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Measure Frictional Sorting

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Abstract

This paper constructs an index to measure sorting in a two-sided market with matching friction. The index uses the cumulative distribution function (cdf) of the matching partners for each individual and incorporates matching failures as a mass point at the lowest type of the partners. The area between this cdf and the cdf in the efficient allocation without friction represents the reduction in sorting. The sorting index is equal to one minus the aggregate reduction in sorting normalized by the amount under uniform matching. In contrast to most measures of sorting in the literature, the sorting index captures the effect of matching failures and compares the matching partner's type with the ideal type instead of the individual's own type. Relative to this sorting index, the correlation coefficient of matched types tends to over-estimate sorting when sorting is positive. The over-estimation increases sharply when matching failures increase or when the market becomes more imbalanced.

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1 Introduction

How to measure the strength of sorting in a two-sided market with matching friction? The answer to this question remains elusive. Although there are various measurements of sorting in theory and practice (see Chiappori et al., 2024), these measurements examine only matched individuals. They ignore the effect of matching failures on sorting. This paper constructs a sorting index to incorporate this effect. I compare the index with some other methods of ranking sorting and, in particular, with the coefficient correlation of matched types and positive quadrant dominance.

When matching is frictionless, the theoretical literature on sorting inspired by Becker (1973) has focused on perfect positive and perfect negative sorting. It is apparent how to compare these two polar cases. However, realistic markets have matching friction that makes the matching patterns deviate from perfect sorting. Some individuals fail to match and, for each matched type, the matching partners can be mixed. In the labor market, for example, there are unemployed workers and unfilled vacancies. Moreover, workers of the same skill are employed by firms that differ in size, the capital stock or firm productivity. Chiappori et al. (2024) review the methods of ranking sorting in the literature. These methods use the observed distribution of matches and compare how the matching partner's type (in the quantile) is close to an individual's own type. As such, they do not adequately capture how matching friction affects sorting and social efficiency.

To illustrate these problems, consider the correlation coefficient of matched types, which is commonly used to quantify sorting in the labor market (Abowd et al., 1999) and the marriage market (Greenwood et al., 2003). First, if the matching probabilities fall uniformly for all individuals, the distribution of matched types is unchanged, and so is the correlation coefficient. Similarly, the correlation coefficient is equal to one even if the only matches are the highest type with the highest type and the lowest type with the lowest type while all other individuals are unmatched. Second, when the market is imbalanced, the socially efficient allocation is not to match an individual with someone at the same quantile but rather with some "ideal" type that depends on the market tightness. In this case, a higher coefficient of correlation does not necessarily indicate a more desirable matching pattern. As a related feature, a uniform shift of the matching partner's type for all individuals on one side of the market does not affect the correlation coefficient. However, the shift affects social efficiency and should be reflected by a sorting index.

In a typical market, an unmatched individual does not stay unmatched permanently. For example, an unemployed worker will eventually find a job that will generate a positive joint surplus. Because this gain is unrealized when a worker is unemployed, sorting is weaker than the socially desirable one. Moreover, the market tightness and the measure of unmatched individuals change in the business cycle and during technological progress such as the one studied by Lindenlaub (2017). To correctly evaluate how these changes affect sorting, the first step is to design a sorting index to include unmatched as well as matched individuals.

The two-sided market described in section 2 has a continuum of types on each side. An individual's type is the quantile in which the individual lies. An allocation or matching pattern is the expected probability that an individual matches with someone at or above a given quantile on the other side. The cumulative distribution function (cdf) of the matching partners for the individual is one minus this expected probability. This distribution includes the probability of failing to match as a mass point at the quantile zero of the matching partner's type. The efficient allocation without frictions is perfect positive sorting that matches each individual at quantile q only with an ideal type $p(q)$ that increases in q. If an individual is on the long side of the market, the ideal type is strictly lower than the individual's own type as long as the individual is not the highest type. In this case, there is a cutoff quantile on the long side of the market below which the individuals are not matched in the efficient allocation.

For each type, I measure the reduction in sorting under a matching pattern relative to the efficient pattern by the area between the cdf's of the matching partners under the two patterns. The aggregate reduction in sorting is the sum of the reductions in sorting over all types in the market, including the unmatched ones. I normalize this reduction in sorting by the corresponding quantity under uniform matching where each individual who should be matched in the efficient allocation draws a partner uniformly from the other side of the market. The sorting index, denoted as σ, is one minus the normalized reduction in sorting. The sorting index is inversely related to the weighted average distance between the matching partner's type and the ideal type. Because the reference is the ideal partner's type, the sorting index is a measure of how close a matching pattern is to the efficient allocation. The imbalance of the market affects σ by changing the ideal type. Matching failures affect the sorting index by changing the cdf of the types of the matching partners.

As an illustration, I compute the sorting index in positive block matching studied by Smith (2006) and Burdett and Coles (1997). Under this matching pattern, all individuals who should be matched in the efficient allocation are divided into k blocks, and individuals match exclusively inside each block randomly. The sorting index is equal to $\sigma = 1 - \frac{1}{k}$. When there is only one block ($k = 1$), matching is uniform and the sorting index is equal to 0. As the number of blocks increases, the sorting index increases. In the limit $k \to \infty$, the matching pattern approaches the efficient allocation of perfectly positive sorting, and the index approaches 1.

An important component of the sorting index is sorting in the upper quadrant. For any individual type q and the ideal partner's type $p(q)$, the upper quadrant $(q, p(q))$ consists of all types on the same side as the individual that are greater than or equal to q and of all types on the other side of the market that are greater than or equal to $p(q)$. The upper quadrant sorting function is the probability that these two groups of types are matched with each other. By summing this function over all types in the market, I construct the upper quadrant sorting index, denoted as σ_U. The upper quadrant sorting index is the only component of the sorting index that captures who matches with whom. The other components reflect only the overall matching failures and the imbalance of the market.

The connection between σ and σ_U is tight in a class of matching patterns termed uniform matching failures. In this class, all types that should be matched in the efficient allocation have the same probability of failing to match. Block matching belongs in this class, with zero probability of failing to match. With uniform matching failures, the difference between σ and σ_U is a constant that depends only on two factors: the market tightness and the uniform probability of matching failures. Given these two factors, any change in the matching pattern in the class affects the two sorting indices in the same way. As σ, the upper quadrant sorting index falls when the probability of failing to match increases for all types.

I compare the sorting index with the coefficient of correlation. Because the correlation coefficient ignores unmatched individuals and compares the matching partner's type with an individual's own type, it over-estimates the extent of positive sorting when sorting is significantly positive. Specifically, with uniform matching failures, the correlation coefficient exceeds the upper quadrant sorting index when the latter is above a threshold. With examples, I illustrate that the over-estimation also exists when matching failures are not uniform. In one example, the matching partner's type is equal to the individual's own type. However, because of the imbalance of the market, the ideal type differs from the individual's own type. When the market tightness falls to increase the imbalance, σ and σ_U fall sharply and can even become negative, but the correlation coefficient remains at 1. In another example, an individual's matching partners are distributed around an individual's own type. A shift in this distribution or an increase in the mixing of matches induces σ and σ_U to decrease more significantly than the correlation coefficient does.[1]

Section 5 examines additional issues. One is to relate the sorting index to positive quadrant dominance (PQD) studied by Lehmann (1966) and Anderson and Smith (2024). PQD is a partial ordering constructed from the joint distribution of matched types. In general, PQD does not imply a larger sorting index, because PQD ignores unmatched types and compares the matching partner's type with an individual's own type. However, with uniform matching failures, PQD is sufficient for the upper quadrant sorting index to be larger. The reverse is not true. In positive block matching, the upper quadrant sorting index increases when the number of blocks increases from k (≥ 2) to $k+1$, but PQD cannot rank the two matching patterns. Moreover, section 5 extends the sorting index to incorporate discrete types and multi-dimensional types.

The sorting index proposed in this paper is related to various measures in the literature (see Chiappori et al., 2024, for a review). The normalization of the sorting index by the value in uniform matching is similar to Liu and Lu (2006) and Shen (2020). The use of

[1] When sorting is significantly negative, the correlation coefficient can be lower than the sorting index.

perfect positive sorting as the upper bound on the sorting index has the same flavor as in Fernandez and Rogerson (2001), who calculate the sorting index as the weight of a matching pattern between perfect sorting and uniform matching. The reference to perfect positive sorting and/or uniform matching is also common in other measures of sorting. For example, the odds ratio computes the relative occurrence of perfect matches to mixed matches (Siow, 2015, Chiappori et al., 2017). The type-specific likelihood ratio computes the likelihood of perfect matches to uniform matches (Eika et al., 2019). The normalized trace computes the fraction of perfect matches in all matches (Cheremukhin et al., 2024, Choi and Kim, 2025). As discussed earlier, all of these measures ignore unmatched individuals.[2]

Section 2 describes a two-sided market with matching friction and characterizes the efficient allocation without matching friction. Section 3 constructs the sorting index and studies its properties. Section 4 compares the sorting index with the correlation coefficient. Section 5 relates the sorting index to PQD and extends the analysis. Section 6 concludes and the appendices provide proofs.

2 Two-Sided Matching

Consider an economy that lasts for one period. The two sides of the market are labeled s (stayers) and v (visitors), respectively. Denote a side of the market as $\tau \in \{v, s\}$, and the opposite side as $-\tau$. There is a continuum of traders on each side of the market. The measure of traders on the side τ is n_τ, with $n_v = 1$ and $n_s = \theta$. To reduce clutter, I assume that the market tightness for the side s satisfies $\theta \leq 1$. If $\theta \geq 1$, the analysis remains valid after switching the labels v and s. The market is balanced if $\theta = 1$. If $\theta < 1$, the side s is the short side.

The type of a trader on the side τ is $y_\tau \in \mathcal{Y}_\tau \subseteq \mathbb{R}_+$. The cumulative distribution function (cdf) of y_τ is F_τ with a continuous density F'_τ. Section 5.2 will incorporate distributions with mass points, e.g., discrete types. Denote $q_\tau = F_\tau(y_\tau)$ as the quantile q_τ of the type

[2]Bartoclucci et al. (2018) propose an index of sorting that is inversely related to the variations in the matching partners' types for individuals. This index is about the purity of matching. For example, both perfect positive sorting and perfect negative sorting have zero variation in matching partners' types.

distribution on the side τ. Because y_τ is distributed continuously, q is uniformly distributed in $[0,1]$. Since the quantile is invariant to monotone transformations of y, such as rescaling, the analysis below will use q_τ as a trader's type instead of y_τ. I add an artificial type $-\varepsilon < 0$ as the matching partner's type if a trader is unmatched, where ε is sufficiently close to 0. Denote the extended set of types as $\mathcal{Q} = [0,1] \cup \{-\varepsilon\}$.

Matches (trades) are bilateral. The joint surplus function of a match is such that the socially efficient allocation with frictionless matching is perfect positive sorting in (q_v, q_s) (see the characterization later). A sufficient condition for this efficient outcome is that the joint surplus function is supermodular in (q_v, q_s) (Becker, 1973). I take the efficient allocation as given to focus on how to measure the sorting strength of frictional matching patterns.

The two sides of the market may go through the processes of meetings and acceptance before forming matches. For example, traders may have multilateral meetings and use a criterion to select the matching partner from the meetings, e.g., Gale and Shapley (1962). To measure sorting, it suffices to represent a matching pattern or an allocation by the expected matching outcome, $\{r_\tau(q_\tau, q) : (q_\tau, q) \in \mathcal{Q}^2\}_{\tau=v,s}$. The function $r_\tau(q_\tau, q)$ is the probability that trader q_τ on the side τ expects to succeed in matching with someone in $[q, 1]$, where the expectation is taken over the meetings that trader q_τ will receive.[3] For all $q_\tau \in \mathcal{Q}$, $r_\tau(q_\tau, q)$ is (weakly) decreasing in q for all $q \in [0,1]$, with $r_\tau(q_\tau, 1) = 0$ for all $q_\tau < 1$. Trader q_τ fails to match with the probability $[1 - r_\tau(q_\tau, 0)]$. Because a trader is either matched or unmatched, $r_\tau(q_\tau, -\varepsilon) = 1$ for all $q_\tau \in \mathcal{Q}$. I assume that $r_\tau(q_\tau, q)$ is right-differentiable in q. This encompasses the case where trader q_τ has positive probability of matching with only one type. Let $r'_{\tau,2}(q_\tau, q)$ denote the right-side derivative of r_τ with respect to q.

For any $(q_\tau, q) \in [0,1]^2$, let the upper quadrant (q_τ, q) refer to the rectangle containing all (q'_τ, q') such that $q_\tau \le q'_\tau \le 1$ and $q \le q' \le 1$. Denote $R_\tau(q_\tau, q)$ as the probability that a trader on the side τ has a type greater than or equal to q_τ and is matched with some type

[3] The matching outcome r_τ is the ex ante allocation. If there are a large number of replicas of each type or a large number of repetitions of the same matching pattern, then r_τ is also the ex post allocation averaged over the replicas or repetitions.

greater than or equal to q. Because q_τ is uniformly distributed in $[0,1]$, then

$$R_\tau(q_\tau, q) = \int_{q_\tau}^{1} r_\tau(q', q)\mathrm{d}q' \text{ for } (q_\tau, q) \in [0,1]^2 . \tag{2.1}$$

I refer to R_τ as the upper quadrant matching function on the side τ.[4] Clearly, $R_\tau(q_\tau, q)$ is a decreasing function of (q_τ, q). The overall matching probability for traders at or above q_τ on the side τ is $R_\tau(q_\tau, 0)$. (2.1) implies $r_\tau(q_\tau, q) = -R'_{\tau,1}(q_\tau, q)$, where the subscript 1 indicates that the derivative is with respect to the first argument.

Because matching is bilateral, matches must be measure preserving. For all $(q_v, q_s) \in [0,1]^2$, the size of the group of traders in $[q_v, 1]$ on the side v who are matched with partners in $[q_s, 1]$ is equal to $n_v R_v(q_v, q_s)$. The size of the group of traders in $[q_s, 1]$ on the side s who are matched with partners in $[q_v, 1]$ is equal to $n_s R_s(q_s, q_v)$. Measure preservation requires these sizes on the two sides of the market to be equal to each other as an accounting identity for all (q_v, q_s). Since $\frac{n_s}{n_v} = \theta$, this requirement is:

$$R_v(q_v, q_s) = \theta R_s(q_s, q_v), \quad \forall (q_v, q_s) \in [0,1]^2 . \tag{2.2}$$

Note that R_v and R_s are defined only for $(q_v, q_s) \in [0,1]^2$, and so the above identity does not extend to $(q_v, q_s) \in \mathcal{Q}^2$.

2.1 Perfect positive sorting

The efficient allocation without matching friction serves as the benchmark against which to measure sorting. Matching is pure if every type is matched with a unique type. In perfect positive sorting, this unique partner is the "ideal" type in an efficient allocation without frictions. I indicate this matching pattern with the superscripts "ps". Formally, perfect positive sorting is an increasing and measure-preserving function $p : \mathcal{Q} \to \mathcal{Q}$ such that

$$r_v^{ps}(q_v, q) = \mathbb{1}\left(q < p(q_v)\right) \quad \forall (p_v, q) \in \mathcal{Q}^2, \tag{2.3}$$

where $\mathbb{1}(A) = 1$ if A is true and 0 otherwise. That is, each type q_v is matched only with the ideal type $p(q_v)$ on the side s, provided $p(q_v) \geq 0$, and $p(q_v)$ is increasing in q_v. Measure

[4] Because q_τ is a random variable in $[0,1]$ and the matching partner's type q is a random variable conditional on q_τ, $R_\tau(q_\tau, q)$ is the joint survival function of the two random variables.

preservation is the requirement (2.2). For any $A \in \mathbb{R}$, denote

$$A^{+} \equiv max\left\{A, 0\right\}.$$

Let p^{-1} be the inverse of p. For $(q_v, q) \in [0,1]^2$, substituting r_v^{ps} into (2.1) yields the upper quadrant matching function on the side v in perfect positive sorting as

$$R_v^{ps}(q_v, q) = 1 - max\left\{q_v, p^{-1}(q)\right\} = 1 - q_v - \left(p^{-1}(q) - q_v\right)^{+}. \tag{2.4}$$

Similarly, $R_s^{ps}(q_s, q) = 1 - q_s - (p(q) - q_s)^{+}$ for $(q_s, q) \in [0,1]^2$. Recall the notation $\theta = \frac{n_s}{n_v}$. Define

$$q_{v0} \equiv \left(p^{-1}(0)\right)^{+} = 1 - \theta. \tag{2.5}$$

q_{v0} is the lowest matched type on the side v. Clearly, $q_{v0} > 0$ if $\theta < 1$. Traders lower than q_{v0} are not matched and, by the notation, they are matched with the artificial type $-\varepsilon$. Traders at or above q_{v0} are matched with traders on the side s. Since p is measure-preserving, $R_v^{ps}(q_v, q_s) = \theta R_s^{ps}(q_s, q_v)$ for all $q_s = (p(q_v))^{+}$. If $p(q_v) \geq 0$, setting $q_s = p(q_v)$ in the measure-preserving requirement (2.2) yields $p(q_v) = 1 - \frac{1}{\theta}(1 - q_v)$. If $p(q_v) < 0$, then $p(q_v) = -\varepsilon$. The perfect positive sorting function is:

$$p(q_v) = \begin{cases} 1 - \frac{1}{\theta}(1 - q_v), & \text{if } q_v \geq q_{v0} \\ -\varepsilon, & \text{osherwise.} \end{cases} \tag{2.6}$$

Clearly, $p(1) = 1$ and $p(q_{v0}) = 0$. The inverse of $p(q)$ is $p^{-1}(q_s) = 1 - \theta(1 - q_s)$, which is non-negative for all $q_s \in [0,1]$. Trader q_s on the side s is matched with type $p^{-1}(q_s)$. $p^{-1}(1) = 1$ and $p^{-1}(0) = q_{v0}$.

