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The Stochastic Matrix and Linear Programming

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Abstract: We take a stochastic matrix (or Markov matrix) and place the matrix in a linear programming framework. The dual program is in a sense a novel "completion" of the stochastic matrix formulation. We identify the primal linear program (LP) as a "quantity" program (based on a key eigenvalue) and the dual program as a "price" program (turning on an eigenvalue of the transpose matrix). Our approach is to present detailed numerical examples, examples based on particular 3 x 3 stochastic matrices. We do not present new types of evolution "descending from" a stochastic matrix. The linear programming framework provides a novel way to envisage a stochastic matrix and its transpose.

Key words: stochastic matrices; linear programming; the transpose.

0.1 Introduction

We report on the links between linear programming and the structure of stochastic matrices. The dual linear program stands as a "price" problem and the primal problem stands as a "quantity" problem. A stochastic matrix or Markov matrix figures centrally in some analyses of Darwinian evolution.[1] We however restrict our attention here to aspects of the stochastic matrix *per se*. A stochastic matrix B is square, *n x n*, and has non-negative entries, with each column having *n* entries that sum to unity. Matrix *B* has *n* eigenvectors, one of which, namely x^*, has the property $Bx^* = x^*$. x^* can be referred to as a steady state vector. The transpose matrix of B, namely, B^T, has *n* distinct eigenvectors, one of which has *n* identical entries. These two distinct eigenvectors play a central role in the solution to a primal and dual pair of linear programs. We set out two 3 x 3 matrices possessing the properties of stochastic matrices and observe the roles of the distinct eigenvectors for each example. Our exposition of linear programs and stochastic matrices here serves as a supplement to the valuable exposition of the stochastic matrix set out online (U of Georgia, (textbooks.math.gatech.edu/ila/stochastic -matrices.html.))

0.2 Analysis

We work first with the following 3x3 example of a stochastic matrix, introduced online in (U of Georgia reference).

$$A = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}$$

A has three eigenvectors, (0.6674, 0.5721, 0.4767), (0.707, 0.00, -0.707), and (-0.2673, 0.802, -0.534); and three corresponding eigenvalues (1, -0.2, 0.1) (obtained with Maple software, Eigenvectors). The first eigenvector, (0.6674, 0.5721, 0.4767), is, for our stochastic matrix, special, as becomes apparent below. The transpose matrix to A is

$$A^T = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$$

A^T has the same eigenvalues as matrix A and has eigenvectors (0.577,0.577,0.577),

[1] See for example Nowak (2006; pp. 23-24) for such a formalization of evolution by natural selection. Lawler, (2006, page 60) takes up two problems in population dynamics. Kerner and Aldrovandi (2009) construct sequences of random variables that they "aggregate" into models based on stochastic matrices. See our later section.

(0.688, -0.2294, -0.688); and (-0.408, 0.816, -0.408) (obtained with Maple software, Eigenvectors).

0.3 A Linear program

We turn to a linear program with a primal program built around matrix A. Consider solving for non-negative vector (x_1, x_2, x_3) that maximizes $c_1x_1 + c_2x_2 + c_3x_3$ subject to

$$\begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

for $(c_1, c_2, c_3) = (0.577, 0.577, 0.577)$ and (D_1, D_2, D_3) = (0.6674, 0.5721, 0.4767). This linear program (LP) has solution x_1=0.6673, x_2= 0.5724, x_3=0.4765 (solved with Maple software, "LPSolver"). The value of the objective function is 0.99025.

Our dual linear programming problem is: solve for non-negative vector (p_1, p_2, p_3) that maximizes $D_1p_1 + D_2p_2 + D_3p_3$ subject to

$$\begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.4 & 0.4 & 0.2 \\ 0.5 & 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \leq \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

for (c_1, c_2, c_3) and (D_1, D_2, D_3) defined above. This dual program has solution p_1=0.577, p_2= 0.5769, p_3=0.577. The value of the objective function is 0.99025.

Our central observation is that two certain eigenvectors of the stochastic matrix and its transpose "nest" naturally in a linear program, both primal and dual. The solution to each LP has each of these eigenvectors appear in the solution.

0.4 A Second 3 x 3 example

We work with the following 3 x 3 example of a stochastic matrix, introduced online in (U of Georgia reference).

$$B = \begin{bmatrix} 0.1 & 0.2 & 0.8 \\ 0.2 & 0.2 & 0.1 \\ 0.7 & 0.6 & 0.1 \end{bmatrix}$$

B has three eigenvectors: (0.6734, 0.2551, 0.6938), (0.7538, -0.1053, -0.6486), and

(-0.6373, 0.7607, -0.1234); and three corresponding eigenvalues (0.999, -0.6162, 0.0162). The transpose matrix to B is

$$B^T = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.8 & 0.1 & 0.1 \end{bmatrix}$$

B^T has the same eigenvalues as matrix B and has eigenvectors (0.57735, 0.57735, 0.57735), (0.5905, 0.3791, -0.6486), and (-0.6373, 0.7607, -0.1234).

