object and morphism in the other, while preserving identities and compositions. A contravariant functor works as above, except for the fact that it reverses the direction of morphisms and the order of compositions. An endofunctor is any functor that maps a category to itself. A fixed point of an endofunctor is an object in the category that is isomorphic to its transformation by the functor. An algebra of an endofunctor is an ordered pair consisting of an object together with a morphism from the transformation of the object by the functor to that object, and an initial algebra is an algebra whose object defines a unique homeomorphic morphism from that object, to the object of any other algebra. By Lambek's lemma, see MacLane (1978) (14), an initial algebra is a minimal fixed point of a functor; there exists a morphism from the initial algebra to any other fixed point that respects identifications via the respective isomorphisms. As such the initial algebra provides with the minimal structure that satisfies a recursive definition, and can thus provide a model for Universal Choice Spaces. Dually, a terminal co-algebra of an endofunctor is a maximal fixed point; there exists a unique morphism from it to any other fixed point of the functor that respects identifications via the respective isomorphisms. As such it provides with maximal structures that satisfy co-recursions, providing thus a model for Universal Type Spaces. A fixed point that is simultaneously isomorphic to an initial algebra and a terminal co-algebra is termed canonical.

Given the above, the decision theoretic framework of the present note can be summarized as follows: X is a space upon which the uncertainty of a phenomenon of interest is structured. $\Delta(X)$ is then the set of probability distributions on (some algebra of subsets of) X, which constitutes the original choice space. The decision maker has preferences over $\Delta(X)$ represented by a Bernoulli utility function $u\colon X \to \mathbb{R}$. $U(X)$ is the space of the possible suchlike utilities over X. At this stage, ambiguity over $U(X)$ is introduced, and it is represented by $\Delta(U(X))$; if $U(X)$ is convex, then choice of a probability distribution $\mathbb{P}$ in $\Delta(U(X))$, corresponds to choice of a utility in $U(X)$ via $\int_{u\in U(X)} ud\mathbb{P}(u)$, w.r.t. which the original decision problem is resolvable. Hence $\Delta(U(X))$ is the choice space of the second order problem. Continuing the regress it is obtained that $(\Delta U)^m(X)$-with $(\Delta U)^m$, $m > 0$ denoting m-fold composition-corresponds to the choice space available to the decision maker at the $m + 1^{\text{th}}$ order of the regress. The question is whether this regress converges to a suitable kind of limit, and if this limit can be considered as a fixed-point of the infinite regress. If the latter is true, the fixed-point can be characterized as the the Universal Choice Space, that represents the collection of all available decision procedures to the decision maker at every level of uncertainty/ambiguity.

The analysis resolves this issue by first choosing appropriately the category upon which the regress is unfold. Second, Δ and U are constructed as endofunctors on this category. Third, another functor that depends on both Δ and U is constructed, and the infinite regress described above is represented as a sequence of self-iterations of this functor. Then, via a Banach-type fixed point theorem for categories, the functor is shown to have a canonical fixed point that is simultaneously characterized as the colimit of the sequence of self-iterations initialized at a suitable object. This is the Universal Choice Space sought, which can be represented as the collection of finite dimensional vectors of probability distributions; the first component of such a vector is some member of $\Delta(X)$, and the m^{th} component is a member of $(\Delta U)^m$, $m > 0$. The analysis differs to that of Vassilakis (1991) (23) and Lipman (1991) (12) in that the decision procedures here do not take the form of trees, but are rather represented by vectors. Furthermore, the analysis extends Vassilakis (1991) (23) by not requiring the finite structures supposed there, and also avoids the considerations of sequences on ordinals larger than ω (the first infinite ordinal) that appear in Lipman (1991) (12).

The remaining note is structured as follows: the following section presents the categorical details, the functorial constructions and the result. The final section presents a characterization of the multi-utility decision procedures of Ok and Weaver (2023) (17) as particular parts of the Universal Choice Space and briefly discusses possible extensions of the current results.