If the side v is the long side (i.e., $\theta < 1$), then $p(q_v) < q_v$ for all $q_v < 1$. That is, for every trader on the side v except $q_v = 1$, the ideal partner's type is strictly lower than the trader's own type. This is an implication of measure preservation. The measure of traders in $[q_v, 1]$ on the side v is equal to $1 - q_v$. If they are matched with traders in $[p(q_v), 1]$ on the side s, the measure of the matching partners is $\theta[1 - p(q_v)]$. For these measures on the two sides to be equal to each other when $\theta < 1$, it must be the case that $p(q_v) < q_v$ for $q_v < 1$. The flip side of this result is that for every trader q_s on the side s except $q_s = 1$, the ideal partner's type $p^{-1}(q_s)$ is strictly higher than q_s.

3 Measurement of Sorting

3.1 The sorting index

For trader q_v, the cdf of the matching partner's type in perfect positive sorting is $1 - r_v^{ps}(q_v, q) = \mathbb{1}\left(q \geq (p(q_v))^+\right)$. For any q_v with $p(q_v) \in (0,1)$, Figure 1 depicts this cdf by the red line. The black line is the cdf in an arbitrary pattern with frictional matching, $[1 - r_v(q_v, \bullet)]$. Note that the distribution in perfect positive sorting does not first-order stochastically dominate the arbitrary distribution, and so such dominance is not a good measure of sorting.

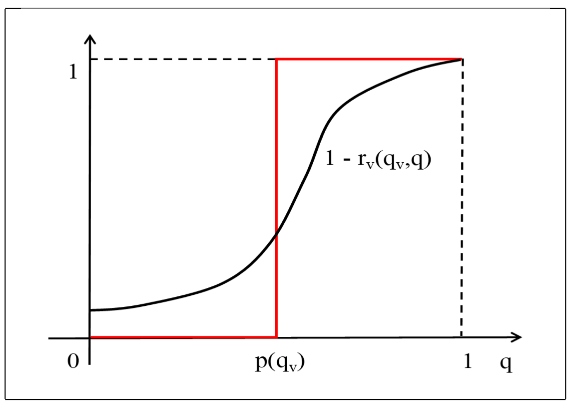

Figure 1: Cumulative distributions of matching partners of trader q_v: Perfect positive sorting (red); frictional matching (black).

The reduction in sorting: Figure 1 suggests that one should measure the strength of sorting relative to perfect positive sorting. Consider q_v such that $p(q_v) \geq 0$. For $q < p(q_v)$, the cdf in frictional matching lies above that in perfect positive sorting. Trader q_v is matched with some traders below $p(q_v)$ with positive probability in frictional matching, which does not occur in perfect positive sorting. For $q > p(q_v)$, the cdf in frictional matching lies below that in perfect positive sorting. Trader q_v is matched with some traders at or above $p(q_v)$ with positive probability in frictional matching, which does not occur in perfect positive sorting. Frictional matching reduces positive sorting for trader q_v by the absolute value of the areas between the two distribution functions. This reduction in positive sorting is:

$$\delta_v(q_v) \equiv \int_0^1 |r_v(q_v, q) - r_v^{ps}(q_v, q)| \, \mathrm{d}q. \tag{3.1}$$

Using the notation $(p(q_v))^+$ to allow for the possibility of $p(q_v) = -\varepsilon$, I break the integral into two parts corresponding to the two areas in Figure 1:

$$\delta_v(q_v) = \int_0^{(p(q_v))^+} [1 - r_v(q_v, q)] \, \mathrm{d}q + \int_{(p(q_v))^+}^1 r_v(q_v, q)\mathrm{d}q. \tag{3.2}$$

In the case $p(q_v) = -\varepsilon$, the first integral in (3.2) is equal to zero.

The quantity $\delta_v(q_v)$ is linked to an efficiency loss that arises from trader q_v being matched with types other than the ideal type. To establish this link, note that (3.1) computes the areas between the two distribution functions in Figure 1 by integrating over q on the horizontal axis. Equivalently, one can compute the areas by integrating on the vertical axis. Formally, Appendix A.1 proves that

$$\delta_v(q_v) = [1 - r_v(q_v, 0)] \, |p(q_v)| + \int_0^1 |q - p(q_v)| \, \mathrm{d}\, [1 - r_v(q_v, q)] + O(\varepsilon), \tag{3.3}$$

where the operator d is with respect to q, and $O(\varepsilon)$ is a quantity in the order of magnitude ε. (3.3) expresses $\delta_v(q_v)$ as the weighted average distance between the matching partner's type and the ideal partner's type, where the weights are the distribution of matches. Specifically, with the probability $[1 - r_v(q_v, 0)]$, trader q_v is unmatched in the examined matching pattern but is matched with $p(q_v)$ in perfect positive sorting. The absolute difference between the two is $|p(q_v)| + O(\varepsilon)$. If trader q_v is matched with $q \geq 0$, the absolute difference in the matching partner's type between the two matching patterns is $|q - p(q_v)|$. Because any deviation of the matching partner's type from the ideal type reduces efficiency, $\delta_v(q_v)$ captures this efficiency loss associated with trader q_v.

By aggregating $\delta_v(q_v)$ over q_v, I obtain a measure of how frictional matching reduces positive sorting on the side v relative to perfect positive sorting. Because q_v is uniformly distributed in $[0, 1]$, this measure is:

$$\Delta_v \equiv \int_0^1 \delta_v(q_v)\mathrm{d}q_v. \tag{3.4}$$

For the side s, define $\delta_s(q_s)$ and Δ_s analogously to (3.1) and (3.4), with the subscript s replacing v, and the function $p^{-1}(.)$ replacing $p(.)$. The overall reduction in sorting in the market is the weighted average of Δ_v and Δ_s, where the weights are the measure of traders on each side. This overall reduction in sorting is:

$$\Delta = \frac{n_v \Delta_v + n_s \Delta_s}{n_v + n_s} = \frac{\Delta_v + \theta \Delta_s}{1 + \theta}. \tag{3.5}$$

By construction, $\Delta = 0$ in perfect positive sorting. Another reference of the matching pattern is uniform matching, where every trader's matching partner is uniformly drawn from the other side. Such a matching pattern has neither positive nor negative sorting, and will be examined in the next subsection. It makes sense to normalize the sorting index so that the index is equal to zero in uniform matching. The following definition does so.

Definition 1. *Denote Δ^e as the reduction in sorting in uniform matching relative to perfect positive sorting. In any matching pattern with Δ, the sorting index is $\sigma(\Delta) \equiv 1 - \frac{\Delta}{\Delta^e}$. Sorting is positive if $\sigma > 0$.*

The sorting index is inversely related to the weighted average distance between the matching partner's type and the ideal type. Note the two important features of this distance. First, the sorting index compares the matching partner's type with the ideal partner's type, $p(q_v)$, not with a trader's own type q_v. The reference to the ideal type is important when the market is imbalanced ($\theta < 1$), because $p(q_v) < q_v$ for all $q_v < 1$ in this case. Second, the sorting index takes into account not only matched types, but also unmatched ones. In particular, the first term of $\delta_v(q_v)$ in (3.3) captures explicitly the efficiency loss from matching failures. If a matching pattern fails to generate a match for trader q_v ($\geq q_{v0}$), it deviates from the efficient allocation and tends to reduce sorting.

The sorting index has the following properties:

(i) Invariance to the scale of the original type y. Because the quantiles of the original type remain unchanged after monotone transformations of the original type, the sorting index is invariant to such transformations of the original type.

(ii) Transitivity. For any three matching patterns with $(\Delta_1, \Delta_2, \Delta_3)$, if $\sigma(\Delta_1) \geq \sigma(\Delta_2)$ and

$\sigma(\Delta_2) \geq \sigma(\Delta_3)$, then $\sigma(\Delta_1) \geq \sigma(\Delta_3)$. If either of the first two inequalities is strict, the last inequality is also strict.

(iii) Linearity and convexity. The sorting index is a linear function of Δ. Take any two sorting patterns whose sorting indices are $\sigma(\Delta_1)$ and $\sigma(\Delta_2)$, and construct $\Delta_3 = b\Delta_1 + (1-b)\,\Delta_2$, where $b \in [0,1]$. Then, $\sigma(\Delta_3) = b\sigma(\Delta_1) + (1-b)\,\sigma(\Delta_2)$.

(iv) Dependence on the scale of the matching probabilities. Chiappori et al. (2024) propose that a measure of sorting should be invariant to the scale of the measures of matches. The sorting index σ violates this axiom of scale invariance. Consider any matching pattern A, $\left\{r_\tau^A(q_\tau, q) : (q_\tau, q) \in \mathcal{Q}^2\right\}_{\tau=v,s}$. Suppose that the matching probabilities in another pattern B are lower than in pattern A by a factor $b \in (0,1)$. That is, $r_\tau^B(q_\tau, q) = br_\tau^A(q_\tau, q)$ for all $q_v \in [q_{v0}, 1]$ and $q \in [0,1]$, and so pattern B is a scaled version of pattern A. The increase in the failure probability to match from pattern A to pattern B increases Δ and, hence, reduces the sorting index σ. In general, for any sorting index to capture the negative effect of matching failures on sorting, the index must violate scale invariance.

3.2 Block matching and the bounds on the sorting index

To illustrate the index σ and to calculate Δ^e, I examine positive block matching that encompasses perfect positive sorting and uniform matching as special cases.

Positive block matching: As in perfect positive matching, traders $q_v < q_{v0}$ are not matched. Other traders are divided into k equal intervals (blocks) indexed by $i \in \{1, 2, ..., k\}$. The ith block consists of $q_s \in [\frac{i-1}{k}, \frac{i}{k})$ and $q_v \in [q_{v0} + \frac{i-1}{k}\theta, q_{v0} + \frac{i}{k}\theta)$. The last block $i = k$ also includes $q_s = 1$ and $q_v = 1$. The traders match exclusively inside each block, and a trader's matching partner is uniformly drawn from the other side in the block. This matching pattern emerges as the equilibrium outcome in some matching markets. Smith (2006) calls the pattern block segregation, and Burdett and Coles (1997) call the pattern marriage classes.[5] In the limit $k \to \infty$, positive block matching becomes perfect positive sorting. At the other end, $k = 1$, matching is uniform.

[5]The probability of failing to match is positive in Smith (2006) and Burdett and Coles (1997). Section 3.4 will incorporate the failure probability into positive block matching.

Figure 2 depicts positive block matching with $\theta = 1$, in which case $q_{v0} = 0$ and $p(q) = q$. The left panel depicts the blocks of (q_s, q_v). In the right panel, the blue line depicts the cdf of matching partners for a trader q_v in the ith block, and the red vertical line is the ideal partner's type in perfect positive sorting. The two shaded triangles are the reductions in positive sorting for a trader in the ith block relative to perfect positive sorting. For uniform matching, the entire square $[0, 1]^2$ in the left panel is one block, and the cdf in the right panel is the 45-degree line (not drawn). Appendix A.2 proves the following lemma:

Lemma 1. *In positive block matching with k blocks,*

$$\Delta = \frac{\theta}{3k}. \tag{3.6}$$

When $k = 1$, the value of Δ is $\Delta^e \equiv \frac{\theta}{3}$. For any $k \geq 1$, the sorting index is $\sigma = 1 - \frac{1}{k}$.

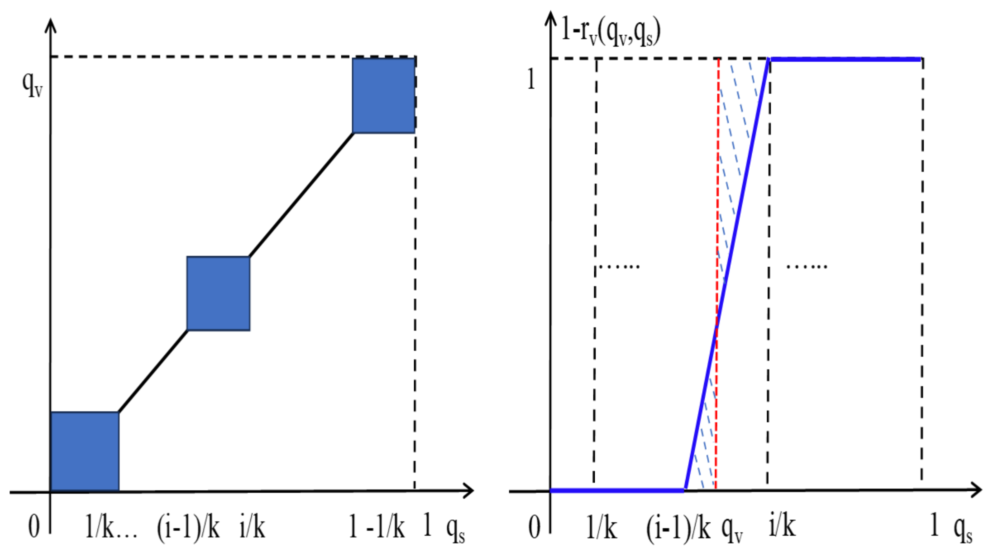

Figure 2: Positive block matching

Uniform matching yields $\sigma(\Delta^e) = 0$. As the number of blocks increases, positive sorting increases because the range of a trader's matching partner's types narrows down closer to the trader's own type. As k increases from 1 to ∞, the sorting index increases from 0 to 1.

By construction, the upper bound on the sorting index is $\sigma(0) = 1$, which is the value in perfect positive sorting. Perfect negative sorting provides the lower bound on σ. Negative block sorting consists of blocks on the negative diagonal in the left panel of Figure 1 (not depicted). The limit $k \to \infty$ of negative block sorting is perfect negative sorting. Appendix A.3 characterizes perfect negative sorting and proves the following lemma:

Lemma 2. *The sorting index in perfect negative sorting is:*

$$\sigma = \begin{cases} -\frac{1}{2} - \frac{3(1-\theta)\left(4\theta-\theta^2-1\right)}{2\theta^2(1+\theta)}, & \text{if } \frac{1}{2} \leq \theta \leq 1 \\ -\frac{1}{2} - \frac{9(1-\theta)}{2(1+\theta)}, & \text{if } \theta < \frac{1}{2}. \end{cases}$$

If $\theta = 1$, then $\sigma = -\frac{1}{2}$ in perfect negative sorting.

In a balanced market, σ is bounded below by $-\frac{1}{2}$. If the market is imbalanced, σ can be smaller than $-\frac{1}{2}$. For example, when $\theta = \frac{1}{2}$, perfect negative sorting yields $\sigma = -2$.[6]

3.3 Relate σ to upper quadrant sorting

For any arbitrary matching pattern, the following proposition links σ to the upper quadrant matching function (see Appendix B.1 for a proof):

Proposition 1. *In any matching pattern, the sorting index is:*

$$\begin{aligned} \sigma = \; & \frac{6}{\theta^2} \int_{q_{v0}}^1 R_v(q, p(q))\, dq + \frac{5\theta-7}{2(1+\theta)} \\ & - \frac{3}{\theta(1+\theta)} \left[\theta \int_0^1 R_s(q, 0)\, dq + \int_0^1 R_v(q, 0)\, dq - 2 \int_0^{q_{v0}} R_v(q, 0)\, dq \right]. \end{aligned} \tag{3.7}$$

The sorting index σ in (3.7) has three groups of terms. The first term is the integral of matches in upper quadrants $(q, p(q))$. This term captures how well the traders in the upper quadrants are matched with partners at or above their ideal types. It involves upper quadrant matching on both sides of the market, although only R_v appears in the term. A similar integral for the side s is expressed in terms of R_v with the identity (2.2) and combined with the integral for the side v. The second term of σ reflects the imbalance of the two sides

[6] The index $\sigma(\Delta)$ satisfies $\sigma(0) = 1$ and $\sigma(\Delta^e) = 0$. To have the additional property $\sigma(\Delta^{ns}) = -1$, where Δ^{ns} is the value of Δ in perfect negative sorting, $\sigma(\Delta)$ must be nonlinear. An example is $\sigma(\Delta) = \frac{\Delta^e - \Delta}{\Delta^e - b\Delta}$, where $b = \frac{2\Delta^e}{\Delta^{ns}} - 1$.

of the market. The third term involves the integrals of $R_s(q,0)$ and $R_v(q,0)$ that capture matching probabilities. These integrals depend on whether traders are matched with anyone but not on whom they are matched with. The three components of σ are not independent. The imbalance of the market affects all three terms through θ, and match failures affect the first and the third terms.