We turn to a linear program with a primal program built around matrix B. Consider solving for non-negative vector (x_1, x_2, x_3) that maximizes $c_1x_1+c_2x_2 + c_3x_3$ subject to

$$\begin{bmatrix} 0.1 & 0.2 & 0.8 \\ 0.2 & 0.2 & 0.1 \\ 0.7 & 0.6 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

for $(c_1, c_2, c_3) = (0.57735, 0.57735, 0.57735)$ and (D_1, D_2, D_3) = (0.6734, 0.2551, 0.6938). This LP has solution x_1=0.6722, x_2= 0.2565, x_3=0.6936. The value of the objective function is 0.9366349.

Our dual linear programming problem is: solve for non-negative vector (p_1, p_2, p_3) that maximizes $D_1p_1+D_2p_2 + D_3p_3$ subject to

$$\begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.2 & 0.2 & 0.6 \\ 0.8 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \leq \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

for (c_1, c_2, c_3) and (D_1, D_2, D_3) defined above. This dual program has solution p_1=0.57735, p_2= 0.57735, p_3=0.57735. The value of the objective function is 0.936635.

0.5 Kerner and Aldrovandi on Sequences of Random Variables

Kerner and Aldrovandi (2009) present a model that gets "summarized" in a stochastic matrix. Their goal was to link processes, a premodel, to a summarizing framework, namely a stochastic matrix. The premodel was a collection of stochastic sequences constructed with two random variables, *L* and *S*. The way that the chains were constructed and related to each other is difficult to summarize in brief, but there emerges a "summary model"

$$\begin{bmatrix} \boldsymbol{M_{LL}} & \mathbf{1} \\ \boldsymbol{M_{SL}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_L} \\ \boldsymbol{x_S} \end{bmatrix} = \begin{bmatrix} \boldsymbol{D_L} \\ \boldsymbol{D_S} \end{bmatrix} \text{ with } \boldsymbol{M_{LL}} + \boldsymbol{M_{SL}} = \mathbf{1} \text{ and } \boldsymbol{x_S} = \mathbf{1} - \boldsymbol{x_L}.$$

The normalized probabilities of the limit distribution are $\boldsymbol{D_L = 1/(1 + M_{SL})}$ and $\boldsymbol{D_S = M_{SL}/(1 + M_{SL})}$. The model can be "summarized" as solving for $\boldsymbol{x_L}$ and $\boldsymbol{x_S}$ that minimize $\boldsymbol{c_L x_L + c_S x_S}$ subject to

$$\begin{bmatrix} \boldsymbol{M_{LL}} & \boldsymbol{1} \\ \boldsymbol{M_{SL}} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_L} \\ \boldsymbol{x_S} \end{bmatrix} \geq \begin{bmatrix} \boldsymbol{D_L} \\ \boldsymbol{D_S} \end{bmatrix} \quad \text{for } \boldsymbol{c_L = c_S = 1}.$$

From our analysis above with numerical examples, we know that $(\boldsymbol{D_L}, \boldsymbol{D_S})$ here is an eigenvector of the matrix $\begin{bmatrix} \boldsymbol{M_{LL}} & \boldsymbol{1} \\ \boldsymbol{M_{SL}} & \boldsymbol{0} \end{bmatrix}$. The solution is $\boldsymbol{x_L = 1/(1 + M_{SL})}$ and $\boldsymbol{x_S = M_{SL}/(1 + M_{SL})}$; that is, $\boldsymbol{x_L = D_L}$ and $\boldsymbol{x_S = D_S}$.

The dual linear program is: solve for $\boldsymbol{z_L}$ and $\boldsymbol{z_S}$ that maximize $\boldsymbol{D_L z_L + D_S z_S}$ subject to

$$\begin{bmatrix} \boldsymbol{M_{LL}} & \boldsymbol{M_{SL}} \\ \boldsymbol{1} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{z_L} \\ \boldsymbol{z_S} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{c_L} \\ \boldsymbol{c_S} \end{bmatrix}$$

We continue to have $\boldsymbol{c_L = c_S = 1}$. $(\boldsymbol{c_L}, \boldsymbol{c_S})$ is an eigenvector of $\begin{bmatrix} \boldsymbol{M_{LL}} & \boldsymbol{M_{SL}} \\ \boldsymbol{1} & \boldsymbol{0} \end{bmatrix}$. The maximizing values are $\boldsymbol{z_L = z_S = 1}$.

0.6 Concluding Remarks

We observe how a stochastic matrix and its transpose fit naturally into a primal and dual linear program. The link from the matrices to the linear program turns on key eigenvectors, one for the stochastic matrix and the other for the transpose of the stochastic matrix. This is interesting linear algebra and has good expositional content for the subject of stochastic matrices. Our contribution is providing of supplementary material to the excellent expositional work in (U of Georgia, (textbooks.math.gatech.edu/ila/stochastic - matrices.html)).

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