2 Categorical framework and result

In what follows $\lambda \in (0, 1)$ is a fixed real number. The analysis considers the CMS enriched category $\mathbf{CompMet}_{\mathrm{CMS}}$. This consists of objects that are compact metric spaces, with uniformly bounded (by 1) hom-sets formed by complete metric spaces w.r.t. the uniform metric, featuring non-expansive composition as referenced in AMM18. Referring to AMM18 (see Notation 5.5) as well as America and Rutten (1989) (4), and Par. III of Adamek and Reiterman (1994) (2), the category of embeddings $\mathbf{CompMet}^{E}_{\mathrm{CMS}}$ is also considered. The objects there are the objects of $\mathbf{CompMet}_{\mathrm{CMS}}$, for which there exist pairs (f, g_f), where $f : X \to Y$, $g : Y \to X$ are morphisms within $\mathbf{CompMet}_{\mathrm{CMS}}$, such that $g_f \circ f = \mathrm{id}_X$. Those pairs constitute the morphisms in $\mathbf{CompMet}^{E}_{\mathrm{CMS}}$. The hom-sets of $\mathbf{CompMet}^{E}_{\mathrm{CMS}}$ are equipped with the max-sup metric $d((f, g_f), (h, g_h)) := \max\left\{\sup_{x\in X} d_X(f(x), h(x)), \sup_{y\in Y} d_Y(g_f(y), g_h(y))\right\}$, under which they remain complete. The Remark of Par. III of Adamek and Reiterman (1994) (2) says that the category of embeddings is also CMS-enriched. The analysis finally considers the product category $\mathcal{G} := (\mathbf{CompMet}^E_{\text{CMS}})^2$; the previous, along with example 5.1.(d) of AMM18 show that it is also CMS-enriched via the max-metric.

Δ denotes the probability functor on $\mathbf{CompMet}_{\text{CMS}}$; $\Delta(X)$ is the set of Borel probability measures on X, an arbitrary object of $\mathbf{CompMet}^E_{\text{CMS}}$. $\Delta(X)$ is equipped with the $\lambda/2$-scaled 1-Wasserstein metric based on the distance function d_X equipping X: for $\mathbb{P}, \mathbb{Q} \in \Delta(X)$, $d^\lambda_{1-W}(\mathbb{P}, \mathbb{Q}; d_X) := \lambda/2 \min_{\gamma \in \Gamma(\mathbb{P},\mathbb{Q})} \int_X d_X(z, z^\star) d\gamma(z, z^\star)$, where $\Gamma(\mathbb{P}, \mathbb{Q})$ denotes the set of Borel probability distributions on $X \times X$ that have respective "marginals" $\mathbb{P}$, $\mathbb{Q}$-see Gao, Chen, and Kleywegt (2024) (9). The metric, being equivalent to the standard form of the 1-Wasserstein metric (this is derived from above by reversing the scalling, i.e. setting $\lambda = 2$), topologizes weak convergence-see Rahimian and Mehrotra (2019) (20). Given that X is compact the property is inherited by $\Delta(X)$, see Theorem 15.11 of Aliprantis and Border (1999) (3), thereby Δ maps compact metric spaces to compact metric spaces. If $f \in \text{Hom}(X, Y)$, then $\Delta(f) \in \text{Hom}(\Delta(X), \Delta(Y))$, with $\Delta(f)(\mathbb{P}) := \mathbb{P} \circ f^{-1}$, and due to the definition of the 1-Wasserstein metric above, the change of variables property of the Lebesgue integral and the non-expansive property of f, $\Delta(f)$ is non-expansive. Obviously $\Delta(\text{id}_X)(\mathbb{P}) = \text{id}_{\Delta(X)}(\mathbb{P})$, and $\Delta(g \circ f)(\mathbb{P}) = \mathbb{P} \circ (f^{-1} \circ g^{-1}) = \Delta(g)(\mathbb{P} \circ f^{-1}) = (\Delta(g) \circ \Delta(f))(\mathbb{P})$, and $\sup_{f,h,\mathbb{P},\mathbb{Q}} d^\lambda_{1-W}(\mathbb{P} \circ f^{-1}, \mathbb{Q} \circ h^{-1}; d_X) \leq \sup_{f,g} \sup_x d_X(f(x), g(x)) \leq 1$. Applying this reasoning to each element of an arbitrary pair in $\text{Hom}(X, Y)$ in $\mathbf{CompMet}^E_{\text{CMS}}$, implies that Δ is a well defined as a covariant endo-functor when restricted on $\mathbf{CompMet}^E_{\text{CMS}}$. Furthermore, and since for arbitrary $f_1, f_2 : X \to Y$, the inequalities $\sup_{\mathbb{P} \in \Delta(X)} d^\lambda_{1-W;d_X}(\mathbb{P} \circ f_1^{-1}, \mathbb{P} \circ f_2^{-1}) \leq \lambda \sup_{x \in X} d_X(f_1(x), f_2(x))$, and $\sup_{\mathbb{P} \in \Delta(Y)} d^\lambda_{1-W;d_Y}(\mathbb{P} \circ g^{-1}_{f_1}, \mathbb{P} \circ g^{-1}_{f_2}) \leq \lambda \sup_{y \in Y} d_Y(g_{f_1}(y), g_{f_2}(y))$ hold, Δ is also λ-locally contracting in the sense of Definition 5.2 in AMM18 (1) (also referenced in America and Rutten (1989) (4)); the functor contracts by a factor λ the sup-distance between every pair of morphisms. Hence:

Lemma 1. *Δ is a λ-locally contracting endofunctor on* $\mathbf{CompMet}^E_{\text{CMS}}$.

Additionally, $U_\lambda(X)$ denotes the set of Bernoulli utilities defined on X, that are d_X-Lipschitz continuous with Lipschitz coefficient bounded above by λ. Notice that d_X-Lipschitz Bernoulli utilities always exist due to Proposition 4.1 of Ok and Weaver (2023) (17), the fact that every compact metric space is separable, and the fact that Lipschitz continuous functions $X \to \mathbb{R}$, e.g. $d_X(\cdot, x_0)$ for arbitrary $x_0 \in X$. Then, invariance of the Bernoulli utilities w.r.t. scalling, and the compactness of (X, d_X), imply that the set $U_\lambda(X)$ is not empty, as well as a complete and totally bounded metric space when equipped with the uniform metric. The set $\text{Hom}(U_\lambda(X), U_\lambda(Y))$, admits the metric $\max\left\{\sup_{u \in U_\lambda(X), y \in Y} d_Y(f^\star(u)(y), h^\star(u)(y)), \sup_{u \in U_\lambda(Y), x \in X} d_X(g_{f^\star}(u)(x), g_{h^\star}(u)(x))\right\}$, which is bounded by 1, and w.r.t. which the hom-set is complete due to the completeness of the target spaces. For any non-expansive map $f : X \to Y$, $U_\lambda(f)(u) := u \circ g_f : U_\lambda(X) \to U_\lambda(Y)$ is non-expansive since $\lambda < 1$, while $U_\lambda(\mathrm{id}_X) = \mathrm{id}_{U_\lambda(X)}$, and, if $v \in U_\lambda(Z)$, and $h : Y \to Z$, $U_\lambda(h \circ f)(v) = (v \circ h) \circ f = U_\lambda(f) \circ U_\lambda(h)$. Applying this reasoning to each element of an arbitrary pair in $\mathrm{Hom}(X, Y)$ in $\mathbf{CompMet}^E_{\mathrm{CMS}}$ establishes that U_λ is a well defined covariant endo-functor on $\mathbf{CompMet}^E_{\mathrm{CMS}}$. Given that we have $\sup_{u\in U_\lambda(X)} \sup_{y\in Y} |u(g_{f_1}(y)) - u(g_{f_2}(y))| \leq \lambda \sup_{y\in Y} d_Y(g_{f_1}(y), g_{f_2}(y))$, along with $\sup_{u\in U_\lambda(Y)} \sup_{x\in X} |u(f_1(x)) - u(f_2(x))| \leq \lambda \sup_{x\in X} d_X(f_1(x), f_2(x))$, it follows that the functor U_λ is λ-locally contracting as well.