To isolate upper quadrant sorting in (3.7), I examine the function R_v in the upper quadrant $(q, p(q))$. For all $q \geq q_{v0}$, $p(q) \geq 0$, and so $R_v(q, p(q))$ is well-defined. For $q < q_{v0}$, $R_v(q, p(q))$ is not defined because $p(q) < 0$. In this case, the upper quadrant $(q, p(q))$ includes all traders at or above q on the side v, including those who are unmatched. The measure of such traders is $1 - q$. This motivates the following definition:

$$\bar{R}_v(q) \equiv \begin{cases} R_v(q, p(q)), & \text{if } q_{v0} \leq q \leq 1 \\ 1 - q, & \text{if } 0 \leq q < q_{v0}. \end{cases} \tag{3.8}$$

Define $\bar{R}_s$ similarly with s replacing v, 0 replacing q_{v0}, and $p^{-1}(q)$ replacing $p(q)$. The weighted average of $\bar{R}_v$ and $\bar{R}_s$ over all $q \in [0,1]$ on the two sides of the market is:

$$U \equiv \frac{1}{1+\theta} \int_0^1 \left[\bar{R}_v(q) + \theta \bar{R}_s(q)\right] \mathrm{d}q. \tag{3.9}$$

Let $\bar{R}_v^{ps}$ and U^{ps} denote the corresponding items in perfect positive sorting. Because perfect positive sorting matches every type q with the ideal type with probability one, it yields an upper bound on $\bar{R}_v$:

$$\bar{R}_v(q) \leq \bar{R}_v^{ps}(q) = 1 - q \text{ for all } q \in [0,1], \tag{3.10}$$

where the equality comes from (2.4). I normalize U by subtracting the corresponding function for uniform matching, U^e, so that the resulting index is zero in uniform matching. This leads to the following definition of the upper quadrant sorting index:

$$\sigma_U \equiv \frac{U - U^e}{U^{ps} - U^e} \tag{3.11}$$

Appendix B.2 calculates U^e and U, and proves the following proposition:

Proposition 2. *In any matching pattern, the upper quadrant sorting index is:*

$$\sigma_U = \frac{6}{\theta^2} \int_{q_{v0}}^1 \bar{R}_v(q)\, dq - 2. \tag{3.12}$$

The first term in the sorting index in (3.7) is equal to $\sigma_U + 2$. *Moreover,* $\sigma_U \in [-2, 1]$.

The index σ_U captures the essential component of σ that reflects who matches with whom. By (3.12), if one matching pattern (A) has a larger function $\bar{R}_v(.)$ than another matching pattern (B), then (3.12) implies that pattern A generates a larger index σ_U than pattern B for any given θ. The reverse is not true. There are examples of positive block matching in which one matching pattern generates a larger σ_U than another pattern, but the two patterns are not ranked by $\bar{R}_v$. Corollary 2 in section 5.1 will provide such an example. It will also link σ_U to positive quadrant dominance in Anderson and Smith (2024).

3.4 Uniform matching failures

As σ, the index σ_U reflects matching failures and uses the ideal matching type as the reference, because σ_U is constructed from the function $\bar{R}_v$. To compare the two indices in detail, I restrict the form of matching failures. A matching pattern has uniform matching failures if it satisfies (i)-(iii) below:

$$\left.\begin{array}{l}\text{(i) } r_v(q_v,0)=0 \text{ for all } q_v<q_{v0};\\ \text{(ii) } r_v(q_v,0)=a\in(0,1] \text{ for all } q_v\geq q_{v0};\\ \text{(iii) } r_s(q_s,0)=a \text{ for all } q_s\in[0,1].\end{array}\right\} \tag{3.13}$$

The probability of failing to match is $(1-a)$. Condition (i) requires that the traders on the side v with $q_v<q_{v0}$ should not be matched, as in perfect positive sorting. Conditions (ii) and (iii) require that all traders other than those with $q_v<q_{v0}$ should have the same matching probability.[7] Note that (i) and (iii) imply that $r_s(q_s,q_v)=a$ for all $q_v<q_{v0}$.

This class contains a large set of matching patterns, because (3.13) imposes no restriction on the matching pattern between $q_v>q_{v0}$ and $q_s>0$. All positive block matching patterns belong in this class, with the additional feature $a=1$. If $\theta=1$, in addition, the class also contains all negative block matching patterns. Any arbitrary mixture of perfect positive sorting and perfect negative sorting belongs in the class.

The set of matching patterns with uniform matching failures is closed under scaling of the allocation. To see this, let matching pattern A have a uniform probability $\left(1-a^A\right)$

[7]Given (i) in (3.13), if the matching failure probability is constant on both sides for all $q_v\geq q_{v0}$, this constant must be the same on the two sides in order to satisfy measure preservation. To see this, suppose that the matching probability is a_v on the side v and a_s on the side s. Then $R_v(q,0)=a_v\,(1-q)$ for $q\geq q_{v0}$ and $R_s(q,q_{v0})=a_s\,(1-q)$ for $q\geq 0$. The identity $R_v(q_{v0},0)=\theta R_s(0,q_{v0})$ requires $a_v(1-q_{v0})=\theta a_s$. Because $q_{v0}=1-\theta$, this requires $a_v=a_s$.

of matching failures, with the allocation $\left\{r_\tau^A(q_\tau, q) : (q_\tau, q) \in \mathcal{Q}^2\right\}_{\tau=v,s}$. Let $b \in [0,1]$ be a constant, and let pattern B have $r_\tau^B(q_\tau, q) = b r_\tau^A(q_\tau, q)$ for all $q_v \in [q_{v0}, 1]$ and $q \in [0,1]$. Then pattern B has a uniform probability $\left(1 - a^B\right)$ of matching failures, where $a^B = b a^A$.

One can scale block matching patterns by a constant $a \in [0,1)$ to incorporate uniform matching failures. That is, each trader in each block is matched with the probability a and the matching partner is uniformly drawn from the other side in the block. In particular, one can scale the pattern of perfect positive sorting as $r_\tau(q_\tau, q) = a r_\tau^{ps}(q_\tau, q)$ for all $q_v \in [q_{v0}, 1]$ and $q \in [0,1]$, where r_τ^{ps} is given by (2.3). In this modified pattern, each type q_v $(\geq q_{v0})$ is matched only with the type $p(q_v)$ and with the probability a. It can be verified that this matching pattern yields $\sigma_U = 3a - 2$, which is the upper bound on σ_U in the class with the uniform probability a of matching failures.

Under (3.13), the functions $R_v(q,0)$ and $R_s(q,0)$ are:

$$\left.\begin{array}{l} \text{for } q \geq q_{v0}: \ R_v(q,0) = \int_q^1 r_v(q_v, 0) \mathrm{d}q_v = (1-q)\, a; \\ \text{for } q < q_{v0}: \ R_v(q,0) = R_v(q_{v0}, 0) = \theta a; \\ \text{for all } q \in [0,1]: \ R_s(q,0) = (1-q)\, a. \end{array}\right\} \tag{3.14}$$

Propositions 1 and 2 imply the following corollary (the proof is omitted):

Corollary 1. *With uniform matching failures, $\sigma_U \leq 3a - 2$ and*

$$\sigma - \sigma_U = \frac{3\,(3\theta - 1)}{2\,(1+\theta)}\,(1-a)\,. \tag{3.15}$$

Index two patterns with uniform matching failures by A and B. If $r_\tau^B(q_\tau, q) = b r_\tau^A(q_\tau, q)$ for all $q_v \in [q_{v0}, 1]$ and $q \in [0,1]$, with $b \in [0,1]$, then $a^B = b a^A$ and $\sigma_U^B = b\left(\sigma_U^A + 2\right) - 2$.

If $a = 1$, there is no matching failure for $q_v \geq q_{v0}$, in which case $\sigma = \sigma_U$. Even if $a < 1$, a change in the matching pattern in the class, other than a change in (θ, a), changes the two indices by the same amount. In this class, a matching pattern generates stronger sorting if and only if it has a larger upper quadrant sorting index. To appreciate this result, suppose $\theta = 1$ so that the class contains all positive and all negative block matching patterns, possibly with match failures. As the matching pattern changes from perfect positive sorting to perfect negative sorting, the sorting index σ decreases from 1 to $-\frac{1}{2}$. The upper quadrant sorting index σ_U captures all these changes in σ. Moreover, a uniform increase in the failure probabilities to match reduces the index σ_U.

4 Compare the Sorting Indices with the Correlation Coefficient

4.1 Correlation of matched types

The most common measure of sorting in the empirical literature is the correlation coefficient of matched types. To compare this alternative measure with (σ, σ_U), I calculate the statistics of matched types and relate them to the upper quadrant matching function. Denote the joint cdf of matched types as $\mathcal{M}(q_v, q_s)$, and the marginal distribution of matched types on the side τ as $H_\tau(q_\tau)$. The joint distribution of matches is:

$$\begin{aligned}&\mathcal{M}(q_v, q_s) = Pr(\hat{q}_v \leq q_v, \hat{q}_s \leq q_s|\text{matched}) \\ &= \int_0^{q_s} Pr(\hat{q}_v \leq q_v|\text{matched with } \hat{q}_s = q')\mathrm{d}H_s(q').\end{aligned} \tag{4.1}$$

To compute the elements in (4.1), let $q \in [0, 1]$ be an arbitrary type on the side s. The matching probability for trader q is $r_s(q, 0)$. Conditional on being matched, trader q is matched with traders at or above q_v with the probability $\frac{r_s(q,q_v)}{r_s(q,0)}$. Thus, conditional on being matched with trader q, traders on the side v are distributed according to:

$$Pr(\hat{q}_v \leq q_v|\text{matched with } q) = 1 - \frac{r_s(q, q_v)}{r_s(q, 0)}. \tag{4.2}$$

Similarly, the measure of matches on the side s is $\int_0^1 r_s(\hat{q}, 0)\mathrm{d}\hat{q}$. For traders in $[0, q]$ on the side s, the measure of matches is $\int_0^q r_s(\hat{q}, 0)\mathrm{d}\hat{q}$. Thus, the (marginal) distribution of matched types on the side s is:

$$H_s(q) \equiv Pr(\hat{q} \leq q|\text{matched}) = \frac{\int_0^q r_s(\hat{q}, 0)\mathrm{d}\hat{q}}{\int_0^1 r_s(\hat{q}, 0)\mathrm{d}\hat{q}} = 1 - \frac{R_s(q, 0)}{R_s(0, 0)}. \tag{4.3}$$

After changing s into v in (4.3), I obtain $H_v(q)$. Appendix C.1 substitutes (4.2) and (4.3) into (4.1) to obtain $\mathcal{M}(q_v, q_s)$. The appendix also proves the following proposition to link the statistics of the matched types to the upper quadrant matching function:

Proposition 3. *The joint distribution of matched types is:*

$$\mathcal{M}(q_v, q_s) = 1 - \frac{1}{R_v(0,0)} \left[R_v(0, q_s) + R_v(q_v, 0) - R_v(q_v, q_s) \right]. \tag{4.4}$$

The mean and the variance of matched types on the side v are, respectively:

$$\mathbb{E}q_v = \frac{1}{R_v(0,0)} \int_0^1 R_v(q, 0)\, dq \tag{4.5}$$

$$Var(q_v) = \frac{2}{R_v(0,0)} \int_0^1 q R_v(q, 0)\, dq - (\mathbb{E}q_v)^2. \tag{4.6}$$

On the side s, the mean and the variance obey the above formulas after changing v into s. The covariance of matched types is:

$$Cov(q_v, q_s) = \frac{1}{R_v(0,0)} \int_0^1 \int_0^1 R_v(q_v, q_s)\, dq_s\, dq_v - \mathbb{E}q_v \mathbb{E}q_s. \tag{4.7}$$

The correlation coefficient of matched types, denoted as ρ, follows the standard formula.

Equation (4.4) relates the distribution of matches in the lower quadrant, $\mathcal{M}(q_v, q_s)$, to the matching probability in the upper quadrant, $R_v(q_v, q_s)$, and the marginal distributions of matches. To understand this relationship, let us divide matched pairs in the square $[0,1]^2$ into four quadrants by q_v and q_s. The joint distribution $\mathcal{M}(q_v, q_s)$ is the probability that matches lie in the lower quadrant, $[0, q_v] \times [0, q_s]$. Matches lie in the upper quadrant, $[q_v, 1] \times [q_s, 1]$, with the probability $\frac{R_v(q_v,q_s)}{R_v(0,0)}$. The probability that matches lie in the quadrant, $[q_v, 1] \times [0, q_s]$, is the marginal distribution of matches on the side s, $H_s(q_s)$, minus $\mathcal{M}(q_v, q_s)$. Similarly, the probability that matches lie in the quadrant, $[0, q_v] \times [q_s, 1]$, is the marginal distribution, $H_v(q_v)$, minus $\mathcal{M}(q_v, q_s)$. Summing up the matching probabilities in the four quadrants to one yields (4.4).

The contrasts of ρ with (σ, σ_U): The correlation coefficient ρ considers only matched traders and uses a trader's own type as the reference to compare with the matching partner's type. In contrast, the sorting indices (σ, σ_U) consider unmatched types as well as matched types, and measure sorting relative to the ideal matching type. Because of these differences, the correlation coefficient mis-measures sorting in several ways. First, the correlation coefficient is invariant to scaling of the matching probabilities, and hence does not capture well how match failures affect sorting. The correlation coefficient is equal to one even if the only matches are $q_v = 0$ with $q_s = 0$ and $q_v = 1$ with $q_s = 1$. Second, a high correlation coefficient does not necessarily indicate a desirable matching pattern or reflect changes in the matching pattern. A parallel shift in all traders' matching partner on one side should affect sorting, but it does not affect ρ. Third, the correlation coefficient does not reflect well the imbalance of the market.

The class with uniform matching failures offers an environment to compare ρ with (σ, σ_U). In this class, σ is tightly related to σ_U, and the upper bound on σ_U is $3a - 2$ (see Corollary 1). The following proposition shows that, when σ_U exceeds a cutoff, ρ tends to over-estimate positive sorting relative to σ_U (see Appendix C.2 for a proof):

Proposition 4. *Consider the matching patterns in (3.13) that satisfy $\sigma_U \geq \sigma_0$ for $\sigma_0 \in (-2, 3a-2]$. Assume that there does not exist a constant a such that $r_\tau(q_\tau, q) = a r_\tau^{ps}(q_\tau, q)$ almost surely for $(q_\tau, q) \in [0,1]^2$ and for both $\tau = v$ and $\tau = s$. A sufficient condition for $\rho > \sigma_U$ is:*

$$\sigma_0 \geq -2 + \frac{3a}{16}\left\{4 + 3a + \left[3\left(3a^2 + 8a + 16\right)\right]^{\frac{1}{2}}\right\}. \tag{4.8}$$

The bound on σ_0 in (4.8) lies in $(-2, 3a-2]$ and is increasing in a.

The lower bound on σ_0 in (4.8) and the upper bound on σ_U (i.e., $3a-2$) both fall sharply below 1 as the matching failure probability $(1-a)$ increases above 0. At $a = 0.8$, the bound is equal to 0.241, while the upper bound on σ_U is 0.4. In this case, ρ exceeds σ_U for all matching patterns in the class if $\sigma_U \in [0.241, 0.4]$. Note that the condition (4.8) is a sufficient, but not necessary, condition for ρ to exceed σ_U. There are cases in which ρ exceeds σ_U for all $a \in [0,1]$. For example, when positive block matching is modified to incorporate uniform matching failures, the correlation coefficient is $\rho = 1 - \frac{1}{k^2}$, the upper quadrant sorting index is $\sigma_U = \left(3 - \frac{1}{k}\right)a - 2$, where k is the number of blocks. In this example, $\rho > \sigma_U$ for all $a \in [0,1]$ and all $1 < k < \infty$.

The inequality $\rho > \sigma_U$ can be reversed if sorting is significantly negative. For example, when $\theta = a = 1$, perfectly negative sorting yields $\sigma_U = \sigma = -\frac{1}{2}$ and $\rho = -1$.

The ensuing subsections give additional examples to illustrate the contrasts between ρ and (σ, σ_U). All of them lie outside the class of uniform matching failures.

4.2 Match failures and the imbalance of the market

A realistic market has match failures and imbalance between the two sides. These failures and imbalance vary over the business cycle and during structural changes of the economy. The sorting measures (σ, σ_U) capture how these features affect sorting, but ρ does not. To illustrate, consider a matching pattern in which each trader q on the side v is matched exclusively with trader q on the side s with a constant probability $a\theta < 1$, where $0 < a \leq 1$. Each trader q on the side s is matched only with trader q on the side v with a constant probability, which is equal to a by measure preservation.[8] If $\theta < 1$, this matching pattern does not belong in uniform matching failures in (3.13). It violates condition (i) in (3.13) and has different matching failure probabilities between the two sides.

Similar to perfect positive sorting, this matching pattern is pure. That is, each trader matches with a unique type that is increasing in the trader's own type. The matching pattern is not perfect in two aspects. First, the matching partner of trader q is q on the other side, instead of $p(q)$ in perfect positive sorting. If the market is imbalanced (i.e., $\theta < 1$), then $q > p(q)$. In particular, traders in $[0, q_{v0})$ on the side v are matched with traders in $[0, q_{v0})$ on the side s in this matching pattern, but they should not be matched in perfect positive sorting. Second, every type on the side v can experience match failure. In contrast, all types in $[q_{v0}, 1]$ are matched with certainty in perfect positive sorting. A decrease in a increases match failures even if the market is balanced. A decrease in θ increases both match failures and the imbalance in the market. If $\theta = 1$, then $q_{v0} = 0$, in which case the matching pattern belongs in the class (3.13). If $a = 1$ and $\theta = 1$, the matching pattern coincides with perfect positive sorting.

[8] Let a_s be the matching probability of each trader on the side s. For any $(q_v, q_s) \in [0,1]^2$, the measure of traders in $[q_v, 1]$ on the side v who are matched with traders in $[q_s, 1]$ is $a\theta n_v (1 - q_v)(1 - q_s)$. Calculating the matches from the side s, this measure is equal to $a_s n_s (1 - q_s)(1 - q_v)$. Measure preservation requires the two calculations to yield the same result. Thus, $a_s = \frac{a\theta n_v}{n_s} = a$.

For any given $q_v \in [0,1]$, the function r_v is:

$$r_v(q_v, q) = \begin{cases} a\theta, & \text{if } q < q_v \\ 0, & \text{if } q \geq q_v. \end{cases}$$

The upper quadrant matching function on the side v is:

$$R_v(q_v, q) = a\theta \left(1 - max\left\{q_v, q\right\}\right).$$

For $q < q_{v0}$, $p(q) = -\varepsilon$. For $q \geq q_{v0}$, $p(q) = 1 - \frac{1}{\theta}(1-q) \leq q$. (3.7) and (3.12) yield:

$$\sigma = \frac{(6a+5)\theta - 7}{2(1+\theta)}, \quad \sigma_U = 3a\theta - 2.$$

The correlation coefficient of matched types is $\rho = 1$. If either $\theta < 1$ or $a < 1$, then $\rho > max\left\{\sigma, \sigma_U\right\}$.

The result $\rho = 1$ is not surprising. Because each type q is matched only with type q on the other side, matched types are perfectly correlated. Since unmatched types are irrelevant for ρ, then $\rho = 1$ holds independently of the extent of match failures captured by $1-a$. Also, ρ is independent of how imbalanced the market is. Even if θ is close to 0, the correlation coefficient remains at 1.

The effect of match failures: For any given θ, the sorting indices σ and σ_U are decreasing in match failures represented by $1-a$. This is intuitive. A match failure is a match with the artificial type, $-\varepsilon$, which is lower than the ideal type given by perfect positive sorting. Match failures can cause large differences between ρ and the sorting indices. When the market is balanced, $\sigma = \frac{3a-1}{2}$ and $\sigma_U = 3a-2$. As $(1-a)$ increases from 0 to 1, σ falls from 1 to $-\frac{1}{2}$, and σ_U falls from 1 to -2. Because ρ remains at 1, the over-estimation of positive sorting by ρ relative to σ increases from 0 to $\frac{3}{2}$.

The effect of market imbalance: For any given a, the sorting indices σ and σ_U decrease in the imbalance in the market represented by $1-\theta$, for several reasons. First, θ changes the matching probabilities. As θ decreases, the interval $[0, q_{v0})$ widens. More types at the lower end on the side v should be unmatched in perfect positive sorting, but they are matched with positive probabilities in the examined matching pattern. Second, as θ decreases, the matching probability for each trader in the interval $[q_{v0}, 1]$ decreases, but this probability should remain at 1 in perfect positive sorting. Third, as θ decreases, the matching partner's type decreases for any $q \geq q_{v0}$ in perfect positive sorting, but it remains unchanged in the matching pattern in the current example.

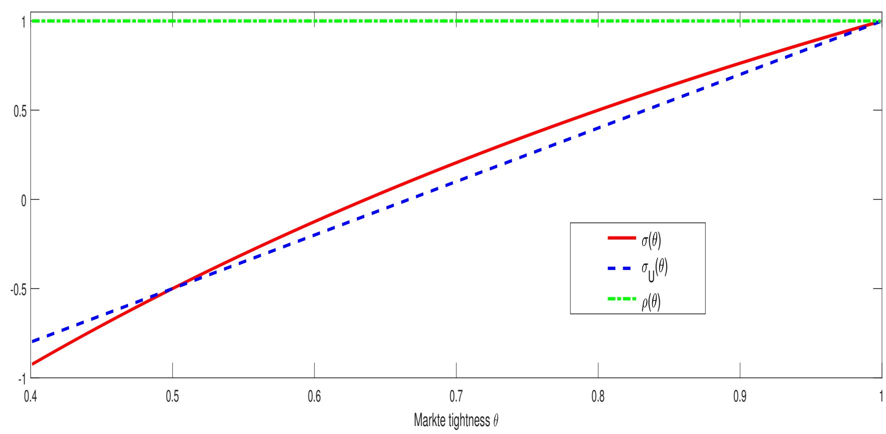

Figure 3: σ, σ_U and ρ as functions of the market tightness θ

Figure 3 depicts (σ, σ_U, ρ) as functions of θ with $a = 1$. When θ falls, the sorting index σ decreases significantly, ρ remains at 1, and so the difference between the two widens. For example, when θ falls from 1 to 0.8, σ decreases from 1 to 0.5. Note that $\theta = 0.8$ is a realistic value of the labor market tightness (see Pissarides, 2009). The contrasts between σ ad ρ become starker when θ falls further. At $\theta = 0.5$, sorting becomes significantly negative with $\sigma = -0.5$, but ρ remains at $\rho = 1$. In contrast to the large difference of ρ from σ, the upper quadrant sorting index σ_U is close to σ. If either $\theta = 1$ or $\theta = \frac{1}{2}$, then $\sigma_U = \sigma$. For $\theta \in \left(\frac{1}{2}, 1\right)$, $\sigma > \sigma_U$, but the maximum difference between σ and σ_U in this interval of θ is less than 0.11.

4.3 Shifts and Mixes in Matches

In a typical market, the matching partners are mixed. To illustrate that ρ can still over-estimate sorting with mixed matches, consider a balanced market ($\theta = 1$) so that the ideal matching partner for each trader q is $p(q) = q$. In this example, let r_v be:

$$r_v(q_v, q) = \begin{cases} a, & \text{if } 0 \leq q < (1-c)\, q_v \\ ae^{b[(1-c)q_v - q]}, & \text{if } q \in [(1-c)\, q_v, q_v + c\,(1-q_v)) \\ 0, & \text{otherwise.} \end{cases}$$

where $b \geq 0$, $c \in \left[0, \frac{1}{2}\right]$, and $0 < a \leq 1 - c$. For trader q_v, the matching partners range from a distance cq_v to the left of the trader to a distance $c\,(1-q_v)$ to the right. The matching density for trader q_v is positive only if the partner q lies in this interval. The width of this matching interval is c. The mixing is asymmetric on the two sides of the trader. In the matching interval, the matching density is decreasing in the partner's type q at the rate b. Matching is pure if $b = 0$ or $b \to \infty$. If $b = 0$, the only matching partner for trader q_v is $q_v + c\,(1-q_v)$, who is located to the right of trader q_v. In the limit $b \to \infty$, the only matching partner for trader q_v is $(1-c)\, q_v$, who is located to the left of trader q_v.

The function $r_v(q_v, q)$ satisfies the requirements that it is decreasing in q, $r_v(q_v, 1) = 0$, and $r_v(q_v, q) \in [0, 1]$ for all $(q_v, q) \in [0, 1]^2$. Because $r_v(q_v, 0) = a$ for all q_v, all traders on the side v succeed in trade with the same probability. However, the matching pattern does not belong in the class of uniform matching failures in (3.13), because the matching failure probability on the side s varies with the trader's type. The assumption $0 < a \leq 1 - c$ is needed to ensure $r_s(q, q_v) \leq 1$ for all (q, q_v).

To obtain the upper quadrant matching function on the side v, I divide the different cases in terms of q_v instead of q. Denote

$$B1(q) = max\left\{\frac{q-c}{1-c}, 0\right\}, \quad B2(q) = min\left\{\frac{q}{1-c}, 1\right\}.$$

Note that $B1(q) \leq B2(q)$ for all $q \in [0, 1]$. The function R_v is:

$$R_v(q_v, q) = \begin{cases} a\,(1-q_v)\,, & \text{if } B2(q) \leq q_v \leq 1 \\ a\frac{e^{b[(1-c)B2(q)-q]} - e^{b[(1-c)max\{q_v, B1(q)\} - q]}}{b(1-c)} + a\,[1 - B2(q)]\,, & \text{if } 0 \leq q_v < B2(q) \end{cases}$$

The sorting indices (σ_U, σ) and the correlation coefficient ρ are:

$$\sigma_U = 3a\,(1-c) - 2 + \frac{6a}{b}\left(1 - \frac{1-e^{-bc}}{bc}\right),$$

$$\sigma = \sigma_U + \frac{3}{2}\,(1-a) + \frac{3ac}{4} - \frac{3a}{2b}\left(1 - e^{-bc}\right),$$

$$\rho = \left[1 + \frac{12}{b^2 (1-c)^2} \left(1 - e^{-2bc} - 2bce^{-bc} \right) \right]^{-1/2}.$$

The effect of a shift in the matching partner: In the case $b \to \infty$, each trader q_v is matched only with $q = (1-c)\, q_v$. An increase in c shifts this unique matching type further to the left. Figure 4 depicts (σ, σ_U, ρ) in this case as functions of $c \in [0, 0.2]$. The parameter a is set at $a = 0.8$ to satisfy $a \leq 1 - c$ for all values of c in the interval. At $c = 0$, the matching partner's type is the ideal type. In this case, the indices σ and σ_U are less than 1 because of matching failures ($a < 1$). σ_U is significantly smaller than σ because the matching pattern is asymmetric between the two sides of the market and the matching probabilities are high between each trader and the partners close to and lower than the trader's own type. These matching probabilities increase σ through the last term in (3.7) but are ignored by σ_U. Despite this difference in the magnitude between σ_U and σ, both indices respond to c similarly. As c increases to shift the matching partner's type further to the left for each trader on the side v, the matching type deviates further from the ideal type. The sorting indices σ and σ_U fall. In contrast, the correlation coefficient remains at $\rho = 1$, because each trader is matched with only one type that increases in the trader's own type.

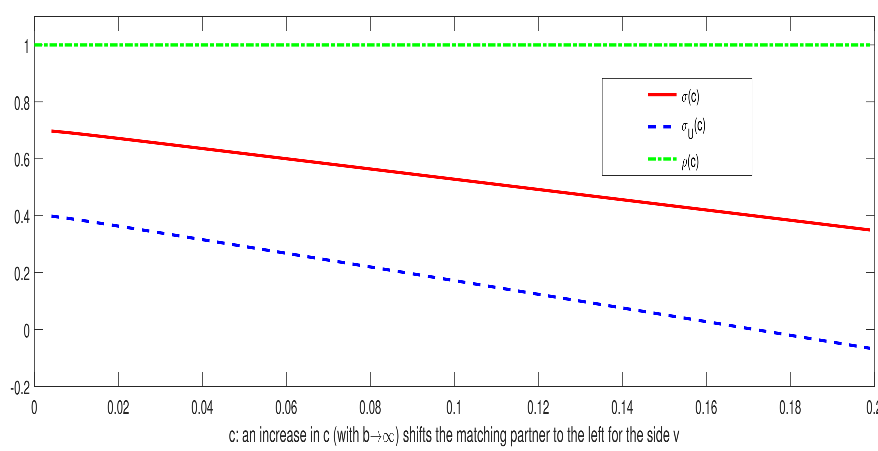

Figure 4: The effects on (σ, σ_U, ρ) of a left shift in the matching partner for the side v

The effect of mixed matches: For any $b \in (0, \infty)$, each trader q_v receives mixed matches. Because an increase in c widens the matching interval for each trader, it makes matches more mixed. With $b = 2$, Figure 5 plots (σ, σ_U, ρ) as functions of $c \in [0, 0.2]$. The parameter a is still fixed at $a = 0.8$ to satisfy $a \leq 1 - c$. An increase in c increases the extent of mixing. All three measures of sorting fall. However, ρ falls by less than σ and σ_U do. Again, ρ over-estimates positive sorting significantly.

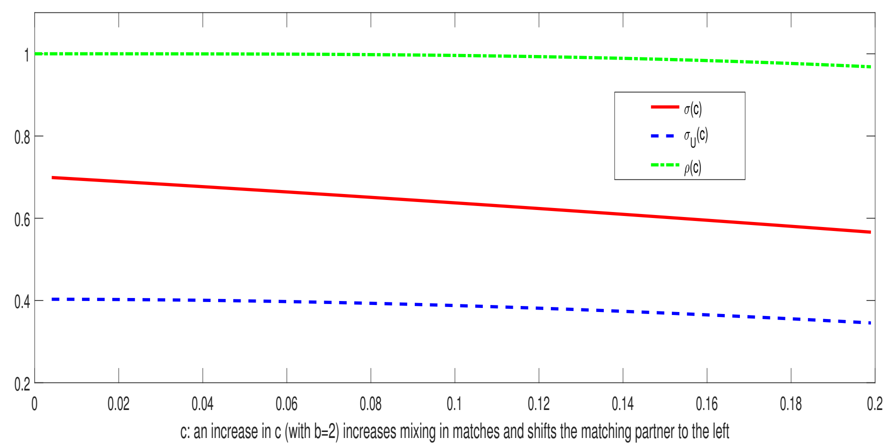

Figure 5: The effect of mixed matches on (σ, σ_U, ρ)

5 Related Issues and Extensions

5.1 Positive quadrant dominance

Anderson and Smith (2024) propose to rank sorting by positive quadrant dominance (PQD) introduced by Lehmann (1966). PQD is a partial ordering based on the joint cdf of matched types, $\mathcal{M}(q_v, q_s)$. A positive (lower) quadrant (q_v, q_s) is the rectangle containing all (q'_v, q'_s) such that $0 \leq q'_v \leq q_v$ and $0 \leq q'_s \leq q_s$. One matching pattern (A) has stronger sorting than another matching pattern (B) according to PQD if $\mathcal{M}_A(q_v, q_s) \geq \mathcal{M}_B(q_v, q_s)$ for all $(q_v, q_s) \in [0, 1]^2$. In common with the correlation coefficient, PQD considers only matched traders. As a result, the differences between ρ and (σ, σ_U) in sections 4.2 and 4.3 also exist to some extent between PQD and (σ, σ_U). PQD does not necessarily imply larger sorting indices (σ, σ_U) with general specifications of matching failures and imbalance in the market.