Lemma 2. *U_λ is a λ-locally contracting endofunctor on $\mathbf{CompMet}^E_{\mathrm{CMS}}$.*

For a fixed $X \in \mathbf{CompMet}^E_{\mathrm{CMS}}$, any decision procedure up to stage m can be represented by a finite dimensional vector of probability measures, say $(\mathbb{P}_n)_{0\leq n\leq m}$, $m \in \omega$, where $\mathbb{P}_0 \in \Delta(X)$, and, $\mathbb{P}_n \in (\Delta \circ U_\lambda)^{(n)}(X)$ for $n > 0$, with $(\Delta \circ U_\lambda)^{(n)}$ denoting the n^{th} self-composition of $\Delta \circ U_\lambda$. Letting m vary in ω, denoting by $\mathbb{P}_y$, the degenerate on the element $y \in Y$ member of $\Delta(Y)$, it is observable that any such vector can be uniquely extended to a sequence by setting $\mathbb{P}_n := \mathbb{P}_{\mathbb{P}_{n-1}}$ for any $n > m$. Thus, the resulting set of decision procedures, say C_{00}, is a compact metric space when equipped with the max between the associated λ scaled 1-Wasserstein metrics, which can then be considered as an object of the product category $\mathcal{G}$.

Then, the functor Ch_λ is considered. This is defined on the objects of $\mathcal{G}$ via the product $\mathrm{Ch}_\lambda(Z, Y) := (\Delta(X), \Delta\circ U_\lambda(Y))$, and on the morphisms by $\mathrm{Ch}_\lambda((f, g_f), (h, g_h)) := ((\mathrm{id}_{\Delta(X)}, \mathrm{id}_{\Delta(X)}), (\Delta \circ U_\lambda(g_f), \Delta \circ U_\lambda(f)))$. Setting $(\Delta \circ U_\lambda)^{(0)} := \mathrm{id}_{\mathcal{G}}$, considering the functor's iteration sequence commencing on (X, X),

$$(X, X) \overset{\mathrm{Ch}_\lambda}{\to} (\Delta(X), \Delta \circ U_\lambda(X)) \overset{\mathrm{Ch}_\lambda}{\to} \cdots \overset{\mathrm{Ch}_\lambda}{\to} (\Delta(X), (\Delta \circ U_\lambda)^{(m)}(X)) \overset{\mathrm{Ch}_\lambda}{\to} \cdots, \quad (1)$$

and since for any $n > 1$, $(\Delta \circ U_\lambda)^{(m-1)}$ is embedded in $(\Delta \circ U_\lambda)^{(m)}$ due to Theorem 15.8 of Aliprantis and Border (1999) (3), it is obtained that at stage m, the set of vectors $(\Delta(X), (\Delta \circ U_\lambda)^{(m)}(X))$ represents every decision procedure up to m; the first part provides information on the initial problem, and the second part summarizes the scope of ambiguity about preferences up to m. The Universal Choice Space sought would then be the colimit-if it exists-of the sequence in (1). The derivation of the existence of this colimit and its identification, would be facilitated by the derivation of the existence and the identification of the initial algebra of the functor-see Vassilakis (1991) (23) as well as America and Rutten (1989) (4). If furthermore C_{00} was proven isomorhic to this initial algebra, it would then have been characterized as the Universal Choice Space sought.