In the class of matching patterns with a uniform probability $(1-a)$ of matching failures, there is a connection between PQD and (σ, σ_U). In this class, σ and σ_U give the same ranking of sorting for any given (θ, a) (see (3.15)). With $R_v(q_v, 0)$ and $R_s(q_s, 0)$ in (3.14) and $\mathcal{M}$ in (4.4), I compute the joint distribution of matched types (for $q_v \geq q_{v0}$) as:

$$\mathcal{M}(q_v, q_s) = q_s - \frac{1}{\theta}(1-q_v) + \frac{1}{\theta a} R_v(q_v, q_s). \tag{5.1}$$

For any given (θ, a), the ranking by $\mathcal{M}$ coincides with the ranking by R_v. Specifically, $\mathcal{M}(q_v, p(q_v)) = 1 - \frac{2}{\theta}(1-q_v) + \frac{1}{\theta a}\bar{R}_v(q_v)$, where $\bar{R}$ is defined by (3.8). In this class, Proposition 2 implies:

$$\sigma_U = \frac{6a}{\theta}\int_{q_{v0}}^{1} \mathcal{M}(q, p(q))\mathrm{d}q - 2. \tag{5.2}$$

For any given (θ, a) in this class, PQD implies a larger σ_U and, hence, a larger σ.[9]

The reverse is not true. Because PQD is a partial ordering, it fails to rank some matching patterns that are clearly ranked by σ_U and σ. Recall that positive block matching has uniform matching failures. Appendix C.3 proves the following corollary:

Corollary 2. *With uniform matching failures in (3.13), PQD implies stronger sorting by σ_U and σ for any given (θ, a). For positive block matching described in section 3.2, an increase in the number of blocks from k to $k+1$ increases (σ, σ_U), but the two matching patterns are not ranked by PQD if $k \geq 2$.*

As an example, let us compare two block matching patterns, $k = 2$ and $k = 3$, and refer to them as Block2 and Block3, respectively. Suppose $\theta = 1$ and $a = 1$, and so $q_{v0} = 0$. Block2 divides the traders on each side into $q \in [0, \frac{1}{2})$ and $q \in [\frac{1}{2}, 1]$. Block 3 has three blocks, $[0, \frac{1}{3})$, $[\frac{1}{3}, \frac{2}{3})$ and $[\frac{2}{3}, 1]$. Denote the joint cdf of matched types as $\mathcal{M}k$ for $k = 2, 3$,

[9]By part (a) in Lemma 1 in Anderson and Smith (2024), PQD implies that the average distance for matched traders, $\mathbb{E}\,|q_s - p(q_v)|$, falls. The result here extends this implication to include the distance $|q_s - p(q_v)|$ for unmatched traders in the class with uniform matching failures.

and evaluate these distributions at $(q_v, q_s) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{2}{3}, \frac{2}{3}\right)$. In Block2, all traders in the block $[0, \frac{1}{2})$ are matched below $\frac{1}{2}$, and so the measure of matches below $\left(\frac{1}{2}, \frac{1}{2}\right)$ is equal to $\frac{1}{2}$. In Block3, a half of the traders in $[\frac{1}{3}, \frac{1}{2})$ are matched with the traders above $\frac{1}{2}$. The measure of matches below $\left(\frac{1}{2}, \frac{1}{2}\right)$ is equal to $\frac{1}{3} + \frac{1}{2} \times \frac{1}{6} = \frac{5}{12}$. Thus,

$$\mathcal{M}2\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} > \frac{5}{12} = \mathcal{M}3\left(\frac{1}{2}, \frac{1}{2}\right).$$

The ranking is reversed for $(q_v, q_s) = \left(\frac{2}{3}, \frac{2}{3}\right)$. In Block3, all traders in $[0, \frac{2}{3})$ are matched below $\frac{2}{3}$, and so the measure of matches below $\left(\frac{2}{3}, \frac{2}{3}\right)$ is equal to $\frac{2}{3}$. In Block2, a half of the traders in $[\frac{1}{2}, \frac{2}{3})$ are matched with the traders above $\frac{2}{3}$. The measure of matches below $\left(\frac{2}{3}, \frac{2}{3}\right)$ is equal to $\frac{1}{2} + \frac{1}{2} \times \frac{1}{6} = \frac{7}{12}$. Thus,

$$\mathcal{M}2\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{7}{12} < \frac{2}{3} = \mathcal{M}3\left(\frac{2}{3}, \frac{2}{3}\right).$$

Because the inequality between $\mathcal{M}2$ and $\mathcal{M}3$ reverses from $(q_v, q_s) = \left(\frac{1}{2}, \frac{1}{2}\right)$ to $\left(\frac{2}{3}, \frac{2}{3}\right)$, Block2 and Block3 are not ranked by PQD.[10]

Intuitively, Block3 should have stronger sorting than Block2. To explain why σ and σ_U yield this intuitive ranking when PQD fails to do so, let us examine σ_U in (5.2). Rather than taking into account all positive quadrants, σ_U focuses on the upper quadrants $(q, p(q))$. Moreover, σ_U is an integral of the measures of matches in the quadrants $(q, p(q))$. Even though Block3 has a lower measure of matches than Block2 in the quadrant $\left(\frac{1}{2}, \frac{1}{2}\right)$, Block 3 has a larger measure of matches than Block 2 in the quadrant $\left(\frac{2}{3}, \frac{2}{3}\right)$, which is more than making up for the deficiency in the lower quadrant. As a result, the total measure of matches in all quadrants $(q, p(q))$ is higher in Block3 than in Block2.

5.2 Discontinuous distribution of types

Let $\Upsilon_\tau(q_\tau)$ be the cdf of $q_\tau = F_\tau(y_\tau) \in [0, 1]$. When types are continuously distributed, Υ_τ is the uniform distribution which is the same for the two sides of the market. This is not true when there are mass points in the type distribution. A mass point in y_τ induces

[10] Because R_v is related to $\mathcal{M}$ by (5.1), the reversal in the ranking between $\mathcal{M}2$ and $\mathcal{M}3$ implies that there is no ranking in the function $\bar{R}_v(q)$ between Block2 and Block3.

a mass point in q_τ, which causes $\Upsilon_\tau(q_\tau)$ to jump up discontinuously at q_τ. Nevertheless, Υ_τ is right-continuous as is required for a cdf. Specifically, when the types are discrete, Υ_τ is a step function. Moreover, Υ_v can be different from Υ_s if the two sides have different mass points or different masses at the same points. I modify the upper quadrant matching function from (2.1) to

$$R_\tau(q_\tau, q) = \int_{q_\tau}^{1} r_\tau(q', q)\mathrm{d}\Upsilon_\tau(q').$$

The accounting identity (2.2) still holds. With discontinuous types, one needs to recompute the efficient allocation, $r_\tau^{ps}(q_\tau, q)$, and the associated sorting function $p(q_\tau)$. Because the distributions Υ_v and Υ_s can differ, the efficient matching pattern may no longer be pure.[11] Then, (3.1) and (3.4) are modified as:

$$\delta_\tau(q_\tau) \equiv \int_0^1 \left| r_\tau(q_\tau, q) - r_\tau^{ps}(q_\tau, q) \right| \mathrm{d}\Upsilon_{-\tau}(q),$$

$$\Delta_\tau \equiv \int_0^1 \delta_\tau(q_\tau)\mathrm{d}\Upsilon_\tau(q_\tau).$$

With these modifications, the definition of the overall reduction in sorting, Δ, is still (3.5). One can use the definition to recalculate the reduction in sorting for uniform matching, Δ^e. The sorting index is $\sigma(\Delta) = 1 - \frac{\Delta}{\Delta^e}$, as in Definition 1.

The upper quadrant matching function on the side v at the ideal type $p(q)$ is still $\bar{R}_v(q)$ in (3.8), and $\bar{R}_s$ is $\bar{R}_s(q) = R_s(q, p^{-1}(q))$. Because the distributions of q may be different on the two sides of the market, the weighted average of $\bar{R}_v$ and $\bar{R}_s$ over all $q \in [0,1]$ is:

$$U \equiv \frac{1}{1+\theta}\left[\int_0^1 \bar{R}_v(q)\mathrm{d}\Upsilon_v(q) + \theta\int_0^1 \bar{R}_s(q)\mathrm{d}\Upsilon_s(q)\right].$$

After recalculating U^{ps} for perfect positive sorting and U^e for uniform matching, the definition of the upper quadrant sorting index σ_U is still (3.11).

As an example, suppose that the original types lie in $\mathcal{Y}_\tau = \{1,2,3,4\}$ for $\tau \in \{v,s\}$. The masses of the types on each side of the market are $\left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}\right\}$. The values of q_τ lie in the set $\left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$. These values form the step function Υ_τ. For any type q_τ, the ideal matching

[11] For example, if there are two types of equal mass on the side v and three types of equal mass on the side s, then each type on the side v is matched with two types with positive probability.

type is $p(q_\tau) = q_\tau$. Assume that there is no matching failure and consider two patterns of positive block matching. Again refer to the two patterns as Block2 and Block3. Block2 has two blocks, $\{1, 2\}$ and $\{3, 4\}$. Block3 has three blocks, $\{1\}$, $\{2, 3\}$ and $\{4\}$. The table below lists the distribution $(1-r_v)$ of matches on the side v on the rows for perfect positive sorting (PS), uniform matching (UM), and the two block matching patterns. Using the masses of the partners (on the side s) as the weights, I compute the weighted average of the absolute differences in $1 - r_v$ between Block2 and PS. This is δ_v listed on the last column in the table for Block2. Using the masses on the side v as the weights, I compute the weighted average of δ_v, which is Δ_v. Similarly, I compute δ_v and Δ_v for Block3. If matching is uniform, the distribution $1 - r_v$ is equal to the cumulative sum of the masses of the partners on the side s. I list this distribution in the table and compute (δ_v, Δ_v).

Table 1. An example of discrete types

	mass	1/3	1/6	1/6	1/3		mass	1/3	1/6	1/6	1/3	
$v \setminus s$	PS: $1 - r_v$						UM: $1 - r_v$					$\Delta_v^e = \frac{11}{36}$
mass	type	1	2	3	4		type	1	2	3	4	δ_v
1/3	4	0	0	0	1		4	1/3	1/2	2/3	1	11/36
1/6	3	0	0	1	1		3	1/3	1/2	2/3	1	9/36
1/6	2	0	1	1	1		2	1/3	1/2	2/3	1	9/36
1/3	1	1	1	1	1		1	1/3	1/2	2/3	1	13/36
$v \setminus s$	Block2: $1 - r_v$					$\Delta_v = \frac{1}{8}$	Block3: $1 - r_v$					$\Delta_v = \frac{1}{36}$
	type	1	2	3	4	δ_v	type	1	2	3	4	δ_v
1/3	4	0	0	1/2	1	1/12	4	0	0	0	1	0
1/6	3	0	0	1/2	1	1/12	3	0	1/2	1	1	1/12
1/6	2	1/2	1	1	1	1/6	2	0	1/2	1	1	1/12
1/3	1	1/2	1	1	1	1/6	1	1	1	1	1	0

In all matching patterns in the table, the two sides of the market are symmetric, and so $\Delta_s = \Delta_v = \Delta$. The sorting index is $\sigma = \frac{13}{22}$ for Block2 and $\sigma = \frac{10}{11}$ for Block3. Moreover, because the market is balanced and there is no matching failure, the upper quadrant sorting index is $\sigma_U = \sigma$. Thus, Block3 has stronger positive sorting than Block2 by σ and σ_U. As in the case of continuous types, PQD cannot rank Block2 and Block3 in this example.

5.3 Multidimensional types

To extend the sorting index σ to multi-dimensional types, suppose that the type has n dimensions, where $n \geq 2$. The original type is a vector $\boldsymbol{y} \in \mathcal{Y} \subseteq R_{+}^{n}$. Let $F_i(y_i)$ be the distribution of the ith dimension of $\boldsymbol{y}$, and $q_i = F_i(y_i)$ the quantile in this dimension. The vector of quantiles corresponding to $\boldsymbol{y}$ is $\boldsymbol{q} = (q_1, ..., q_n)$, which belongs in $[0,1]^n$. For trader $\boldsymbol{q}_\tau$ on the side $\tau \in \{v, s\}$, $r_\tau(\boldsymbol{q}_\tau, \boldsymbol{q})$ is the probability that trader $\boldsymbol{q}_\tau$ is matched with traders at or above $\boldsymbol{q}$ on the other side. Let $r_\tau^{ps}(\boldsymbol{q}_\tau, \boldsymbol{q})$ be the corresponding probability in the efficient allocation without matching friction. Then, the definitions of δ_τ and Δ_τ are:

$$\delta_\tau(\boldsymbol{q}_\tau) \equiv \int_{[0,1]^n} |r_\tau(\boldsymbol{q}_\tau, \boldsymbol{q}) - r_\tau^{ps}(\boldsymbol{q}_\tau, \boldsymbol{q})| \, \mathrm{d}\boldsymbol{q}$$

$$\Delta_\tau \equiv \int_0^1 \delta_\tau(\boldsymbol{q}_\tau) \mathrm{d}\boldsymbol{q}_\tau.$$

With these extensions, Δ is still defined by (3.5), the sorting index σ by Definition 1, and the upper quadrant sorting index by (3.11). The main challenge in the calculation of σ in this case is to compute the efficient allocation with frictionless matching, $r_\tau^{ps}(\boldsymbol{q}_\tau, \boldsymbol{q})$. For an example that undertakes this task, see Lindenlaub (2017). Villani (2009) gives a thorough treatment of the related mathematical literature on the optimal transport problem. With such a benchmark, one can use the above formulas to compute the sorting index.

6 Concluding Remarks

This paper constructs an index to measure sorting in a two-sided market with matching friction. The index uses the cumulative distribution function (cdf) of the matching partners for each individual and incorporates matching failures as a mass point at the lowest type of the partners. The area between this cdf and the cdf in the efficient allocation without friction represents the reduction in sorting. The sorting index is equal to one minus the aggregate reduction in sorting normalized by the amount under uniform matching. A main component of this sorting index is the upper quadrant sorting index, which captures how well individuals are matched with types at or above the ideal type. In contrast to most measures of sorting in the literature, such as the correlation coefficient, the sorting index captures the effect of matching failures and compares the matching partner's type with the ideal type instead of the individual's own type. Relative to this sorting index, the correlation coefficient tends to over-estimate sorting when sorting is positive. The over-estimation increases sharply when matching failures increase or when the market becomes more imbalanced.

The analysis takes the type distribution of individuals as given. In some markets, the types may not be fully observable. For example, workers have unobservable skills and jobs have unobservable qualities. A challenge to measuring sorting in such markets is to use observables, such as wages and mobility patterns, to identify the type distribution (see Eeckhout and Kircher, 2011, Bonhomme et al., 2019). Although the current analysis does not take up this challenge, it suggests that the types should include those of the unmatched individuals as well as matched individuals. For unemployed workers, their past employments and wages may be useful for such identification.

Appendix

A Derivation of (3.3) and Proofs of Lemmas 1 and 2

A.1 Derivation of (3.3)

Denote the right-hand side of (3.3) as $\hat{\delta}_v(q_v)$. I prove $\hat{\delta}_v(q_v) = \delta_v(q_v) + O(\varepsilon)$, where $O(\varepsilon)$ is a quantity of the same magnitude as ε. If $p(q_v) \geq 0$, the integral in $\hat{\delta}_v(q_v)$ becomes:

$$\begin{aligned}
&\int_0^1 |q - p(q_v)| \,\mathrm{d}\left[1 - r_v(q_v, q)\right] \\
&= -\int_0^{p(q_v)} \left[p(q_v) - q\right] \mathrm{d}r_v(q_v, q) - \int_{p(q_v)}^1 \left[q - p(q_v)\right] \mathrm{d}r_v(q_v, q) \\
&= p(q_v) r_v(q_v, 0) - \int_0^{p(q_v)} r_v(q_v, q) \mathrm{d}q + \int_{p(q_v)}^1 r_v(q_v, q) \mathrm{d}q.
\end{aligned}$$

The second equality comes from integration by parts and the use of $r_v(q_v, 1) = 0$. Substituting into $\hat{\delta}_v(q_v)$ yields:

$$\hat{\delta}_v(q_v) = p(q_v) - \int_0^{p(q_v)} r_v(q_v, q) \mathrm{d}q + \int_{p(q_v)}^1 r_v(q_v, q) \mathrm{d}q + O(\varepsilon).$$

If $p(q_v) = -\varepsilon$, then

$$\begin{aligned}
&\int_0^1 |q - p(q_v)| \,\mathrm{d}\left[1 - r_v(q_v, q)\right] = -\int_0^1 (q + \varepsilon) \,\mathrm{d}r_v(q_v, q) \\
&= \varepsilon r_v(q_v, 0) + \int_0^1 r_v(q_v, q) \mathrm{d}q.
\end{aligned}$$

Unifying the two cases, I have:

$$\begin{aligned}
\hat{\delta}_v(q_v) = &\quad (p(q_v))^+ - \int_0^{(p(q_v))^+} r_v(q_v, q) \mathrm{d}q + \int_{(p(q_v))^+}^1 r_v(q_v, q) \mathrm{d}q + O(\varepsilon) \\
= &\quad \int_0^{(p(q_v))^+} \left[1 - r_v(q_v, q)\right] \mathrm{d}q + \int_{(p(q_v))^+}^1 r_v(q_v, q) \mathrm{d}q + O(\varepsilon).
\end{aligned}$$

This is equal to $\delta_v(q_v) + O(\varepsilon)$, where $\delta_v(q_v)$ is given by (3.2).

A.2 Proof of Lemma 1

This appendix examines positive block matching and proves Lemma 1.

The traders in $[0, q_{v0})$ on the side v are not matched. The remaining traders are divided into k equal blocks indexed by $i \in \{1, 2, ..., k\}$. The width of each block is $c_s = \frac{1}{k}$ on the side s, and $c_v = \frac{1-q_{v0}}{k} = \theta c_s$ on the side v. The ith block contains $q_v \in [q_{v0} + (i-1)\, c_v, q_{v0} + i c_v)$ and $q_s \in [(i-1)\, c_s, i c_s)$. The last block $i = k$ also contains $q_v = 1$ and $q_s = 1$. Traders in the ith block on the two sides trade exclusively among themselves. Moreover, the matching partner is uniformly drawn from the same block on the other side. Although the width of types in matches on the side v is weakly smaller than the width of types on the side s, the measure of traders in each block is the same on the two sides; i.e., $c_v n_v = c_s n_s$. This implies that a trader in each block is matched with probability one. The matching partner is uniformly drawn from the same block on the other side.

Let us take any trader q_v in the ith block on the side v and derive the cdf of the trader's matching partners, q. If $q < (i-1)\,c_s$, then $1 - r_v(q_v, q) = 0$, because trader q_v is never matched with any trader below the ith block. If $q \in [(i-1)\,c_s, ic_s)$, then q is in the ith block on the side s. Because the trading partner of q_v is uniformly drawn from the ith block, the partner's type is smaller than or equal to q with the probability $1 - r_v(q_v, q) = \frac{q-(i-1)c_s}{c_s}$. If $q \geq ic_s$, then q is above the ith block. In this case, the trading partner for trader q_v is lower than q with probability 1. Putting these cases together, the cdf of the matching partner of trader q_v in the ith block is:

$$1 - r_v(q_v, q) = \begin{cases} 0, & \text{if } q < (i-1)\,c_s \\ \frac{q-(i-1)c_s}{c_s}, & \text{if } q \in [(i-1)\,c_s, ic_s) \\ 1, & \text{if } q \geq ic_s. \end{cases} \tag{A.1}$$

To compute the sorting index, recall that perfect positive sorting has $p(q_v) = 1 - \frac{1}{\theta}(1 - q_v)$. If $q_v \in [0, q_{v0})$, trader q_v fails to match under both positive block matching and perfect positive sorting. It suffices to compare the distribution of matching partners for $q_v \geq q_{v0}$. In the ith block on the side v, let $z \in [0,1)$ be such that $q_v = q_{v0} + (i-1+z)\,c_v$. That is, z is the trader's relative location in the block. Then,

$$p(q_v) = 1 - \frac{1}{\theta}\left(1 - q_{v0} - (i-1)\,c_v - zc_v\right) = c_s\,(i-1+z)\,.$$

The second equality used $q_{v0} = 1 - \theta$ and $c_v = \theta c_s$. The probability that trader q_v matches with some partner in $[(i-1)\,c_s, p(q_v)]$ is $\frac{p(q_v)-(i-1)c_s}{c_s} = z$. The sum of the areas of the two triangles in the ith block in the right panel of Figure 2 is:

$$\delta_v(q_v) = \frac{1}{2}\left[p(q_v) - (i-1)\,c_s\right] z + \frac{1}{2}\left[ic_s - p(q_v)\right](1-z) = c_s\left(z^2 - z + \frac{1}{2}\right).$$

Integrating $\delta_v(q_v)$ over q_v in the ith block and changing the integration variable from q_v to $z = \frac{q_v - q_{v0}}{c_v} - i + 1$, I have:

$$\int_{q_{v0}+(i-1)c_v}^{q_{v0}+ic_v} \delta_v(q_v)\mathrm{d}q_v = c_v \int_0^1 c_s \left(z^2 - z + \frac{1}{2}\right) \mathrm{d}z = \frac{1}{3}c_v c_s.$$

Because this result is the same for all i, summing over the k blocks yields:

$$\Delta_v = \frac{1}{3}c_v c_s k = \frac{\theta}{3k}.$$

The second equality follows from $c_s = \frac{1}{k}$ and $c_v = \theta c_s$.

Similarly, I can derive $\Delta_s = \frac{\theta}{3k}$. Substituting (Δ_v, Δ_s) into (3.5), I have $\Delta = \frac{\theta}{3k}$. Setting $k = 1$ in (3.6) generates $\Delta^e = \frac{\theta}{3}$. **QED**

A.3 Proof of Lemma 2

This appendix examines perfect negative sorting and proves Lemma 2. Perfect negative sorting is a decreasing and measure preserving function $\nu : \mathcal{Q} \to \mathcal{Q}$ that satisfies:

$$r_v(q_v, q) = \mathbb{1}\left(q < \nu(q_v)\right) \quad \forall (p_v, q) \in \mathcal{Q}^2.$$

That is, each type q_v on the side v is matched only with the type $\nu(q_v)$ on the other side, provided $\nu(q_v) \geq 0$, and $\nu(q_v)$ decreases in the trader's own type q_v. For $(q_v, q) \in [0,1]^2$, (2.1) implies:

$$R_v(q_v, q) = max\left\{\nu^{-1}(q) - q_v, 0\right\} = \left(\nu^{-1}(q) - q_v\right)^+.$$

Similarly, $R_s(q_s, q) = (\nu(q) - q_s)^+$ for For $(q_s, q) \in [0,1]^2$. Denote $q_{va} = (\nu^{-1}(0))^+$. Then $q_{va} = \theta \in (0, 1]$. Setting $q_s = 0$ in (2.2), I have:

$$\nu(q_v) = \begin{cases} 1 - \frac{1}{\theta}q_v, & \text{if } q_v \leq q_{va} \\ -\varepsilon, & \text{osherwise.} \end{cases} \tag{A.2}$$

The inverse is $\nu^{-1}(q_s) = \theta\,(1 - q_s)$.

To compute the index Δ, note that $q_{va} = \theta > q_{v0}$ if and only if $\theta > \frac{1}{2}$. Examine first the case $\frac{1}{2} \leq \theta \leq 1$. In this case, $0 \leq q_{v0} \leq \frac{1}{2} \leq q_{va} \leq 1$. For $q_v \in [q_{v0}, q_{va}]$, the distribution function $1-r_v(q_v,\bullet)$ differs from the one in perfect positive sorting only when q_s lies between $p(q_v)$ and $\nu(q_v)$. One distribution is equal to 0 and the other to 1. For such values of q_v,

$$\delta_v(q_v) = |\nu(q_v) - p(q_v)| = \frac{1}{\theta}|1-2q_v|.$$

If $q_v < q_{v0}$, then for $q_s < \nu(q_v)$, the distribution function $1 - r_v(q_v, q_s)$ is equal to zero under perfect negative sorting, but to 1 under perfect positive sorting. For such values of q_v, $\delta_v(q_v) = \nu(q_v) = 1 - \frac{1}{\theta}q_v$. If $q_v > q_{va}$, then $\delta_v(q_v) = p(q_v) = 1 - \frac{1}{\theta}(1-q_v)$. Thus,

$$\begin{aligned}\Delta_v = & \int_0^{q_{v0}} \left(1 - \tfrac{1}{\theta}q\right) \mathrm{d}q + \int_{q_{va}}^1 \left[1 - \tfrac{1}{\theta}(1-q)\right] \mathrm{d}q \\ & + \tfrac{1}{\theta}\int_{q_{v0}}^{1/2} (1-2q)\,\mathrm{d}q + \tfrac{1}{\theta}\int_{1/2}^{q_{va}} (2q-1)\,\mathrm{d}q \\ = & \left(1 - \tfrac{1}{2\theta}q_{v0}\right) q_{v0} + (1-q_{va})\left[1 - \tfrac{1}{2\theta}(1-q_{va})\right] \\ & + \tfrac{1}{\theta}\left(q_{va}^2 - q_{va} - q_{v0} + q_{v0}^2 + \tfrac{1}{2}\right).\end{aligned}$$

Substituting (q_{v0}, q_{va}), I have:

$$\Delta_v = 2 - \theta - \frac{1}{2\theta}.$$

This is a concave function of θ. For $\theta \in \left[\frac{1}{2}, 1\right]$, the minimum of Δ_v is $\frac{1}{2}$, which is achieved at $\theta = 1$ or $\frac{1}{2}$. The maximum is $2 - \sqrt{2}$, which is achieved at $\theta = \frac{\sqrt{2}}{2}$.

Now examine the case $0 \le \theta < \frac{1}{2}$. In this case, $0 \le q_{va} < \frac{1}{2} < q_{v0} \le 1$. If $q_v \le q_{va}$, then for $q < \nu(q_v)$, the distribution function $1 - r_v(q_v, q)$ is equal to 0 under perfect negative sorting but to 1 under perfect positive sorting. For such values of q_v, $\delta_v(q_v) = \nu(q_v)$. If $q_v \ge q_{v0}$, then for $q < p(q_v)$, the distribution function $1 - r_v(q_v, q)$ is equal to 1 under perfect negative sorting but to 0 under perfect positive sorting. For such values of q_v, $\delta_v(q_v) = p(q_v)$. If $q_{va} < q_v < q_{v0}$, both distributions are equal to 1 for all q.

$$\begin{aligned}\Delta_v = & \int_0^{q_{va}} \left(1 - \tfrac{1}{\theta}q\right) \mathrm{d}q + \int_{q_{v0}}^1 \left[1 - \tfrac{1}{\theta}(1-q)\right] \mathrm{d}q \\ = & \left(1 - \tfrac{1}{2\theta}q_{va}\right) q_{va} + (1-q_{v0})\left[1 - \tfrac{1}{2\theta}(1-q_{v0})\right] = \theta.\end{aligned}$$

For all $(n_v, n_s) > 0$, $\theta \ge 2 - \theta - \frac{1}{2\theta}$.

Putting together the case $\frac{1}{2} \le \theta \le 1$ and the case $\theta < \frac{1}{2}$, I have:

$$\Delta_v = \begin{cases} 2 - \theta - \frac{1}{2\theta}, & \text{if } \frac{1}{2} \le \theta \le 1 \\ \theta, & \text{if } \theta < \frac{1}{2}. \end{cases}$$

To compute Δ_s, note that all traders on the side s are matched with some traders on the side v under both pure positive and perfect negative sorting. For trader q_s, the distribution functions of matching partners under the two types of sorting differ only between $\nu^{-1}(q_s)$ and $p^{-1}(q_s)$. Then,

$$\delta_s(q_s) = \left|p^{-1}(q_s) - \nu^{-1}(q_s)\right| = \left|1 - 2\theta\left(1 - q_s\right)\right|.$$

$1 - 2\theta(1 - q_s) > 0$ if and only if $q_s > 1 - \frac{1}{2\theta}$. If $\frac{1}{2} \leq \theta$ (≤ 1), then $1 - \frac{1}{2\theta} \geq 0$. In this case,

$$\begin{aligned}\Delta_s = &\int_0^{1-\frac{1}{2\theta}} \left[2\theta\left(1-q\right)-1\right] \mathrm{d}q + \int_{1-\frac{1}{2\theta}}^1 \left[1-2\theta\left(1-q\right)\right] \mathrm{d}q \\ = &\left(1-\frac{1}{2\theta}\right)\left(\theta - \frac{1}{2}\right) + \frac{1}{4\theta} = \theta - 1 + \frac{1}{2\theta}.\end{aligned}$$

If $\theta < \frac{1}{2}$, then $1 - \frac{1}{2\theta} < 0$. In this case,

$$\Delta_s = \int_0^1 \left[1 - 2\theta\left(1-q\right)\right] \mathrm{d}q = 1 - \theta.$$

Putting the two cases together yields:

$$\Delta_s = \begin{cases} \theta - 1 + \frac{1}{2\theta}, & \text{if } \frac{1}{2} \leq \theta \leq 1 \\ 1 - \theta, & \text{if } 0 \leq \theta < \frac{1}{2}. \end{cases}$$

Therefore,

$$\Delta = \begin{cases} \frac{1}{1+\theta}\left(\frac{5}{2} - 2\theta - \frac{1}{2\theta} + \theta^2\right), & \text{if } \frac{1}{2} \leq \theta \leq 1 \\ \frac{\theta}{1+\theta}\left(2 - \theta\right), & \text{if } 0 \leq \theta < \frac{1}{2}. \end{cases}$$

Because $\Delta^e = \frac{\theta}{3}$, the sorting index is:

$$\sigma = \begin{cases} 1 - \frac{3}{\theta(1+\theta)}\left(\frac{5}{2} - 2\theta - \frac{1}{2\theta} + \theta^2\right), & \text{if } \frac{1}{2} \leq \theta \leq 1 \\ 1 - \frac{3}{1+\theta}\left(2 - \theta\right), & \text{if } 0 \leq \theta < \frac{1}{2}. \end{cases}$$

Re-organizing the result yields the expression for σ in the lemma. If $\theta = 1$, then $\Delta = \frac{1}{2}$ and $\sigma = -\frac{1}{2}$. **QED**

B Proofs of Proposition 1 and Proposition 2

B.1 Proof of Proposition 1

Recall that $p(q_{v0}) = 0$ and $p^{-1}(0) = q_{v0} = 1 - \theta$. Substituting δ_v from (3.2) into (3.4), I get:

$$\begin{aligned}\Delta_v = &\int_0^1 \int_0^{(p(q_v))^+} \left[1 - r_v(q_v, q_s)\right] \mathrm{d}q_s \mathrm{d}q_v + \int_0^1 \int_{(p(q_v))^+}^1 r_v(q_v, q_s) \mathrm{d}q_s \mathrm{d}q_v \\ = &\int_0^1 \left(p(q_v)\right)^+ \mathrm{d}q_v + \int_0^1 \int_0^1 r_v(q_v, q_s) \mathrm{d}q_s \mathrm{d}q_v - 2\int_0^1 \int_0^{(p(q_v))^+} r_v(q_v, q_s) \mathrm{d}q_s \mathrm{d}q_v.\end{aligned}$$

Since $(p(q_v))^+ = 1 - \frac{1}{\theta}(1 - q_v)$ if $q_v \geq q_{v0}$, and $(p(q_v))^+ = 0$ otherwise, I have the first equality below :

$$\int_0^1 (p(q_v))^+ \mathrm{d}q_v = \int_{q_{v0}}^1 \left[1 - \frac{1}{\theta}(1 - q_v)\right] \mathrm{d}q_v = (1 - q_{v0}) \left[1 - \frac{1 - q_{v0}}{2\theta}\right] = \frac{\theta}{2}.$$

The last equality substitutes $q_{v0} = 1 - \theta$.

For the second term of Δ_v above, I change the order of the two integrals to obtain the first equality below:

$$\begin{aligned} &\int_0^1 \int_0^1 r_v(q_v, q_s)\mathrm{d}q_s\mathrm{d}q_v = \int_0^1 \int_0^1 r_v(q_v, q_s)\mathrm{d}q_v\mathrm{d}q_s \\ &= \int_0^1 R_v(0, q)\mathrm{d}q = \theta \int_0^1 R_s(q, 0)\mathrm{d}q. \end{aligned}$$

The second equality uses the definition of R_v in (2.1) and changes the notation q_s to q. The last equality uses the identity (2.2).