Essentially, a contractivity property of the Ch_λ functor, inherited by Lemmata 1 and 2, leads to a stronger result than the existence of an initial algebra, via the application of the Banach type functorial fixed point theorem of Adamek and Reiterman (1994) (2). The theorem says that a terminal co-algebra also exists and it is isomorphic to the initial algebra. Those are also identified with C_{00}. This is the main result:

Theorem 1. *The space C_{00} is the canonical fixed point of Ch_λ on $\mathcal{G}$, as well as the colimit of the sequence in (1).*

Proof. Lemmata 1 and 2 directly imply that the composite functor $\Delta \circ U_\lambda$ is a λ-locally contracting well-defined endofunctor on $\mathbf{CompMet}^E_{\mathrm{CMS}}$. Then, and since Ch_λ is constant in the first argument, the functor is a well-defined λ-locally contracting endofunctor on $\mathcal{G}$. The existence of a canonical fixed point would be then obtained by Theorem 3 of Adamek and Reiterman (1994) (2), since $(\mathbf{CompMet}^2_{\mathrm{CMS}})^E \cong \mathcal{G}$, as long as $\mathbf{CompMet}^2_{\mathrm{CMS}}$ has inverse limits for contracting sequences (see the Definition in Par. II of Adamek and Reiterman (1994) (2)). This is due to that $\mathbf{CompMet}^2_{\mathrm{CMS}}$ has a terminal object-the self-product on the one point space as the terminal object in $\mathbf{CompMet}^2_{\mathrm{CMS}}$, and it is connected since $\mathbf{CompMet}_{\mathrm{CMS}}$ is; via constant maps. It suffices that $\mathbf{CompMet}^2_{\mathrm{CMS}}$ has inverse limits, for which it also suffices that $\mathbf{CompMet}_{\mathrm{CMS}}$ has inverse limits since then the product respects them.

$\mathbf{CompMet}^E_{\mathrm{CMS}}$ has inverse limits since for any ω^{op}-sequence in this category, represented by the set of non-expansive morphisms $\{X_{n+1} \overset{h_n}{\to} X_n\}_{n\in\omega}$, the product $\prod_{n\in\omega} X_n$ equipped with the maximum metric, is compact. The inverse limit is defined as the subset $\mathrm{IL} = \{(x_0, x_1, ...) \in \prod_{n\in\omega} X_n \mid x_n = h_n(x_{n+1}), n \in \omega\}$, equipped with the set of projections $\{\mathrm{IL} \overset{\pi_n}{\to} X_n\}_{n\in\omega}$, where $\pi_n(x) = x_n$, which is obviously non-expansive, since the following two conditions are satisfied by the construction. First, $\pi_n = h_n \circ \pi_{n+1}$ must hold for all $n \in \omega$. It is easy to see to see that this holds by the definitions. The second condition is that for any other set of non-expansive morphisms $\{X \overset{f_n}{\to} B_n\}_{n\in\omega}$ such that $f_n = h_n \circ f_{n+1}$ for all $n \in \omega$, where A is an object in the category, a unique morphism $f\colon X \to \mathrm{IL}$ such that $f_n = \pi_n \circ f$ for all $n \in \omega$ must exist. The proof of latter condition follows easily if we define f in the only possible way, that is $f(y) = (f_0(y), f_1(y), ...)$, and notice that $f(y) \in \mathrm{IL}$ due to the way each morphism f_n is defined. Due to the definition of the max-metric and the properties of the associated functions f is readily seen to be non-expansive, and the uniform distances between any such functions are bounded by 1. This shows that $\mathbf{CompMet}_{\mathrm{CMS}}$ has inverse limits, which then implies that Ch_λ has a canonical fixed point. Finally, the identifications implied by Theorem 15.8 of Aliprantis and Border (1999) (3) as well as the proof of Corollary 3.15 of America and Rutten (1989) (4) and their subsequent Remark (see also Definition in Section II of Adamek and Reiterman (1994) (2), and Theorem 1 of Adamek and Reiterman (1994) (2)) imply that C_{00} is constructed as the colimit (see Ch.3.III of Mac Lane (1978) (14)) of the sequence in (1), and additionally that $\mathrm{Ch}_\lambda(C_{00}) \cong C_{00}$ establishing the result. $\square$