For the third term of Δ_v above, the fact that $p(q_v) > 0$ if and only if $q_v > q_{v0}$ implies the first equality below:

$$\begin{aligned} &\int_0^1 \int_0^{(p(q_v))^+} r_v(q_v, q_s)\mathrm{d}q_s\mathrm{d}q_v = \int_{q_{v0}}^1 \int_0^{p(q_v)} r_v(q_v, q_s)\mathrm{d}q_s\mathrm{d}q_v \\ &= \int_{q_{v0}}^1 \int_{p(q_{v0})}^{p(q_v)} r_v(q_v, q_s)\mathrm{d}q_s\mathrm{d}q_v = \frac{1}{\theta} \int_{q_{v0}}^1 \int_{q_{v0}}^{q_v} r_v(q_v, p(q))\mathrm{d}q\mathrm{d}q_v \end{aligned}$$

The second equality uses $0 = p(q_{v0})$. For the third equality, I set $q_s = p(q) = 1 - \frac{1}{\theta}(1 - q)$ in the inner integral to change the integration variable from q_s to $q = p^{-1}(q_s)$. In the last double integral, the integration region is $\{(q_v, q) : q_{v0} \leq q_v \leq 1, q_{v0} \leq q \leq q_v\}$. This region is equivalent to $\{(q, q_v) : q_{v0} \leq q \leq 1, q \leq q_v \leq 1\}$. Using this equivalence to change the order of integration, I obtain the first equality below:

$$\begin{aligned} &\int_0^1 \int_0^{(p(q_v))^+} r_v(q_v, q_s)\mathrm{d}q_s\mathrm{d}q_v \\ &= \frac{1}{\theta} \int_{q_{v0}}^1 \int_q^1 r_v(q_v, p(q))\mathrm{d}q_v\mathrm{d}q = \frac{1}{\theta} \int_{q_{v0}}^1 R_v(q, p(q))\mathrm{d}q. \end{aligned}$$

The second equality uses the definition of R_v in (2.1) and then uses the identity (2.2). Substituting the above results into the expression for Δ_v above, I obtain Δ_v as

$$\Delta_v = \frac{\theta}{2} + \theta \int_0^1 R_s(q, 0)\mathrm{d}q - \frac{2}{\theta} \int_{q_{v0}}^1 R_v(q, p(q))\mathrm{d}q \tag{B.1}$$

Similarly, I can derive Δ_s as

$$\begin{aligned} \Delta_s = \; & 1 - \frac{\theta}{2} + \frac{1}{\theta} \int_0^1 R_v(q, 0)\mathrm{d}q \\ & - \frac{2}{\theta} \int_0^{q_{v0}} R_v(q, 0)\mathrm{d}q - 2\theta \int_0^1 R_s(q, p^{-1}(q))\mathrm{d}q. \end{aligned}$$

The additional integral over $q \in [0, q_{v0}]$ did not appear in (B.2) because the analogous term is an integral from 0 to $p(q_{v0}) = 0$, which is equal to 0. Using $0 = p(q_{v0})$, I rewrite the last integral in Δ_s as in the first equality below:

$$\begin{aligned}&\textstyle\int_0^1 R_s(q_s, p^{-1}(q_s))\mathrm{d}q_s = \int_{p(q_{v0})}^1 R_s(q_s, p^{-1}(q_s))\mathrm{d}q_s \\ &= \textstyle\frac{1}{\theta}\int_{q_{v0}}^1 R_s(p(q), q)\mathrm{d}q = \frac{1}{\theta^2}\int_{q_{v0}}^1 R_v(q, p(q))\mathrm{d}q.\end{aligned}$$

The second equality sets $q_s = p(q) = 1 - \frac{1}{\theta}(1-q)$ to change the integration variable from q_s to $q = p^{-1}(q_s)$. The third equality comes from the identity (2.2). Substituting the above result into Δ_s above yields Δ_s as:

$$\Delta_s = 1 - \frac{\theta}{2} + \frac{1}{\theta}\int_0^1 R_v(q,0)\mathrm{d}q - \frac{2}{\theta}\int_0^{q_{v0}} R_v(q,0)\mathrm{d}q - \frac{2}{\theta}\int_{q_{v0}}^1 R_v(q,p(q))\mathrm{d}q \tag{B.2}$$

Substituting Δ_v from (B.1) and Δ_s from (B.2) for Δ_s, the definition of Δ yields:

$$\begin{aligned}\Delta = \ & \textstyle\frac{\theta(3-\theta)}{2(1+\theta)} - \frac{2}{\theta}\int_{q_{v0}}^1 R_v(q,p(q))\mathbf{d}q \\ & \textstyle+\frac{1}{1+\theta}\left[\theta\int_0^1 R_s(q,0)\mathbf{d}q + \int_0^1 R_v(q,0)\mathbf{d}q - 2\int_0^{q_{v0}} R_v(q,0)\mathbf{d}q\right].\end{aligned} \tag{B.3}$$

Substituting Δ into the definition of σ yields (3.7). **QED**

B.2 Proof of Proposition 2

To derive U, I use the definition of $\bar{R}_v$ in (3.8) to compute:

$$\int_0^1 \bar{R}_v(q)\mathrm{d}q = \int_{q_{v0}}^1 \bar{R}_v(q)\mathrm{d}q + \int_0^{q_{v0}} (1-q)\,\mathrm{d}q = \int_{q_{v0}}^1 \bar{R}_v(q)\mathrm{d}q + \frac{1}{2}\left(1-\theta^2\right).$$

The second equality uses $q_{v0} = 1-\theta$. For $\bar{R}_s$, I compute:

$$\theta\int_0^1 R_s(q, p^{-1}(q))\mathrm{d}q = \int_{p(q_{v0})}^1 R_v(p^{-1}(q), q)\mathrm{d}q = \frac{1}{\theta}\int_{q_{v0}}^1 R_v(q, p(q))\mathrm{d}q.$$

The first equality uses the identity (2.2) and $p(q_{v0}) = 0$. The second equality sets $q' = p^{-1}(q)$ to change the integration variable from q to q', and then changes the notation q' to q. Substituting the above results into the definition (3.9) yields:

$$U = \frac{1}{\theta}\int_{q_{v0}}^1 \bar{R}_v(q)\mathrm{d}q + \frac{1-\theta}{2}. \tag{B.4}$$

Substituting the expression for $\bar{R}_v^{ps}$ from (3.10) into (B.4), it is easy to calculate $U^{ps} = \frac{1}{2}$. Also, the upper bound in (3.10) implies $U \leq U^{ps}$, and so $\sigma_U \leq 1$.

To compute U^e, I examine uniform matching (i.e., positive block matching with one block). Traders in $[q_{v0}, 1]$ on the side v and traders in $[0, 1]$ on the side s are uniformly matched. The width of the interval of types in matches is $1 - q_{v0} = \theta$ on the side v and 1 on the side s. Consider any $q_v \in [q_{v0}, 1]$. The probability for trader q_v to match with a trader in $[q_s, 1]$ is:

$$r_v^e(q_v, q_s) = 1 - q_s.$$

The superscript e indicates uniform matching. For such (q_v, q_s), the function R_v is:

$$R_v^e(q_v, q_s) = \int_{q_v}^{1} r_v^e(q, q_s)\mathrm{d}q = (1 - q_s) \int_{q_v}^{1} \mathrm{d}q = (1 - q_s)(1 - q_v).$$

Because $p(q_v) = 1 - \frac{1}{\theta}(1 - q_v)$ for all $q_v \in [q_{v0}, 1]$, the above result for R_v yields:

$$\bar{R}_v^e(q_v) = R_v^e(q_v, p(q_v)) = \frac{1}{\theta}(1 - q_v)^2.$$

Because $q_{v0} = 1 - \theta$, then

$$\int_{q_{v0}}^{1} \bar{R}_v^e(q)\mathrm{d}q = \frac{1}{\theta}\int_{q_{v0}}^{1} (1 - q_v)^2 \mathrm{d}q_v = \frac{\theta^2}{3}.$$

Substituting into (B.4) yields

$$U^e = \frac{\theta}{3} + \frac{1 - \theta}{2} = \frac{3 - \theta}{6}. \tag{B.5}$$

Substituting U from (B.4), U^e from the above, and $U^{ps} = \frac{1}{2}$ into (3.11), I obtain σ_U in (3.12). **QED**

C Proofs of Propositions 3 and 4 and Corollary 2

C.1 Proof of Proposition 3

The marginal distribution of matched types on the side s is $H_s(q)$ given by (4.3). Because $R'_{s,1}(q, 0) = -r_s(q, 0)$, the density function is $H'_s(q) = \frac{r_s(q,0)}{R_s(0,0)}$. Substituting this result and

(4.2) into (4.1), I obtain the first equality below:

$$\begin{aligned}&\mathcal{M}(q_v,q_s)=\textstyle\int_0^{q_s}\left[1-\frac{r_s(q_s',q_v)}{r_s(q_s',0)}\right]\frac{r_s(q_s',0)}{R_s(0,0)}\mathrm{d}q_s'\\&=\tfrac{1}{R_s(0,0)}\textstyle\int_0^{q_s}\left[r_s(q_s',0)-r_s(q_s',q_v)\right]\mathrm{d}q_s'\\&=\tfrac{1}{R_s(0,0)}\left\{\textstyle\int_0^{1}\left[r_s(q_s',0)-r_s(q_s',q_v)\right]\mathrm{d}q_s'-\int_{q_s}^{1}\left[r_s(q_s',0)-r_s(q_s',q_v)\right]\mathrm{d}q_s'\right\}\\&=1-\tfrac{1}{R_s(0,0)}\left[R_s(0,q_v)+R_s(q_s,0)-R_s(q_s,q_v)\right].\end{aligned}$$

The second equality is rewriting. The third equality breaks the integration into two intervals. The last equality uses the definition of R_s. Using the identity (2.2) to substitute $R_s(q_s,q_v)$ by $\frac{1}{\theta}R_v(q_v,q_s)$ for all q_v, the above result for $\mathcal{M}(q_v,q_s)$ becomes (4.4). Note that the marginal distributions of matched types on the two sides are $H_v(q_v)=\mathcal{M}(q_v,1)$ and $H_s(q_s)=\mathcal{M}(1,q_s)$, which are consistent with (4.3) because $R_\tau(1,q)=R_\tau(q,1)=0$ for $\tau\in\{v,s\}$ and all q.

The mean of matched types on the side v is:

$$\mathbb{E}q_v=\int_0^1 qH_v'(q)\mathrm{d}q=-\frac{1}{R_v(0,0)}\int_0^1 qR_{v,1}'(q,0)\mathrm{d}q=\frac{1}{R_v(0,0)}\int_0^1 R_v(q,0)\mathrm{d}q.$$

The last equality integrates by parts and uses $R_v(1,0)=0$. The result above is (4.5). The variance of matched types on the side v is:

$$\begin{aligned}&Var(q_v)=-\textstyle\int_0^1\left(q-\mathbb{E}q_v\right)^2\mathrm{d}\left[1-H_v(q)\right]\\&=\left(\mathbb{E}q_v\right)^2+\tfrac{2}{R_v(0,0)}\textstyle\int_0^1\left(q-\mathbb{E}q_v\right)R_v(q,0)\mathrm{d}q=\tfrac{2}{R_v(0,0)}\textstyle\int_0^1 qR_v(q,0)\mathrm{d}q-\left(\mathbb{E}q_v\right)^2.\end{aligned}$$

The second equality comes from integrating by parts and uses $H_v(0)=0$. The third equality comes from (4.5). This verifies (4.6). By switching v to s, I obtain $\mathbb{E}q_s$ and $Var(q_s)$. Using the identity $R_s(q,q_v)=\frac{1}{\theta}R_v(q_v,q)$ for all (q_v,q), I can rewrite the results for $\mathbb{E}q_s$ and $Var(q_s)$ in terms of R_v instead of R_s.

To derive $Cov(q_v,q_s)$, note that the derivative of the joint distribution of matched types, $\mathcal{M}(q_v,q_s)$, with respect to q_v is $\frac{-1}{R_v(0,0)}\left[r_v(q_v,q_s)-r_v(q_v,0)\right]$. Applying the operator d to q_s, I have:

$$\frac{-1}{R_v(0,0)}\mathrm{d}\left[r_v(q_v,q_s)-r_v(q_v,0)\right]=\frac{-1}{R_v(0,0)}\mathrm{d}r_v(q_v,q_s).$$

Thus, the covariance of matched types is:

$$Cov(q_v,q_s)=-\frac{1}{R_v(0,0)}\int_0^1\left(q_v-\mathbb{E}q_v\right)\int_0^1\left(q_s-\mathbb{E}q_s\right)\mathrm{d}r_v(q_v,q_s)\mathrm{d}q_v. \tag{C.1}$$

The operator d on r_v is with respect to q_s. Integrating by parts, I compute the inner integral:

$$\int_0^1 (q_s - \mathbb{E}q_s)\, \mathrm{d}r_v(q_v, q_s) = r_v(q_v, 0)\mathbb{E}q_s - \int_0^1 r_v(q_v, q_s)\mathrm{d}q_s.$$

I have used $r_v(q_v, 1) = 0$. Substituting this result into (C.1) yields:

$$Cov(q_v, q_s) = \frac{1}{R_v(0,0)} \int_0^1 (q_v - \mathbb{E}q_v) \left[\int_0^1 r_v(q_v, q_s)\mathrm{d}q_s - r_v(q_v, 0)\mathbb{E}q_s\right] \mathrm{d}q_v.$$

Since $r_v(q_v, q_s) = -R'_{v,1}(q_v, q_s)$, then

$$\begin{aligned}
&\textstyle\int_0^1 (q_v - \mathbb{E}q_v)\, r_v(q_v, 0)\mathrm{d}q_v = -\int_0^1 (q_v - \mathbb{E}q_v)\, R'_{v,1}(q_v, 0)\mathrm{d}q_v \\
&= -\mathbb{E}q_v R_v(0,0) + \textstyle\int_0^1 R_v(q_v, 0)\mathrm{d}q_v = -\mathbb{E}q_v R_v(0,0) + \theta \int_0^1 R_s(0, q_v)\mathrm{d}q_v \\
&= -\mathbb{E}q_v R_v(0,0) + \textstyle\int_0^1 R_v(q_v, 0)\mathrm{d}q_v = 0.
\end{aligned}$$

Similarly, the first equality below uses $r_v(q_v, q_s) = -R'_{v,1}(q_v, q_s)$ and changes the order of integration:

$$\begin{aligned}
&\textstyle\int_0^1 \int_0^1 (q_v - \mathbb{E}q_v)\, r_v(q_v, q_s)\mathrm{d}q_s\mathrm{d}q_v = -\int_0^1 \int_0^1 (q_v - \mathbb{E}q_v)\, R'_{v,1}(q_v, q_s)\mathrm{d}q_v\mathrm{d}q_s \\
&= -\textstyle\int_0^1 \left[\mathbb{E}q_v R_v(0, q_s) - \int_0^1 R_v(q_v, q_s)\mathrm{d}q_v\right] \mathrm{d}q_s \\
&= -\mathbb{E}q_v \mathbb{E}q_s R_v(0,0) + \textstyle\int_0^1 \int_0^1 R_v(q_v, q_s)\mathrm{d}q_s\mathrm{d}q_v.
\end{aligned}$$

The second equality integrates by parts and uses $R_v(1, q_s) = 0$. The third equality uses the expression for $\mathbb{E}q_s$ and changes the order of two integrals. The last equality uses (4.5). Substituting these results into the above expression for $Cov(q_v, q_s)$, I obtain (4.7).[12] **QED**

C.2 Proof of Proposition 4

When the matching pattern has uniform matching failures in (3.13), R_v and R_s satisfy (3.14). Using Proposition 3, I compute the statistics of matched types as follows:

$$\mathbb{E}q_s = \frac{1}{2}, \quad \mathbb{E}q_v = 1 - \frac{\theta}{2},$$

$$Var(q_s) = \frac{1}{12}, \quad Var(q_v) = \frac{\theta^2}{12},$$

$$\rho = \frac{12}{\theta} Cov(q_v, q_s) = \frac{12}{\theta^2 a} B - \frac{3}{\theta}(2 - \theta), \tag{C.2}$$

[12]Because $\frac{R_v(q_v,q_s)}{R_v(0,0)} = \frac{R_s(q_s,q_v)}{R_s(0,0)}$ by (2.2), the result (4.7) shows that the covariance is symmetric between q_v and q_s, as it should, although the symmetry is not apparent in the earlier formula (C.1).

where B temporarily denotes:

$$B \equiv \int_0^1 \int_0^1 R_v(q_v, q_s) \mathrm{d}q_s \mathrm{d}q_v. \tag{C.3}$$

I derive a lower bound on B and then link it to σ_U. If $q_v < q_{v0}$, then (i) in (3.13) implies $r_v(q_v, q_s) = r_v(q_v, 0) = 0$ for all q_s. For such q_v, the following result holds:

$$R_v(q_v, q_s) = \int_0^1 r_v(q', q_s) \mathrm{d}q' = R_v(0, q_s) = \theta R_s(q_s, 0) = \theta a \left(1 - q_s\right).$$

The first equality uses the fact that $\int_0^{q_v} r_v(q', q_s) \mathrm{d}q' = 0$ for all $q_v < q_{v0}$. The second equality is the definition of $R_v(0, q_s)$. The third equality uses the identity (2.2), and the last equality uses (3.14). Breaking the outer integral in B into two intervals, I have:

$$\begin{aligned} B &= \int_0^{q_{v0}} \int_0^1 R_v(q_v, q_s) \mathrm{d}q_s \mathrm{d}q_v + \int_{q_{v0}}^1 \int_0^1 R_v(q_v, q_s) \mathrm{d}q_s \mathrm{d}q_v \\ &= \tfrac{a}{2} \theta \left(1 - \theta\right) + \int_{q_{v0}}^1 \int_0^1 R_v(q_v, q_s) \mathrm{d}q_s \mathrm{d}q_v. \end{aligned}$$

The second equality substitutes the previous result for $R_v(q_v, q_s)$ with $q_v < q_{v0}$, integrates, and uses $q_{v0} = 1 - \theta$. For the last double integral above, I break the inner integral on q_s into two intervals, subtract and add the same term in each interval:

$$\begin{aligned} & \int_{q_{v0}}^1 \int_0^1 R_v(q_v, q_s) \mathrm{d}q_s \mathrm{d}q_v \\ = \ & \int_{q_{v0}}^1 \int_0^{p(q_v)} \left[R_v(q_v, q_s) - \bar{R}_v(q_v)\right] \mathrm{d}q_s \mathrm{d}q_v \\ & + \int_{q_{v0}}^1 \int_{p(q_v)}^1 \left[R_v(q_v, q_s) - \bar{R}_v(p^{-1}(q_s))\right] \mathrm{d}q_s \mathrm{d}q_v \\ & + \int_{q_{v0}}^1 \int_0^{p(q_v)} \bar{R}_v(q_v) \mathrm{d}q_s \mathrm{d}q_v + \int_{q_{v0}}^1 \int_{p(q_v)}^1 \bar{R}_v(p^{-1}(q_s)) \mathrm{d}q_s \mathrm{d}q_v. \end{aligned}$$

Let us examine the first two double integrals on the right-hand side. Recall that $R_v(q_v, q_s)$ is decreasing in (q_v, q_s). Because $R_v(q_v, q_s) \geq R_v(q_v, p(q_v))$ for all $q_s \leq p(q_v)$, the first double integral on the right-hand side above is positive. It is equal to 0 only if $R_v(q_v, q_s) = R_v(q_v, p(q_v))$ for all $(q_v, q_s) \in [q_{v0}, 1] \times [0, p(q_v)]$. By the definition of R_v, this requirement is equivalent to $r_v(q_v, q_s) = 0$ almost surely for all $(q_v, q_s) \in [q_{v0}, 1] \times [0, p(q_v)]$. In this case, each type q_v is matched only with $p(q_s)$ and with some probability a. That is, $r_v(q_v, q_s) = a r_v^{ps}(q_v, q_s)$ for all $q_v \geq q_{v0}$ and $q_s \in [0, 1]$. By the identity (2.2), the integrand in the second double integral on the right-hand side is equal to $\theta \left[R_s(q_s, q_v) - R_s(q_s, p^{-1}(q_s))\right]$. Because $R_s(q_s, q_v) \geq R_s(q_s, p^{-1}(q_s))$ for all $q_s \geq p(q_v)$, this integral is also positive. It is equal to 0 only if $R_s(q_s, q_v) = R_s(q_s, p^{-1}(q_s))$ for all $(q_s, q_v) \in [0, 1] \times [q_{v0}, p^{-1}(q_s)]$. This requires

$r_s(q_s, q_v) = 0$ almost surely for all $(q_s, q_v) \in [0,1] \times [q_{v0}, p^{-1}(q_s)]$. That is, each q_s is matched only with $p^{-1}(q_s)$ and with some probability a. Thus, the first two double integrals on the right-hand side above are positive. They are strictly positive under the assumption in the proposition that $r_\tau(q_\tau, q) \neq a r_\tau^{ps}(q_\tau, q)$ for a positive measure of $(q_\tau, q) \in [0,1]^2$. Therefore, B is bounded as

$$B > \frac{a}{2}\theta(1-\theta) + \int_{q_{v0}}^{1}\int_{0}^{p(q_v)} \bar{R}_v(q_v)\mathrm{d}q_s\mathrm{d}q_v + \int_{q_{v0}}^{1}\int_{p(q_v)}^{1} \bar{R}_v(p^{-1}(q_s))\mathrm{d}q_s\mathrm{d}q_v. \tag{C.4}$$

I compute the two double integrals in (C.4). Note that $p(q) = \frac{q-q_{v0}}{\theta}$ because $q_{v0} = 1-\theta$. The first double integral in (C.4) is:

$$\int_{q_{v0}}^{1}\int_{0}^{p(q_v)} \bar{R}_v(q_v)\mathrm{d}q_s\mathrm{d}q_v = \frac{1}{\theta}\int_{q_{v0}}^{1} (q-q_{v0})\,\bar{R}_v(q)\mathrm{d}q.$$

For the outer integral on q_v in the second double integral in (C.4), I integrate by parts, with $(q_v - q_{v0})$ and $\int_{p(q_v)}^{1} \bar{R}_v(p^{-1}(q_s))\mathrm{d}q_s$ as the parts, and using $\bar{R}_v(1) = 0$:

$$\int_{q_{v0}}^{1}\int_{p(q_v)}^{1} \bar{R}_v(p^{-1}(q_s))\mathrm{d}q_s\mathrm{d}q_v = \frac{1}{\theta}\int_{q_{v0}}^{1} (q_v-q_{v0})\,\bar{R}_v(q_v)\mathrm{d}q_v.$$

Substituting the above results into (C.4) generates:

$$B > \frac{2}{\theta}\int_{q_{v0}}^{1} (q-q_{v0})\,\bar{R}_v(q)\mathrm{d}q + \frac{a}{2}\theta(1-\theta). \tag{C.5}$$

Substituting this lower bound on B into (C.2), I have:

$$\rho > \frac{24}{\theta^3 a}\int_{q_{v0}}^{1} (q-q_{v0})\,\bar{R}_v(q)\mathrm{d}q - 3. \tag{C.6}$$

I choose the function $\bar{R}_v(q)$ to minimize the integral in (C.6). The choice of $\bar{R}_v(q)$ must be feasible. For all $(q, q_s) \in [0,1]^2$, $0 \leq r_v(q, q_s) \leq r_v(q, 0) = a$, and so $0 \leq R_v(q, q_s) \leq (1-q)\,a$. This implies $0 \leq \bar{R}_v(q) \leq (1-q)\,a$ for all $q \geq q_{v0}$, which is the feasibility constraint on $\bar{R}_v(q)$. Another constraint is $\sigma_U \geq \sigma_0$, where $\sigma_0 \in (-2, 3a-2]$. With σ_U in (3.12), this constraint is the last one in the following minimization problem:

$$\begin{aligned} & min_{\{\bar{R}_v(q)\}} \int_{q_{v0}}^{1} (q-q_{v0})\,\bar{R}_v(q)\mathrm{d}q \\ \text{s.t.}\quad & 0 \leq \bar{R}_v(q) \leq (1-q)\,a \quad \text{for all } q \in [q_{v0}, 1], \\ & \theta\int_{q_{v0}}^{1} \bar{R}_v(q)\mathrm{d}q \geq \frac{\theta^3}{6}(\sigma_0+2). \end{aligned}$$

There is an additional constraint that $\bar{R}_v(q)$ must be a decreasing function. As shown below, this constraint is not binding. Let λ be the Lagrangian multiplier of the last constraint above. Assume that $0 < \lambda \leq 1$, which will be verified later. Then, $q_{v0} + \lambda\theta \leq 1$. It is straightforward to verify that the solution to the minimization problem is:

$$\bar{R}_v(q) = \begin{cases} (1-q)\,a, & \text{if } q_{v0} \leq q < q_{v0} + \lambda\theta \\ 0, & \text{if } q_{v0} + \lambda\theta \leq q \leq 1. \end{cases}$$

This solution is a decreasing function of q, indeed. Substituting the solution into the last constraint of the minimization problem yields:

$$\frac{\theta^2}{6}(\sigma_0 + 2) = a \int_{q_{v0}}^{q_{v0}+\lambda\theta} (1-q)\,\mathrm{d}q = \frac{a\theta^2}{2}\left[1 - (1-\lambda)^2\right].$$

This solves:

$$\lambda = 1 - \left(1 - \frac{\sigma_0 + 2}{3a}\right)^{\frac{1}{2}}. \tag{C.7}$$

For any $\sigma_0 \in (-2, 3a - 2]$, this solution is a real number that satisfies $0 < \lambda \leq 1$. The minimum of the objective function is:

$$\int_{q_{v0}}^{q_{v0}+\lambda\theta} (q - q_{v0})\,a\,(1-q)\,\mathrm{d}q = \frac{a\theta^3}{6}\left[1 - 3(1-\lambda)^2 + 2(1-\lambda)^3\right].$$

Substituting the above bound on the minimum into (C.6), I have:

$$\rho > 1 - 12(1-\lambda)^2 + 8(1-\lambda)^3. \tag{C.8}$$

Because $\lambda > 0$, the constraint $\sigma_U \geq \sigma_0$ is binding. Then, $\sigma_U = \sigma_0 = -2 + 3a\left[1 - (1-\lambda)^2\right]$ by (C.7). With this value of σ_U, (**??**) implies:

$$\rho - \sigma_U > 3(1-a) - 3(4-a)(1-\lambda)^2 + 8(1-\lambda)^3 \tag{C.9}$$

Since $\lambda \in (0,1)$, then $(1-\lambda)^3 \geq (1-\lambda)^4$, and so

$$\rho - \sigma_U > 3(1-a) - 3(4-a)(1-\lambda)^2 + 8(1-\lambda)^4. \tag{C.10}$$

The expression on the right-hand side of (C.10) is a quadratic function of $(1-\lambda)^2$. It is equal to $3(1-a) \geq 0$ at $(1-\lambda)^2 = 0$, and to -1 (< 0) at $(1-\lambda)^2 = 1$. The smaller root of the function for $(1-\lambda)^2$ lies in $(0,1)$. The expression on the right-hand side of (C.10) is positive if and only if $(1-\lambda)^2$ is less than or equal to this smaller root, i.e.,

$$(1-\lambda)^2 \leq \frac{1}{16}\left\{3(4-a) - \left[3\left(3a^2+8a+16\right)\right]^{\frac{1}{2}}\right\}.$$

Substituting λ from (C.7) into this condition yields (4.8). It can be verified that the bound on σ_0 in (4.8) lies in $(-2, 3a-2]$ and is increasing in a. It is equal to 1 at $a=1$, to 0.241 at $a=0.8$, and to -0.745 at $a=0.5$. **QED**

C.3 Proof of Corollary 2

With uniform matching failures, the analysis prior to Corollary 2 has shown that PQD implies larger values of σ and σ_U. The remainder of this appendix proves that PQD cannot rank sorting between any two positive block matching patterns with k and $k+1$ blocks, where $k \geq 2$. For simplicity, consider the case $\theta = 1$ and $a = 1$, in which $q_{v0} = 0$ and $p(q) = q$ for all $q \in [0,1]$. For any $i \in \{1, 2, ..., k\}$ and any $q_v \in [\frac{i-1}{k}, \frac{i}{k})$, (A.1) in Appendix A.2 implies:

$$r_v(q_v, q) = \begin{cases} 1, & \text{if } q < \frac{i-1}{k} \\ i - kq, & \text{if } q \in [\frac{i-1}{k}, \frac{i}{k}) \\ 0, & \text{if } q \geq \frac{i}{k}. \end{cases} \tag{C.11}$$

Let q_v lie in the ith block and denote the location of q_v in the ith block as $z = kq_v - i + 1$. Then $z \in [0,1)$. If $i = k$, then $z = 1$ is also possible. Using (C.11) and $p(q) = q$, I derive:

$$\bar{R}_v(q) = \frac{1}{k}(1-z)^2 + 1 - \frac{i}{k} \text{ for } q = \frac{i-1+z}{k}. \tag{C.12}$$

For any $k \geq 2$, I prove that there exist regions of q in which the ranking in $\bar{R}_v(q)$ between matching patterns with k and $k+1$ blocks reverses as q changes in those regions. By (5.1), $\mathcal{M}(q,q)$ cannot be ranked between block matching patterns with k and $k+1$ blocks. That is, PQD cannot rank these two matching patterns.

For any $k \geq 2$, consider block matching patterns with k and $k+1$ blocks. Let $j \in \{k, k+1\}$. When there are j blocks, let i_j be the block in which q lies, the location of q in the i_jth block be $z_j = jq - i_j + 1$, and the function $\bar{R}_v(q)$ be $\bar{R}_{v,j}(q)$. (C.12) yields:

$$\bar{R}_{v,j}(q) = \frac{1}{j}(1-z_j)^2 + 1 - \frac{i_j}{j} \text{ for } q = \frac{i_j - 1 + z_j}{j}.$$

I prove that either $i_{k+1} = i_k$ or $i_{k+1} = i_k + 1$. Because i_k and i_{k+1} are integers, it suffices to prove that $i_k \leq i_{k+1} \leq i_k + 1$. Because $\frac{i_j - 1 + z_j}{j} = q$ for both $j = k$ and $j = k + 1$, then,

$$i_{k+1} - 1 = \frac{k+1}{k}\left(i_k - 1 + z_k\right) - z_{k+1}. \tag{C.13}$$

If $q = 1$, then $i_k = k$ and $z_k = 1$. In this case, $i_{k+1} = k + 2 - z_{k+1} \geq k + 1$. Because $i_{k+1} \leq k + 1$ must hold when the number of blocks is $k + 1$, then $i_{k+1} = k + 1 = i_k + 1$. Suppose $q < 1$. Then $z_k < 1$. Since $z_{k+1} \geq 0$, $z_k < 1$ and $i_k \leq k$, then (C.13) implies:

$$i_{k+1} - 1 < i_k + \frac{i_k}{k} \leq i_k + 1.$$

That is, $i_{k+1} < i_k + 2$. Because i_k and i_{k+1} are integers, the strict inequality is satisfied if and only if $i_{k+1} \leq i_k + 1$. Similarly, because $z_k \geq 0$ and $z_{k+1} \leq 1$, (C.13) implies:

$$i_{k+1} - 1 \geq \frac{k+1}{k}\left(i_k - 1\right) - 1 \geq i_k - 2.$$

The second inequality is strict if $i_k > 1$. If $i_k = 1$ and $z_k > 0$, the first inequality above is strict. Thus, if $i_k > 1$ or $z_k > 0$, then $i_{k+1} > i_k - 1$. Because i_k and i_{k+1} are integers, the strict inequality implies $i_{k+1} \geq i_k$. If $i_k = 1$ and $z_k = 0$, then $q = 0$. In this case, $i_{k+1} = 1 = i_k$. Therefore, $i_k \leq i_{k+1} \leq i_k + 1$ in all cases.

Consider first the case $i_{k+1} = i_k$ $(= i)$. The relationship (C.13) becomes:

$$z_{k+1} = \frac{k+1}{k}\left(i - 1 + z_k\right) - i + 1.$$

Substituting into $\bar{R}_{v,k+1}$, I compute:

$$\bar{R}_{v,k+1}(q) - \bar{R}_{v,k}(q) = \frac{1}{k+1}\left(i - \frac{k+1}{k}\left(i - 1 + z_k\right)\right)^2 - \frac{1}{k}\left(1 - z_k\right)^2 + \frac{i}{k\left(k+1\right)}.$$

This difference is increasing in z_k. At $z_k = 0$, this difference is equal to:

$$-\frac{i-1}{k\left(k+1\right)}\left(1 - \frac{i-1}{k}\right).$$

This is strictly negative if $1 < i < k + 1$. Because $k \geq 2$, the case $i > 1$ exists. Thus, $\bar{R}_{v,k+1}(q) < \bar{R}_{v,k}(q)$ for $q = \frac{i-1}{k}$. At $z_k = 1$, the difference in $\bar{R}_v$ is strictly positive. That is, $\bar{R}_{v,k+1}(q) > \bar{R}_{v,k}(q)$ for $q = \frac{i}{k}$. The ranking in $\bar{R}_v(q)$ between the two matching patterns reverses as q increases in the ith block.

Next consider the case $i_{k+1} = i_k + 1$ $(= i + 1)$. The relationship (C.13) becomes:

$$z_{k+1} = \frac{k+1}{k}\left(i - 1 + z_k\right) - i.$$

Substituting into $\bar{R}_{v,k+1}$, I compute:

$$\bar{R}_{v,k+1}(q) - \bar{R}_{v,k}(q) = \frac{1}{k+1}\left(1 + i - \frac{k+1}{k}\left(i - 1 + z_k\right)\right)^2 - \frac{1}{k}\left(1 - z_k\right)^2 - \frac{k-i}{k\left(k+1\right)}.$$

This difference is decreasing in z_k. At $z_k = 1$, this difference is equal to:

$$-\frac{i}{k\left(k+1\right)}\left(1 - \frac{i}{k}\right).$$

This is strictly negative if $i < k$. Thus, $\bar{R}_{v,k+1}(q) < \bar{R}_{v,k}(q)$ for $q = \frac{i}{k}$. At $z_k = 0$, the difference in $\bar{R}_v$ is strictly positive for $i < k + 1$. That is, $\bar{R}_{v,k+1}(q) > \bar{R}_{v,k}(q)$ for $q = \frac{i-1}{k}$. Again, the ranking in $\bar{R}_v(q)$ between the two matching patterns reverses as q increases in the ith block. **QED**

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