3 Discussion

Theorem 1 says that C_{00} is the up to isomorphism unique Banach-type fixed point of the choice functor, establishing it as the Universal Choice Space over the fixed X. Canonicality of the fixed point implies that it is the initial algebra of the endofunctor; C_{00} thus corresponds to the most economical way to construct the collection of the available decision procedures. It simultaneously is the terminal co-algebra of Ch_λ, hence also representing the maximal dynamics of the infinite regress. The local contractivity of the functor involved makes those two minimal and maximal structures isomorphic. It also implies that even though no finite structures are involved in the constructions above, in contrast with Vassilakis (1991) (23) where a finite language and its finite power set is used in the formation of the infinite regresses there, no ordinal numbers greater than ω are needed for the establishment of the existence of the choice space as an inverse limit of the infinite regress, as opposed to the constructions in Lipman (1991) (12). The fact that vNM preferences are invariant w.r.t. rescalling of their Bernoulli components, implies that the result holds also for the more general preference spaces of Lipschitz continuous Bernoulli utilities.

The characterization of the Universal Choice Space above implies a subsequent characterization of the multi-utility orders in Ok and Weaver (2023) (17) as a conservative resolution of ambiguity about preferences at $m = 1$, as well as their extension to larger stages of the regress. Given $U_\lambda(X)$, and $\mathbb{P}_1, \mathbb{P}_2 \in \Delta(X)$, $\mathbb{P}_1$ dominates $\mathbb{P}_2$ w.r.t. to the U_λ utilities if and only if $\int_X u d\mathbb{P}_1 \geq \int_X u d\mathbb{P}_2$, $\forall u \in U_\lambda(X)$. Given that $U_\lambda(X)$ is uniformly bounded, the envelope function $v_X(x) := \sup_{u \in U_\lambda} u(x)$ is an element of the hom-set, since $|v_X(x) - v_X(x')| \leq \sup_{u \in U_\lambda} |u(x) - u(x')| \leq \lambda d_X(x, x')$. Also, due to that $\int_X v_X d\mathbb{P} \geq \sup_{u \in U_\lambda} \int_X u(x) d\mathbb{P}$ for any $\mathbb{P} \in \Delta(X)$, choice via the envelope v_X, is equivalent to choice to the U_λ utilities and represents the most conservative decision procedure on $\Delta(X)$. Thereby, the resolution of ambiguity for preferences at $m = 1$ with the U_λ multi-utility ordering, is represented by the subset of C_{00}, identified by the vectors of the form $(\mathbb{P}_0, \mathbb{P}_{u_X}), \mathbb{P}_0 \in \Delta(X)$. Since the construction of the envelope utility is possible in $U_\lambda(Y)$ for any Y in the working category, the multi-utility ordering as a conservative resolution of ambiguity for preferences is generalizable for $m \geq 1$. Thus, vectors of the form $(\mathbb{P}_0, \ldots, \mathbb{P}_{m-1}, \mathbb{P}_{u_{(\Delta \circ U_\lambda)^{(m)}(X)}})$ identify a part of the choice space that represent the decision procedures that are the most conservative at their peak level m, thereby defining multi-utility ordering type resolutions of ambiguity for preferences at stage $m \geq 1$. Those vectors could also represent choices in problems of hierarchical decision making with highly conservative leaders.

The representation of the Universal Choice Space as a Banach-type fixed point was facilitated by the consideration of the $\mathbf{CompMet}^{E}_{\mathrm{CMS}}$ category of embeddings, which allowed covariance in the definition of U_λ. If the analysis is extended to $\mathbf{CompMet}_{\mathrm{CMS}}$, U_λ would need to be defined as a contravariant functor. This change would result in Ch_λ losing its covariance and becoming a pro-functor, meaning it would function as both a covariant and contravariant bi-functor. Further research into this scenario, which involves examining initial algebras and terminal coalgebras for profunctors—as discussed by Lorengian (2021)- (13)—appears promising. It could also facilitate the exploration of the existence of the Universal Choice Space beyond the constraints of contractivity for the involved functors.

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